Parameterized Complexity of Finding Small Degree-constrained Subgraphs⋆

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Abstract

In this article we study the parameterized complexity of problems consisting in finding degree-constrained subgraphs, taking as the parameter the number of vertices of the desired subgraph. Namely, given two positive integers d and k, we study the problem of finding a d-regular (induced or not) subgraph with at most k vertices and the problem of finding a subgraph with at most k vertices and of minimum degree at least d. The latter problem is a natural parameterization of the *d-girth* of a graph (the minimum order of an induced subgraph of minimum degree at least d).

We first show that both problems are fixed-parameter intractable in general graphs. More precisely, we prove that the first problem is $W[1]$-hard using a reduction from Multi-Color Clique. The hardness of the second problem (for the non-induced case) follows from an easy extension of an already known result. We then provide explicit fixed-parameter tractable (FPT) algorithms to solve these problems in graphs with bounded local treewidth and graphs with excluded minors, using a dynamic programming approach. Although these problems can be easily defined in first-order logic, hence by the results of Frick and Grohe [23] are FPT in graphs with bounded local treewidth and graphs with excluded minors, the dependence on k of our algorithms is considerably better than the one following from [23].

Keywords: parameterized complexity, degree-constrained subgraph, fixed-parameter tractable algorithm, $W[1]$-hardness, treewidth, dynamic programming, excluded minors.

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1 Introduction

Problems of finding subgraphs with certain degree constraints are well studied both algorithmically and combinatorially, and have a number of applications in network design (cf. for instance [1, 20, 25, 29, 35]). In this article we consider two natural such problems: finding a small regular (induced or not) subgraph and finding a small subgraph with given minimum degree. We discuss in detail these two problems in Sections 1.1 and 1.2, respectively.

1.1 Finding a small regular subgraph

The complexity of finding regular graphs as well as regular (induced) subgraphs has been intensively studied in the literature [6–8, 11, 24, 30, 31, 35, 36]. One of the first problems of this kind was stated by Garey and Johnson: Cubic Subgraph, that is, the problem of deciding whether a given graph contains a 3-regular subgraph, is NP-complete [11]. More generally, the problem of deciding whether a given graph contains a d-regular subgraph for any fixed degree $d \geq 3$ is NP-complete on general graphs [8] as well as in planar graphs [36] (where in the latter case only $d = 4$ and $d = 5$ were considered, since any planar graph contains a vertex of degree at most 5). For $d \geq 3$, the problem remains NP-complete even in bipartite graphs of degree at most $d+1$ [33]. Note that this problem is clearly polynomial-time solvable for $d \leq 2$. If the regular subgraph is required to be induced, Cardoso *et al.* proved that finding a maximum cardinality d-regular induced subgraph is NP-complete for any fixed integer $d \geq 0$ [7] (for $d = 0$ and $d = 1$ the problem corresponds to Maximum Independent Set and Maximum Induced Matching, respectively).

Concerning the parameterized complexity of finding regular subgraphs, Moser and Thilikos proved that the following problem is $W[1]$-hard for every fixed integer $d \geq 0$ [31]:

> $\geq k$-size d-Regular Induced Subgraph
> **Input:** A graph $G = (V, E)$ and a positive integer k.
> **Parameter:** k.
> **Question:** Does there exist a subset $S \subseteq V$, with $|S| \geq k$, such that $G[S]$ is d-regular?

On the other hand, the authors proved that the following problem (which can be seen as the dual of the above one) is NP-complete but has a problem kernel of size $\mathcal{O}(kd(k+d)^2)$ for $d \geq 1$ [31]:

> $\leq k$-Almost d-Regular Graph
> **Input:** A graph $G = (V, E)$ and a positive integer k.
> **Parameter:** k.
> **Question:** Does there exist a subset $S \subseteq V$, with $|S| \leq k$, such that $G[V \setminus S]$ is d-regular?

Mathieson and Szeider studied in [30] variants and generalizations of the problem of finding a d-regular subgraph (for $d \geq 3$) in a given graph by deleting at most k vertices. In particular, they answered a question of [31], proving that the $\leq k$-Almost d-Regular Graph problem (as well as some variants) becomes $W[1]$-hard when parameterized only by k (that is, it is unlikely that there exists an algorithm to solve it in time $f(k) \cdot n^{\mathcal{O}(1)}$, where $n = |V(G)|$ and f is a function independent of n and d).

Given two integers d and k, it is also natural to ask for the existence of an induced d-regular graph with at most k vertices. The corresponding parameterized problem is defined as follows.

> $\leq k$-SIZE d-REGULAR INDUCED SUBGRAPH (kdRIS)
> **Input:** A graph $G = (V, E)$ and a positive integer k.
> **Parameter:** k.
> **Question:** Does there exist a subset $S \subseteq V$, with $|S| \leq k$, such that $G[S]$ is d-regular?

Note that the hardness of $\leq k$-SIZE d-REGULAR INDUCED SUBGRAPH does not follow directly from the hardness of $\geq k$-SIZE d-REGULAR INDUCED SUBGRAPH as, for instance, the approximability of the problems of finding a densest subgraph on *at least* k vertices or on *at most* k vertices are significantly different [3]. In general, a graph may not contain an induced d-regular subgraph on at most k vertices, while containing a non-induced d-regular subgraph on at most k vertices. This observation leads to the following problem:

> $\leq k$-SIZE d-REGULAR SUBGRAPH (kdRS)
> **Input:** A graph $G = (V, E)$ and a positive integer k.
> **Parameter:** k.
> **Question:** Does there exist a d-regular subgraph $H \subseteq G$, with $|V(H)| \leq k$?

Observe that $\leq k$-SIZE d-REGULAR SUBGRAPH could a priori be easier than its corresponding induced version, as it happens for the MAXIMUM MATCHING (which is in P) and the MAXIMUM INDUCED MATCHING (which is NP-hard) problems.

The two parameterized problems defined above have not been considered in the literature. We prove in Section 2 that both problems are $W[1]$-hard for every fixed $d \geq 3$, by reduction from MULTI-COLOR CLIQUE.

1.2 Finding a small subgraph with given minimum degree

For a finite, simple, and undirected graph $G = (V, E)$ and $d \in \mathbb{N}$, the d-girth $g_d(G)$ of G is the minimum order of an induced subgraph of G of minimum degree at least d. The notion of d-girth was proposed and studied by Erdős *et al.* [18, 19] and Bollobás and Brightwell [5]. It generalizes the usual girth, the length of a shortest cycle, which coincides with the 2-girth. (This is indeed true because every induced subgraph of minimum degree at least two contains a cycle.) Combinatorial bounds on the d-girth can also be found in [4, 27]. The corresponding optimization problem has been recently studied in [1], where it has been proved that for any fixed $d \geq 3$, the d-girth of a graph cannot be approximated within any constant factor, unless P = NP [1]. From the parameterized complexity point of view, it is natural to introduce a parameter $k \in \mathbb{N}$ and ask for the existence of a subgraph with at most k vertices and with minimum degree at least d. The problem can be formally defined as follows.

$\leq k$-SIZE SUBGRAPH OF MINIMUM DEGREE $\geq d$ (kSMDd) **Input:** A graph $G = (V, E)$ and a positive integer k. **Parameter:** k. **Question:** Does there exist a subset $S \subseteq V$, with $\|S\| \leq k$, such that $G[S]$ has minimum degree at least d?

Note that the case $d = 2$ in in P, as discussed above. The special case of $d = 4$ appears in the book of Downey and Fellows [15, page 457], where it is announced that H.T. Wareham proved that kSMD4 is $W[1]$-hard. (However, we were not able to find a proof.) From this result, it is easy to prove that kSMDd is $W[1]$-hard for every fixed $d \geq 4$ (see Section 2). The complexity of the case $d = 3$ remains open (see Section 4). Note that in the kSMDd problem we can assume without loss of generality that we are looking for the existence of an induced subgraph, since we only require the vertices to have degree at least d.

Besides the above discussion, another motivation for studying the kSMDd problem is its close relation to the well studied DENSE k-SUBGRAPH problem [3,14,20,28], which we proceed to explain. The *density* $\rho(G)$ of a graph $G = (V, E)$ is defined as $\rho(G) := \frac{|E|}{|V|}$. More generally, for any subset $S \subseteq V$, we denote its *density* by $\rho(S)$, and define it to be $\rho(S) := \rho(G[S])$. The DENSE k-SUBGRAPH problem is formulated as follows:

DENSE k-SUBGRAPH (DkS) **Input:** A graph $G = (V, E)$. **Output:** A subset $S \subseteq V$, with $\|S\| = k$, such that $\rho(S)$ is maximized.

Understanding the complexity of DkS remains widely open, as the gap between the best hardness result (APX-hardness [28]) and the best approximation algorithm (with ratio $\mathcal{O}(n^{1/3-\varepsilon})$ [20]) is huge. Suppose we are looking for an induced subgraph $G[S]$ of size at most k and with density at least ρ. In addition, assume that S is minimal, i.e,. no subset of S has density greater than $\rho(S)$. This implies that every vertex of S has degree at least $\rho/2$ in $G[S]$. To see this, observe that if there is a vertex v with degree strictly smaller than $\rho/2$, then removing v from S results in a subgraph of density greater than $\rho(S)$ and of smaller size, contradicting the minimality of S. Secondly, if we have an induced subgraph $G[S]$ of minimum degree at least ρ, then S is a subset of density at least $\rho/2$. These two observations together show that, modulo a constant factor, looking for a densest subgraph of G of size at most k is equivalent to looking for the largest possible value of d for which kSMDd returns YES. As the degree conditions are more rigid than the global density of a subgraph, a better understanding of the kSMDd problem could provide an alternative way to approach the DkS problem.

Finally, we would like to point out that the kSMDd problem has practical applications to traffic grooming in optical networks. Traffic grooming refers to packing small traffic flows into larger units then can then be processed as single entities. For example, in a network using both time-division and wavelength-division multiplexing, flows destined to a common node can be aggregated into the same wavelength, allowing them to be dropped by a single optical Add-Drop Multiplexer. The main objective of grooming is to minimize the equipment cost of the network, which is mainly given in Wavelength-Division Multiplexing optical networks by the number of electronic terminations. (We refer, for instance, to [16] for a general survey on grooming.) It has been recently proved by Amini, Pérennes and Sau [2] that the TRAFFIC GROOMING problem in optical networks can be reduced (modulo polylogarithmic factors) to DkS, or equivalently to kSMDd. Indeed, in graph theoretic terms, the problem can be translated into partitioning the edges of a given request graph into subgraphs with a constraint on their number of edges. The objective is then to minimize the total number of vertices of the subgraphs of the partition. Hence, in this context of partitioning a given set of edges while minimizing the total number of vertices, the problems of DkS and kSMDd come into play. More details can be found in [2].

1.3 Presentation of the results

We do a thorough study of the kdRS, the kdRIS, and the kSMDd problems in the realm of parameterized complexity, which is a recent approach to deal with intractable computational problems having some parameters that can be relatively small with respect to the input size. This area has been developed extensively during the last decade (the monograph of Downey and Fellows [15] provides a good introduction, and for more recent developments see the books by Flum and Grohe [22] and by Niedermeier [32]).

For decision problems with input size n and parameter k, the goal is to design an algorithm with running time $f(k)n^{O(1)}$, where f depends only on k. Problems having such an algorithm are said to be *fixed-parameter tractable* (FPT). There is also a theory of parameterized intractability to identify parameterized problems that are unlikely to admit fixed-parameter tractable algorithms. There is a hierarchy of intractable parameterized problem classes above FPT, the important ones being:

$$FPT \subseteq M[1] \subseteq W[1] \subseteq M[2] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq XP.$$

The principal analogue of the classical intractability class NP is $W[1]$, which is a strong analogue, because a fundamental problem complete for $W[1]$ is the k-STEP HALTING PROBLEM FOR NONDETERMINISTIC TURING MACHINES (with unlimited nondeterminism and alphabet size); this completeness result provides an analogue of Cook's Theorem in classical complexity. A convenient source of $W[1]$-hardness reductions is provided by the result stating that k-CLIQUE is complete for $W[1]$. The principal "working algorithmic" way of showing that a parameterized problem is unlikely to be fixed-parameter tractable, is to prove its $W[1]$-hardness using a parameterized reduction (defined in Section 2).

Our results can be classified into two categories:

General graphs: We show in Section 2 that kdRS is not fixed-parameter tractable by showing it to be $W[1]$-hard for any $d \geq 3$ in general graphs. We will see that the graph constructed in our reduction implies also the $W[1]$-hardness of kdRIS. In general, parameterized reductions are quite stringent because of parameter-preserving requirements of the reduction, and require some technical care. Our reduction is based on a new methodology emerging in parameterized complexity, called *multi-color clique edge representation*. This has proved to be useful in showing various problems to be $W[1]$-hard recently [9]. We first spell out step by step the procedure to use this methodology, which can be used as a template for future purposes. Then we adapt this methodology to the reduction for the kSMDd problem. The hardness of kSMDd for $d \geq 4$ follows from an easy extension of a result of H.T. Wareham [15, page 457].

Graphs with bounded local treewidth and graphs with excluded minors: Both the kSMDd and kdRS problems can be easily defined in first-order logic, where the formula only depends on k and d, both being bounded by the parameter. Frick and Grohe [23] have shown that first-order definable properties of graph classes of bounded local treewidth can be decided in time $\mathcal{O}(n^{1+1/\ell})$ for every positive integer ℓ, in particular in time $\mathcal{O}(n^2)$, and first-order model checking is FPT on M-minor-free graphs. This immediately gives us the *classification result* that both problems are FPT in graphs with bounded local treewidth and graphs excluding a fixed graph M as a minor. These classification results can be generalized to a larger class of graphs, namely graphs locally excluding a fixed graph M as a minor, by a recent result of Dawar, Grohe and Kreutzer [12]. These results are by nature very general and can involve huge coefficients (dependence on k). A natural problem arising in this context is then the design of an explicit algorithm for kSMDd for $d \geq 3$ in these graph classes with explicit time complexity, faster than the one coming from the meta-theorem of Frick and Grohe. In Section 3, we provide explicit algorithms for kSMDd, $d \geq 3$, in graphs with bounded local treewidth and graphs excluding a fixed graph M as a minor. In particular, these algorithms apply to planar graphs, graphs of bounded genus, and graphs with bounded maximum degree. For the sake of simplicity, we present the algorithms for the kSMDd problem, but similar algorithms can be applied to the kdRS problem, with the same time bounds. Our algorithms use standard dynamic programming over graphs with bounded treewidth and a few results concerning the clique decomposition of M-minor-free graphs developed by Robertson and Seymour in their graph minor theory [34]. A set of non-trivial observations allow to get improvements in the time complexity of the algorithms. We note that the techniques used in our dynamic programming over graphs with bounded local treewidth are quite generic, and we believe that they can handle variations on degree-constrained subgraph problems with simple changes.

Notations: We use standard graph terminology. Let G be a graph. We use $V(G)$ and $E(G)$ to denote vertex and the edge set of G, respectively. We simply write V and E if the graph is clear from the context. For $V' \subseteq V$, we denote the induced subgraph on V' by $G[V'] = (V', E')$, where $E' = \{\{u, v\} \in E : u, v \in V'\}$. For $v \in V$, we denote by $N(v)$ the neighborhood of v, namely $N(v) = \{u \in V : \{u, v\} \in E\}$. The closed neighborhood $N[v]$ of v is $N(v) \cup \{v\}$. In the same way we define $N[S]$ for $S \subseteq V$ as $N[S] = \cup_{v \in S} N[v]$, and $N(S) = N[S] \setminus S$. We define the degree of vertex v in G as the number of vertices incident to v in G. Namely, $d(v) = |N(v)|$.

2 Fixed-Parameter In-tractability Results

We begin by defining parameterized reductions.

Definition 2.1 *Let Π, Π' be two parameterized problems, with instances (x, k) and (x', k'), respectively. We say that Π is (uniformly many:1)* reducible *to Π' if there is a function Φ, called a* parameterized reduction*, which transforms (x, k) into $(x', g(k))$ in time $f(k)|x|^{\alpha}$, where $f, g : \mathbb{N} \to \mathbb{N}$ are arbitrary functions and α is a constant independent of k, so that $(x, k) \in \Pi$ if and only if $(x', g(k)) \in \Pi'$.*

As mentioned in the introduction, kSMDd is known to be $W[1]$-hard for $d = 4$ [15, page 457]. It can be easily proved that kSMDd is $W[1]$-hard for every $d \geq 4$, by reducing kSMDd to $k\text{SMD}_{d+1}$.

Indeed, let G be an instance of kSMDd, with parameter k. We construct an instance G' of $k\text{SMD}_{d+1}$ from G by adding a vertex u and connecting it to all the vertices of G. We set the parameter to $k+1$. If there is a subset of vertices $S \subseteq V(G)$ of size at most k and with minimum degree at least d, then $S \cup \{u\}$ is a solution to $k\text{SMD}_{d+1}$ in G' (the degree of u is also at least $d+1$ since we can assume that $k \geq d+1$). Conversely, if there is a subset of vertices $S \subseteq V(G')$ of size at most $k+1$ and with minimum degree at least $d+1$, we construct a solution to kSMDd in G as follows.

- if $u \in S$, then $S \setminus \{u\}$ is a solution in G.
- otherwise, if $u \notin S$, let v be an arbitrary vertex in S. Then any connected component of the subgraph induced by $S \setminus \{v\}$ is a solution in G, since $|S \setminus \{v\}| \leq k$ and the degrees of the vertices in $S \setminus \{v\}$ have decreased by at most 1 after the removal of v.

In the remainder of this section we give a $W[1]$-hardness reduction for kdRS. Our reduction is from Multi-Color Clique, which is known to be $W[1]$-complete by a simple reduction from the ordinary Clique [21], and is based on the methodology known as *multi-color edge representation*. The Multi-Color Clique problem is defined as follows.

> Multi-color Clique
> **Input**: An graph $G = (V, E)$, a positive integer k, and a proper k-coloring of $V(G)$.
> **Parameter:** k.
> **Question**: Does there exist a clique of size k in G consisting of exactly one vertex of each color?

Consider an instance $G = (V, E)$ of Multi-color Clique with its vertices colored with the set of colors $\{c_1, \cdots, c_k\}$. Let $V[c_i]$ denote the set of vertices of color c_i. For each edge $e = \{u, v\}$ of G, with $u \in V[c_i]$, $v \in V[c_j]$, and $i < j$, we first replace e with two arcs $e^f = (u, v)$ and $e^b = (v, u)$. By abuse of notation, we also call this digraph G. Let $E[c_i, c_j]$ be the set of arcs $e = (u, v)$, with $u \in V[c_i]$ and $v \in V[c_j]$, for $1 \leq i \neq j \leq k$. An arc $(u, v) \in E[c_i, c_j]$ is called *forward* (resp. *backward*) if $i < j$ (resp. $i > j$). We also assume that for some positive integers N and M, $|V[c_i]| = N$ for all i and $|E[c_i, c_j]| = M$ for all $i \neq j$, i.e., we assume that the color classes of G, and also the arc sets between them, have uniform sizes. For a simple justification of this assumption, we can reduce Multi-color Clique to itself, taking the union of $k!$ disjoint copies of G, one for each permutation of the color sets.

In this methodology, the basic encoding bricks correspond to the arcs of G, which we call **arc gadgets**. We generally have three kinds of gadgets, which we call **selection**, **coherence**, and **match gadgets**. These are engineered together to get an overall reduction gadget for the problem. In an optimal solution to the problem (that is, a solution providing a Yes answer), the selection gadget ensures that *exactly one* arc gadget is selected among arc gadgets corresponding to arcs going from a color class $V[c_i]$ to another color class $V[c_j]$. For any color class $V[c_i]$, the coherence gadget ensures that the out-going arcs from $V[c_i]$, corresponding to the selected arc gadgets, have a common vertex in $V[c_i]$. That is, all the arcs corresponding to these selected arc gadgets *emanate from the same vertex in $V[c_i]$*. Finally, the match gadget ensures that if we have selected an arc gadget corresponding to an arc (u, v) from $V[c_i]$ to $V[c_j]$, then the arc gadget selected from $V[c_j]$ to $V[c_i]$ corresponds to (v, u). That is, *both of e^f and e^b are selected together.* In what follows, we show how to particularize this general strategy to obtain a reduction from MULTI-COLOR CLIQUE to $k d$RS for $d \geq 3$. To simplify the presentation, we first describe our reduction for the case $d = 3$ (in Section 2.1) and then we describe the required modifications for the case $d \geq 4$ in Section 2.2.

2.1 $W[1]$-hardness for the cubic case

In this section we give in detail the construction of all the gadgets for $d = 3$. Recall that an arc $(u, v) \in E[c_i, c_j]$ is forward if $i < j$, and it is backward if $i > j$. We refer the reader to Figure 1 to get an idea of the construction.

Arc gadgets: For each arc $(u, v) \in E[c_i, c_j]$ with $i < j$ (resp. $i > j$) we have a cycle $\mathcal{C}_{e^f}$ (resp. $\mathcal{C}_{e^b}$) of length $3 + 2(k - 2) + 2$, with the set of vertices:

- *selection vertices:* e^f_{s1}, e^f_{s2}, and e^f_{s3} (resp. e^b_{s1}, e^b_{s2}, and e^b_{s3});
- *coherence vertices:* e^f_{ch1r}, e^f_{ch2r} (resp. e^b_{ch1r}, e^b_{ch2r}), for all $r \in \{1, \ldots, k\}$ and $r \neq i, j$; and
- *match vertices:* e^f_{m1} and e^f_{m2} (resp. e^b_{m1} and e^b_{m2}).

Selection gadgets: For each pair of indices i, j with $1 \leq i \neq j \leq k$, we add a new vertex A_{c_i,c_j}, and connect it to all the selection vertices of the cycles $\mathcal{C}_{e^f}$ if $i < j$ (resp. $\mathcal{C}_{e^b}$ if $i > j$) for all $e \in E[c_i, c_j]$. This gadget is called *forward selection gadget* (resp. *backward selection gadget*) if $i < j$ (resp. $i > j$), and it is denoted by $\mathcal{S}_{i,j}$.
That is, we have $k(k - 1)$ clusters of gadgets: one gadget $\mathcal{S}_{i,j}$ for each set $E[c_i, c_j]$, for $1 \leq i \neq j \leq k$.

Coherence gadgets: For each i, $1 \leq i \leq k$, let us consider all the selection gadgets of the form $\mathcal{S}_{i,p}$, $p \in \{1, \cdots, k\}$ and $p \neq i$. For any $u \in V[c_i]$, and any two indices $1 \leq p \neq q \leq k$, $p, q \neq i$, we add two new vertices u_{pq} and u_{qp}, and a new edge $\{u_{pq}, u_{qp}\}$. For every arc $e = (u, v) \in E[c_i, c_p]$, with $u \in V[c_i]$, we pick the cycle $\mathcal{C}_{e^x}$, $x \in \{f, b\}$ depending on whether e is forward or backward, and add two edges of the form $\{e^x_{ch1q}, u_{pq}\}$ and $\{e^x_{ch2q}, u_{pq}\}$. Similarly, for an arc $e = (u, w) \in E[c_i, c_q]$, with $u \in V[c_i]$, we pick the cycle $\mathcal{C}_{e^x}$, $x \in \{f, b\}$, and add two edges $\{e^x_{ch1p}, u_{qp}\}$ and $\{e^x_{ch2p}, u_{qp}\}$.

Match gadgets: For any pair of arcs $e^f = (u, v)$ and $e^b = (v, u)$, we consider the two cycles $\mathcal{C}_{e^f}$ and $\mathcal{C}_{e^b}$ corresponding to e^f and e^b. Now, we add two new vertices e^* and e_*, a *matching edge* $\{e^*, e_*\}$, and all the edges of the form $\{e^f_{m1}, e^*\}$, $\{e^f_{m2}, e^*\}$, $\{e^b_{m1}, e_*\}$ and $\{e^b_{m2}, e_*\}$ where e^f_{m1}, e^f_{m2} are match vertices on $\mathcal{C}_{e^f}$, and e^b_{m1} , e^b_{m2} are match vertices on $\mathcal{C}_{e^b}$.

This completes the construction of the gadgets, and the union of all of them defines the graph $\mathscr{G}_G$ depicted in Figure 1.
We now prove that this construction yields the reduction through a sequence of simple claims.

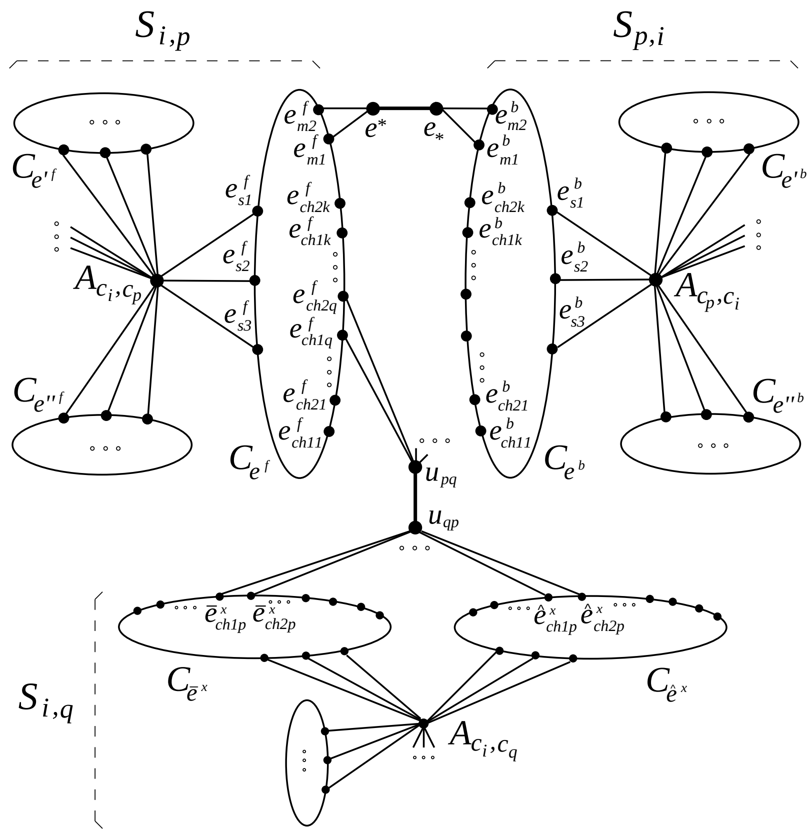

Figure 1: Gadgets used in the reduction of the proof of Theorem 2.5 (we suppose $i < p$).

Claim 2.2 *Let G be an instance of* Multi-color Clique*, and $\mathscr{G}_G$ be the graph we constructed above. If G has a multi-color k-clique, then $\mathscr{G}_G$ has a 3-regular subgraph of size $k' = (3k+1)k(k-1)$.*

Proof Let ω be a multi-color clique of size k in G. For every edge $e \in E(\omega)$, select the corresponding cycles $\mathcal{C}_{e^f}$, $\mathcal{C}_{e^b}$ in $\mathscr{G}_G$. Let us define S as follows.

$$S = \bigcup_{e\in\omega, x\in\{f,b\}} N[V(\mathcal{C}_{e^x})] \ .$$

Note that since ω is a multi-color clique, each vertex of the form A_{c_i,c_j} is adjacent in $\mathscr{G}_G[S]$ with vertices in at most one cycle. It can then be routinely checked that $\mathscr{G}_G[S]$ is a 3-regular subgraph of $\mathscr{G}_G$, as by construction the vertices in the cycles together with their neighbors have degree exactly 3. To verify the size of $\mathscr{G}_G[S]$, note that we have $2 \cdot \binom{k}{2}$ cycles in $\mathscr{G}_G[S]$ and each of them contributes $3k+1$ vertices. Indeed, each cycle contains $2k+1$ vertices, and their neighborhood outside the cycle has size k, as pairs of consecutive coherence and match vertices in the cycle have one common neighbor outside it, and the triple of selection vertices has one common neighbor of the form A_{c_i,c_j}. $\square$

Claim 2.3 *Any* 3*-regular subgraph of* $\mathscr{G}_G$ *contains one of the cycles* $\mathcal{C}_{e^x}$*,* $x \in \{b, f\}$*, corresponding to arc gadgets.*

Proof Note that if such a subgraph of $\mathscr{G}_G$ intersects a cycle $\mathcal{C}_{e^x}$, then it must contain all of its vertices. Further, if we remove all the vertices corresponding to arc gadgets in $\mathscr{G}_G$, then the remaining graph is a forest. These two facts together imply that any 3-regular subgraph of $\mathscr{G}(G)$ should intersect at least one cycle $\mathcal{C}_{e^x}$ corresponding to an arc gadget, hence it must contain $\mathcal{C}_{e^x}$. □

Claim 2.4 *If* $\mathscr{G}_G$ *contains a* 3*-regular subgraph of size* $k' = (3k+1)k(k-1)$*, then* G *has a multi-color* k*-clique.*

Proof Let $H = G[S]$ be a 3-regular subgraph of size k'. Now, by Claim 2.3, S must contain all the vertices of a cycle corresponding to an arc gadget. Furthermore, notice that to ensure the degree condition in H, once we have a vertex of a cycle in S, all the vertices of this cycle and their neighbors are also in S. Without loss of generality, let $\mathcal{C}_{e^f}$ be this cycle, and suppose that it belongs to the gadget $\mathcal{S}_{i,j}$, i.e., $e \in E[c_i, c_j]$ and $i < j$. Notice that by construction, this forces some of the other vertices to belong also to S. Indeed, its match vertices force the cycle $\mathcal{C}_{e^b}$ of $\mathcal{S}_{j,i}$ to be in S. The coherence vertices of $\mathcal{C}_{e^f}$ force S to contain at least one cycle in $\mathcal{S}_{i,l}$, for every $l \in \{1, \cdots, k\}$, $l \neq i$. They in turn force S to contain at least one cycle from the remaining gadgets $\mathcal{S}_{p,q}$ for all $p \neq q \in \{1, \ldots, k\}$. The selection vertices of each such cycle in $\mathcal{S}_{p,q}$ force S to contain $A_{p,q}$. But because of our condition on the size of S ($|S| = k'$), we can select *exactly one* cycle gadget from each of the gadgets $\mathcal{S}_{p,q}$, $p \neq q \in \{1, 2, \cdots, k\}$. Let E' be the set of edges in $E(G)$ corresponding to arc gadgets selected in S. We claim that $G[V[E']]$ is a multi-color clique of size k in G. Here $V[E']$ is a subset of vertices of $V(G)$ containing the end points of the edges in E'. First of all, because of the match vertices, once e^f is in E', e^b is forced to be in E'. To conclude the proof we only need to ensure that all the edges from a particular color class emanate from the same vertex. But this is ensured by the restriction on the size of S and the presence of coherence vertices on the cycles selected in S from $\mathcal{S}_{p,q}$, $p \neq q \in \{1, 2, \cdots, k\}$. To see this, let us take two arcs $e = (u, v) \in (E[c_i, c_p] \cap E')$ and $e' = (u', w) \in (E[c_i, c_q] \cap E')$. Now the four vertices u_{pq}, u_{qp}, u'_{pq}, and u'_{qp} belong to S. If u is different from u', then S has at least two elements more than the expected size k', which contradicts the condition on the size of S. All these facts together imply that $G[V(E')]$ forms a multi-color k-clique in the original graph G. □

Claims 2.2 and 2.4 together yield the following theorem:

Theorem 2.5 k3RS *is* W[1]*-hard.*

We shall see in the next section that the proof of the Theorem 2.5 can be generalized to larger values of d. Note that the 3-regular subgraph constructed in the proof of Theorem 2.5 is a 3-regular *induced* subgraph, so our proof implies the following corollary.

Corollary 2.6 k3RIS *is* W[1]*-hard.*

2.2 $W[1]$-hardness for higher degrees

In this section we generalize the reduction given in Section 2.1 for $d \geq 4$. The main idea is to change the role of the cycles $\mathcal{C}_e$ by $(d-1)$-regular graphs of appropriate size. We show below all the necessary changes in the construction of the gadgets to ensure that the proof for $d = 3$ works for $d \geq 4$.

Arc gadgets for $d \geq 4$***:*** Let us take $\mathcal{C}$ to be a connected $(d-1)$-regular graph of size $(d-1)+(d-1)(k-2)+d$, if it exists (that is, if $(d-1)$ is even or k is odd). If $(d-1)$ is odd and k is even, we take a graph of size $(d-1)+(d-1)(k+2)+d+1$ and with regular degree $d-1$ on the set $\mathcal{C}$ of $(d-1)+(d-1)(k+2)+d$ vertices and degree d on the last vertex v. As before, we replace each edge e with two arcs e^f and e^b. For each arc $e^x \in E[c_i, c_j]$, we add a copy of $\mathcal{C}$, that we call $\mathcal{C}_{e^x}$, with the following vertex set:

- *selection vertices:* $e^x_{s1}, e^x_{s2}, \cdots, e^x_{sd}$;
- *coherence vertices:* $e^x_{ch1r}, \cdots, e^x_{ch(d-1)r}$, for all $r \in \{1, \ldots, k\}$, $r \neq i, j$; and
- *match vertices:* $e^x_{m1}, \cdots, e^x_{m(d-1)}$.

Selection gadgets for $d \geq 4$***:*** Without loss of generality suppose that $x = f$. As before, we add a vertex A_{c_i,c_j}, and for every arc $e^f \in E[c_i, c_j]$ we add all the edges from A_{c_i,c_j} to all the selection vertices of the graph $\mathcal{C}_{e^f}$. We call this gadget $\mathcal{S}_{i,j}$.

Coherence gadgets for $d \geq 4$***:*** Fix an i, $1 \leq i \leq k$. Let us consider all the selection gadgets of the form $\mathcal{S}_{i,p}$, $p \in \{1, \cdots, k\}$ and $p \neq i$. For any $u \in V[c_i]$, and any two indices $p \neq q \leq k$, $p, q \neq i$, we add a new edge $\{u_{pq}, u_{qp}\}$. For every arc $e = (u, v) \in E[c_i, c_p]$, with $u \in V[c_i]$, we pick the graph $\mathcal{C}_{e^x}$, $x \in \{f, b\}$, depending on whether e is forward or backward, and add $d-1$ edges of the form $\{e_{ch1q}, u_{pq}\}, \{e_{ch2q}, u_{pq}\}, \ldots, \{e_{ch(d-1)q}, u_{pq}\}$. Similarly, for an arc $e = (u, w) \in E[c_i, c_q]$, with $u \in V[c_i]$, we pick the graph $\mathcal{C}_{e^x}$, $x \in \{f, b\}$, and add $d-1$ edges of the form $\{e_{ch1p}, u_{qp}\}, \ldots, \{e_{ch(d-1)p}, u_{qp}\}$.

Match gadgets for $d \geq 4$***:*** For the two arcs $e^f = (u, v)$ and $e^b = (v, u)$, we consider the two graphs $\mathcal{C}_{e^f}$ and $\mathcal{C}_{e^b}$ corresponding to e^f and e^b. Now we add a matching edge $\{e^*, e_*\}$ and add all the edges of the form $\{e^f_{m1}, e^*\}, \ldots, \{e^f_{m(d-1)}, e^*\}$ and $\{e^b_{m1}, e_*\}, \ldots, \{e^b_{m1}, e_*\}$, where e^f_{mi}, e^b_{mi} are match vertices of $\mathcal{C}_{e^f}$ and of $\mathcal{C}_{e^b}$, respectively.

This completes the construction of the gadgets, and the union of all of them defines the graph $\mathscr{G}_G$. It is not hard to see that a proof similar to that of Theorem 2.5 shows that G, an instance of multi-color clique, has a multi-color clique of size k if and only if $\mathscr{G}_G$ has a d-regular subgraph of size $k' = dk + 1$. We have the following theorem.

Theorem 2.7 *$k d$RS is* W[1]*-hard for all* $d \geq 3$*.*

Notice that again the d-regular subgraph constructed in the proof of Theorem 2.7 turns out to be an induced subgraph of regular degree d in $\mathcal{G}_G$. As a consequence we obtain the following corollary.

Corollary 2.8 *$k d$RIS is* W[1]*-hard for all* $d \geq 3$*.*

3 FPT Algorithms for Graphs with Bounded Local Treewidth and Graphs with Excluded Minors

In this section, we provide explicit (and fast) algorithms for kSMDd, $d \geq 3$, in graphs with bounded local treewidth (Section 3.1) and in graphs excluding a fixed graph M as a minor (Section 3.2). We first provide the necessary background.

The definition of treewidth, which has become quite standard, can be generalized to take into account the local properties of G, and this is called *local treewidth*. To define it formally, we first need to define the r-neighborhood of vertices of G. The *distance* $d_G(u,v)$ between two vertices u and v of G is the length of a shortest path in G from u to v. For $r \geq 1$, a *r-neighborhood* of a vertex $v \in V$ is defined as $N_G^r(v) = \{u \in V \mid d_G(v,u) \leq r\}$.

The *local treewidth* of a graph G is a function $ltw^G : \mathbb{N} \to \mathbb{N}$ which associates to every integer $r \in \mathbb{N}$ the maximum treewidth of an r-neighborhood of vertices of G, i.e.,

$$ltw^G(r) = \max_{v \in V(G)} \{tw(G[N_G^r(v)])\}.$$

A graph class $\mathscr{G}$ has *bounded local treewidth* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for each graph $G \in \mathscr{G}$ and for each integer $r \in \mathbb{N}$, we have $ltw^G(r) \leq f(r)$. For a given function $f : \mathbb{N} \to \mathbb{N}$, $\mathscr{G}_f$ is the class of all graphs G of local treewidth at most f, i.e., such that $ltw^G(r) \leq f(r)$ for every $r \in \mathbb{N}$. We refer to [17] and [26] for more details.

A graph G contains a graph M has a *minor* if M can be obtained from a subgraph of G by a (possibly empty) sequence of edge contractions or edge deletions. A family of graphs $\mathscr{G}$ *excludes* a graph M as a minor if no graph in $\mathscr{G}$ contains M as a minor. We now provide the basics to understand the structure of the classes of graphs excluding a fixed graph as a minor.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs, and $k \geq 0$ an integer. For $i = 1, 2$, let $W_i \subseteq V_i$ form a clique of size h and let G_i' be the graph obtained from G_i by removing a set of edges (possibly empty) from the clique $G_i[W_i]$. Let $F : W_1 \to W_2$ be a bijection between W_1 and W_2. The *h-clique sum* or the *h-sum* of G_1 and G_2, denoted by $G_1 \oplus_{h,F} G_2$, or simply $G_1 \oplus G_2$ if there is no confusion, is the graph obtained by taking the union of G_1' and G_2' by identifying $w \in W_1$ with $F(w) \in W_2$, and by removing all the multiple edges. The image of the vertices of W_1 and W_2 in $G_i \oplus G_2$ is called the *join* of the sum.

Note that $\oplus$ is not well defined; different choices of G_i' and the bijection F can give different clique sums. A sequence of h-sums, not necessarily unique, which result in a graph G, is called a *clique sum decomposition* or, simply, a *clique decomposition* of G.

Let Σ be a surface with boundary cycles $C_1, \ldots, C_h$. A graph G is *h-nearly embeddable* in Σ, if G has a subset X of vertices of size at most h, called *apices*, such that there are (possibly empty) subgraphs $G_0, \ldots, G_h$ of $G \setminus X$ such that

1. $G \setminus X = G_0 \cup \cdots \cup G_h$;
2. G_0 is embeddable in Σ (we fix an embedding of G_0);
3. $G_1, \ldots, G_h$ are pairwise disjoint;
4. For $1 \leq \cdots \leq h$, let $U_i := \{u_{i_1}, \ldots, u_{i_{m_i}}\} = V(G_0) \cap V(G_i)$, G_i has a path-decomposition $(\{B_{ij}\},\ 1 \leq j \leq m_i)$ of width at most h such that

(a) for $1 \le i \le h$ and for $1 \le j \le m_i$ we have $u_j \in B_{ij}$; and

(b) for $1 \le i \le h$, we have $V(G_0) \cap C_i = \{u_{i_1}, \dots, u_{i_{m_i}}\}$ and the points $u_{i_1}, \dots, u_{i_{m_i}}$ appear on C_i in this order (either walking through the cycles clockwise or counterclockwise).

3.1 Graphs with bounded local treewidth

In order to prove our results, we need the following lemma, which gives the time complexity of finding a smallest induced subgraph of degree at least d in graphs with bounded treewidth.

Lemma 3.1 *Let G be a graph on n vertices with a tree-decomposition of width at most t, and let d be a positive integer. Then in time $\mathcal{O}((d+1)^t(t+1)^{d^2}n)$ we can decide whether there exists an induced subgraph of degree at least d in G and, if such a subgraph exists, find one of the smallest size.*

Proof Let $(T, \mathcal{X})$ be the given tree-decomposition. We assume that T is a rooted tree, and that the decomposition is *nice*, which means the following:

- Each node has at most two children;
- For every node t with exactly two children t_1 and t_2, $X_t = X_{t_1} = X_{t_2}$;
- For every node t with exactly one child s, either $X_t \subset X_s$ and $|X_s| = |X_t| + 1$, or $X_s \subset X_t$ and $|X_t| = |X_s| + 1$.

Note that such a decomposition always exists and can be found in linear time, and in fact we may assume that $|V(T)| = O(n)$. As usual in algorithms based on tree decompositions, we employ a dynamic programming approach based on this decomposition, which at the end either produces a connected subgraph of G of minimum degree at least d and of size at most k, or decides that G does not have any such subgraph.

As the tree decomposition is rooted, we can speak of the subgraph defined by the subtree rooted at node i. More precisely, for any node i of T, let Y_i be the set of all vertices that appear either in X_i or in X_j for some descendant j of i. Denote by $G[Y_i]$ the graph induced by the nodes in Y_i.

Note that if i is a node in the tree and j_1 and j_2 are two children, then Y_{j_1} and Y_{j_2} are disjoint except for vertices in X_i, i.e., $Y_{j_1} \cap Y_{j_2} = X_i$. A $\mathcal{P}$-coloring of the vertices in X_i, for the palette $\mathcal{P} = \{0, 1, \dots, d\}$, is a function $c : X_i \to \mathcal{P}$. The *support* of c is $supp(c) = \{v \in X_i \mid c(v) \neq 0\}$.

For any such $\mathcal{P}$-coloring c of vertices in X_i, let $a(i, c)$ be the minimum size of an induced subgraph $H(i, c)$ of $G[Y_i]$, which has degree $c(v)$ for every $v \in X_i$ with $c(v) \neq d$, and degree at least d on its other vertices. Note that $H(i, c) \cap X_i = supp(c)$. If such a subgraph does not exist, we define $a(i, c) = +\infty$.

We develop recursive formulas for $a(i, c)$. In the base case, i is a leaf of the tree decomposition. Hence $Y_i = X_i$. The size of the minimum induced subgraph with prescribed degrees is exactly $|supp(c)|$ if $G[supp(c)]$ satisfies the degree conditions, and is $+\infty$ if it does not.

In the recursive case, node i has at least one child. We distinguish between three cases, depending on the size of the bag of i and its number of children.

Case (1): i has only one child j and $X_i \subset X_j$.
Then $|X_j| = |X_i|+1$ and $X_i = X_j \setminus \{v\}$ for some vertex v. Also, $Y_i = Y_j$, since X_i does not add any new vertices. Consider a coloring $c : X_i \to \mathcal{P}$. Consider the two colorings $c_0 : X_j \to \mathcal{P}$ and $c_1 : X_j \to \mathcal{P}$ of X_j, defined as follows: $c_0 = c_1 = c$ on X_i, and $c_0(v) = 0$, $c_1(v) = d$. Then we let $a(i, c) = \min\{a(j, c_0), a(j, c_1)\}$.
Case (2): i has only one child j and $X_j \subset X_i$.
Then $|X_j| = |X_i| - 1$ and $X_j = X_i \setminus \{v\}$ for some vertex v. Also, $Y_j = Y_i \setminus \{v\}$. Let c be a coloring of X_i. It is clear that the only neighbors of v in $G[Y_i]$ are already in X_i.

- If $c(v) \geq 1$, for any collection $\mathcal{A}$ of $c(v)$ edges in $G[X_i]$ connecting v to vertices $v_1, \ldots, v_{c(v)}$, with $c(v_i) \geq 1$ (note that such a collection may not exist at all), we consider the coloring $c_{\mathcal{A}}$ of X_j as follows: $c_{\mathcal{A}}(v_i) = c(v_i) - 1$ for any $1 \leq i \leq c(v)$, and $c_{\mathcal{A}}(w) = c(w)$ for any other vertex w. Then we define

$$a(i, c) = \min_{\mathcal{A}}\{a(j, c_{\mathcal{A}})\} + 1\ .$$

- If $c(v) = 0$, we simply define $a(i, c) = a(j, c)$.

Note that there are at most $(t + 1)^{d+1}$ choices for such a collection $\mathcal{A}$.
Case (3): i has two children j_1 and j_2.
Then $X_i = X_{j_1} = X_{j_2}$. Let c be a coloring of X_i, then $supp(c) \subset X_i$ is part of the subgraph we are looking for. For any vertex $v \in X_i$, calculate the degree $deg_{G[X_i]}(v)$. Suppose that v has degree d_1^v, d_2^v in $H \cap G[Y_{j_1}], H \cap G[Y_{j_2}]$ (H is the subgraph we are looking for). These degree sequences should guarantee the degree condition on v imposed by the coloring c. In other words, if $c(v) \leq d-1$ then we should have $d_1^v + d_2^v - d_{G[X_i]} = c(v)$, and if $c(v) = d$, then $d_1^v + d_2^v - d_{G[X_i]} \geq d$. Every such sequence $\mathcal{D} = \{d_1^v, d_2^v \mid v \in X_i\}$ on vertices of X_i determines two colorings $c_1^{\mathcal{D}}$ and $c_2^{\mathcal{D}}$ of X_{j_1} and X_{j_2} respectively. For each such pair of colorings, let H_1 and H_2 be the minimum subgraphs with these degree constraints in $G[Y_{j_1}]$ and $G[Y_{j_2}]$ respectively. Then $H_1 \cup H_2$ satisfies the degree constraints imposed by c. We define

$$a(i, c) = \min_{\mathcal{D}}\{|H| \mid H = H_1 \cup H_2\}$$

for all degree distributions as above. For every vertex we have at most d^2 possible degree choices for d_1^v and d_2^v. We have also $|X_i| \leq t + 1$. This implies that the minimum is taken over at most $(t + 1)^{d^2}$ colorings.
As the size of our tree-decomposition is linear on n, we can determine all the values $a(i, c)$ for every $i \in V(T)$ and every coloring of X_i in time linear in n. Now return the minimum value of $a(i, c)$ computed for all colorings c, for values in the set $\{0, d\}$ assigning at least one non-zero value. The time dependence on t follows from the size of the bags and the choices made using the colorings. □

Lemma 3.1 leads to the following theorem:

Theorem 3.2 *For any $d \geq 3$ and any function $f : \mathbb{N} \to \mathbb{N}$, kSMDd is fixed-parameter tractable on $\mathscr{G}_f$. Furthermore, the algorithm runs in time $\mathcal{O}((d+1)^{f(2k)}(f(2k)+1)^{d^2} n^2)$.*

Proof Let $G = (V, E)$ be a graph in $\mathscr{G}_f$, that is, G has bounded local treewidth and the bound is given by the function f. We first notice that if there exists an induced subgraph $H \subseteq G$ of size at most k and degree at least d, then H can be supposed to be connected. Secondly, if we know a vertex v of H, then H is contained in $N_G^k[v]$, which has diameter at most $2k$. Hence there exists the desired H if and only if there exists $v \in V$ such that H is contained in $N_G^k[v]$. To solve the problem, for each $v \in V$, we find a tree-decomposition of $N_G^k[v]$ of width at most $f(2k)$ in time polynomial in n, and then run the algorithm of Lemma 3.1. $\square$

The function $f(k)$ is known to be $3k$, $C_g gk$, and $b(b-1)^{k-1}$ for planar graphs, graphs of genus g, and graphs of degree at most b, respectively [17,26]. Here C_g is a constant depending only on the genus g of the graph. As an easy corollary of Theorem 3.2, we have the following:

Corollary 3.3 *kSMDd can be solved in $\mathcal{O}((d+1)^{6k}(6k+1)^{d^2}n^2)$, $\mathcal{O}((d+1)^{2C_g gk}(2C_g gk + 1)^{d^2}n^2)$ and $\mathcal{O}((d+1)^{2b(b-1)^{k-1}}(2b(b-1)^{k-1}+1)^{d^2}n^2)$ time in planar graphs, graphs of genus g, and graphs of degree at most b, respectively.*

3.2 M-minor-free graphs

In this section, we consider the class of M-minor-free graphs. We need the following theorem of Robertson and Seymour [34] (see also Demaine *et al.* [14] for an algorithmic version).

Theorem 3.4 ([14, 34]) *For every graph M, there exists an integer h, depending only on the size of M, such that every graph excluding M as a minor can be obtained by clique sums of order at most h from graphs that can be h-nearly embedded in a surface Σ in which M cannot be embedded. Furthermore, such a clique decomposition can be found in polynomial time.*

Let G be an M-minor-free graph, and let $(T, \mathcal{B} = \{B_t\})$ be a clique decomposition of G given by Theorem 3.4. We suppose in addition that T is rooted at a given vertex $r \in V(G)$. We define $A_t := B_t \cap B_{p(t)}$ where $p(t)$ is the unique parent of the vertex t in T, and $A_r = \emptyset$. Let $\hat{B}_t$ be the graph obtained from B_t by adding all the possible edges between the vertices of A_t and also between the vertices of A_s, for each child s of t. In this way, A_t and A_s's will induce cliques in $\hat{B}_t$ (see Figure 2). In addition, G becomes an h-clique sum of the graphs $\hat{B}_t$ according to the above tree T where each $\hat{B}_t$ is h-nearly embeddable in a surface Σ in which M cannot be embedded. Let X_t be the set of apices of $\hat{B}_t$; we have $|X_t| \leq h$ and $\hat{B}_t \setminus X_t$ has linear local treewidth. We denote by G_t the subgraph induced by all the vertices of $B_t \cup \bigcup_s B_s$, for s ranging over all descendants of t in T.

In order to simplify the presentation, in what follows, we will restrict ourselves to the case $d = 3$, but it is quite straightforward to check that the proof extends to all $d \geq 3$. Recall that we are looking for a subset of vertices S, of size at most k, which induces a graph $H = G[S]$ of minimum degree at least three.

Our algorithm consists of two levels of dynamic programming. The top level of dynamic programming runs over the clique decomposition, and within each subproblem of this dynamic programming, we focus on the induced subgraph of the vertices in B_t. Our first level of dynamic programming computes the size of a smallest subgraph of G_t, complying with degree

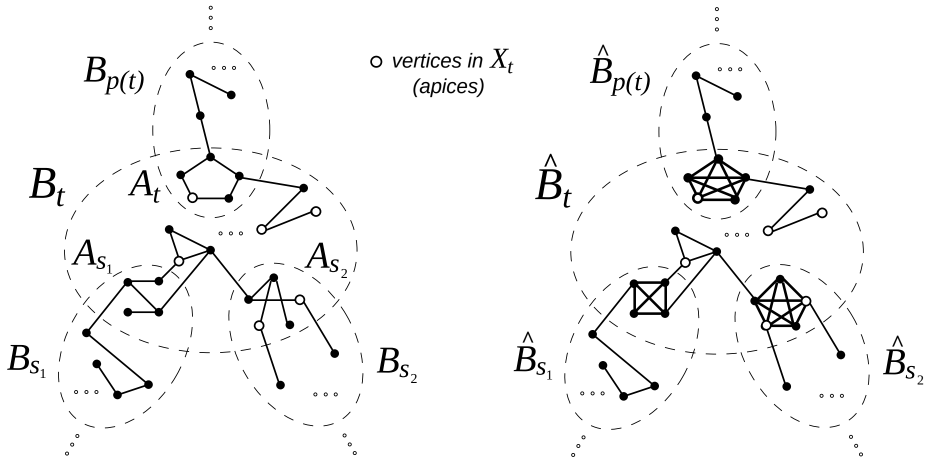

Figure 2: Tree-decomposition of a minor-free graph. The vertices in X_t (i.e., the *apices*) are depicted by $\circ$. Note that B_{s_1} and B_{s_2} could have non-empty intersection (in B_t).

constraints on the vertices of A_t. These constraints, as before, represent the degree of each vertex of A_t in the subgraph $H_t := G_t[S_t]$, i.e., the *trace* of H in G_t, where $S_t = S \cap V(G_t)$. This two-level dynamic programming requires a combinatorial bound on the treewidth as a function of the parameter k for each of the B_t's (after removing the apices X_t from B_t). The next two lemmas are used later to obtain this combinatorial bound.

Lemma 3.5 *Let $H = G[S]$ be a connected induced subgraph of G. Then the subgraph $\hat{B}_t[S \cap B_t]$ is connected.*

The proof of Lemma 3.5 easily follows from the properties of a tree-decomposition and the fact that A_t and A_s's are cliques in $\hat{B}_t$, for s a child of t in T.

Lemma 3.6 *Let $H = G[S]$ be a smallest connected subgraph of G of minimum degree at least three. Then the subgraph $\hat{B}_t[S_t \cap B_t \setminus X_t]$ has at most $3h + 1$ connected components, where h is the integer given by Theorem 3.4.*

Proof Let $C_1, \ldots, C_r$ be the connected components of $L := \hat{B}_t[S_t \cap B_t \setminus X_t]$. We want to prove that $r \leq 3h + 1$. Assume for the sake of a contradiction that $r > 3h + 1$. We will find another solution H' with size strictly smaller than H, which will contradict our assumption that H is of minimum size.

The graph H' is defined as follows. For each vertex $v \in X_t \cap S_t$, let

$$b_v := \min\{d_{H_t}(v), 3\}.$$

Then for each vertex $v \in X_t \cap S_t$, we choose at most b_v connected components of L, covering at least b_v neighbors of v in H_t. We also add the connected component containing all the vertices of $A_t \setminus X_t$ (recall that A_t induces a clique in $\hat{B}_t$). Let A be the union of all the vertices of these connected components. Since $|X_t| \leq h$, A has at most $3h+1$ connected components. Also, since A_s induces a clique in $\hat{B}_t$, for each child s of t such that $A_s \cap A \neq \emptyset$, we have that $A_s \setminus X_t \subset A$. We define H' as follows.

$$H' := G\left[\left(\bigcup_{\{s\ :\ A_s \cap A \neq \emptyset\}} S_s\right) \ \cup\ ((X_t \cup A) \cap S_t)\ \cup\ (S \setminus S_t)\right].$$

Clearly, $H' \subseteq H$. We have that $|H'| < |H|$ because, assuming that $r > 3h+1$, there are some vertices of $H_t \subset H$ which are in some connected component C_i which does not intersect H'.

Thus, it just remains to prove that H' is indeed a solution of kSMD3, i.e., H' has minimum degree at least 3. We prove it using a sequence of four simple claims:

Claim 3.7 *The degree of each vertex $v \in (V(H') \cap X_t)$ is at least 3 in H'.*

Proof This is because each such vertex v has degree at least b_v in H'_t. If $d_v < 3$, then v should be in A_t (if not, v has degree $d_v < 3$ in H, which is impossible), hence v is connected to at least $3 - d_v$ vertices in $S \setminus S_t$. But $S \setminus S_t$ is included in H', and so every vertex of $X_t \cap V(H')$ has degree at least 3 in H'. □

Claim 3.8 *The degree of each vertex in $(H \setminus H_t)$ is at least 3 in H'.*

Proof This follows because $A_t \cap H \subset H'$. □

Claim 3.9 *The degree of each vertex in A is at least 3 in H'.*

Proof Every vertex in A has the same degree in both H' and H. This is because A is the union of some connected components, and no vertex of A is connected to any other vertex in any other component. □

Claim 3.10 *Every other vertex of H' also has degree at least 3.*

Proof To prove the claim we prove that the vertices of $H' \setminus (G[X_t] \cup (H \setminus H_t) \cup A)$ have degree at least 3 in H'. Remember that all these vertices are in some S_s, for some s such that A_s has a non-empty intersection with A. We claim that all these vertices have the same degree in both H and H'. To prove this, note that $H' \cap A_s = H \cap A_s$ for all such s. Indeed, $(A_s \setminus X_t) \subset A$, and so $A_s \subset (A \cup X_t)$. Let u be such a vertex. We can assume that $u \notin X_t$. If $u \in A_s$, then clearly $u \in A$, and we are done. If $u \in (S_s \setminus A)$, then every neighbor of u is in H_s. But $H_s \subset H'$, hence we are also done in this case. □

This concludes the proof of the lemma. □

We define a *coloring* of A_t to be a function $c : A_t \cap S \to \{0, 1, 2, 3\}$. For $i < 3$, $c(v) = i$ means that the vertex v has degree i in the subgraph H_t of G_t that we are looking for, and $c(v) = 3$ means that v has degree at least three in H_t. By $a(t, c)$ we denote the minimum size of a subgraph of G_t with the prescribed degrees in A_t according to c. We describe in what follows the different steps of our algorithm.

Recursively, starting from the leaves of T and moving towards the root, for each node $t \in V(T)$ and for every coloring c of A_t, we compute $a(t, c)$ from the values of $a(s, c)$, where s is a child of t, or we store $a(t, c) = +\infty$ if no such subgraph exists. The steps involved in computing $a(t, c)$ for a fixed coloring c are the following:

(i) We guess a subset $R_t \subseteq X_t \setminus A_t$ such that $R_t \subseteq S_t$. We have at most 2^h choices for R_t.

(ii) For each vertex v in R_t, we guess whether v is adjacent to a vertex of $B_t \setminus (R_t \cup A_t)$, i.e., we test all the 2-colorings $\gamma : R_t \to \{0,1\}$; a coloring has the following meaning: $\gamma(v) = 1$ if and only if v is adjacent to a vertex of $B_t \setminus (R_t \cup A_t)$. The number of such colorings is at most 2^h. Let γ be a fixed coloring. For each of the vertices v in R_t with $\gamma(v) = 1$, we guess one vertex in $B_t \setminus (R_t \cup A_t)$, which we suppose to be in S_t. For each coloring γ, we have at most n^h choices for the new vertices which could be included in S_t. If a vertex has $\gamma(v) = 0$, it is not allowed to be adjacent to any vertex of B_t besides the vertices in $A_t \cup R_t$. Let D_t^γ be the chosen vertices at this level.

(iii) We remove now all the vertices of X_t from B_t. Lemma 3.6 ensures that the induced graph $\hat{B}_t[S_t \cap B_t \setminus X_t]$ has at most $3h+1$ connected components. We then choose these connected components of $\hat{B}_t[S_t \cap B_t \setminus X_t]$ by guessing a vertex from these connected components in $B_t \setminus X_t$. Since we need to choose at most $3h+1$ vertices this way, we have at most $(3h+1)n^{3h+1}$ new choices. Let these newly chosen vertices be F_t^γ and

$$R_t^\gamma = R_t \cup D_t^\gamma \cup F_t^\gamma \cup \{v \in A_t \setminus X_t \mid c(v) \neq 0\}.$$

Let G_t^* be the graph induced by the k-neighborhood (vertices at distance at most k) of all vertices of R_t^γ in $\hat{B}_t \setminus X_t$, i.e., $G_t^* = (\hat{B}_t \setminus X_t)[N^k(R_t^\gamma)]$.

(iv) Each connected component of G_t^* has diameter at most $2k$ in $\hat{B}_t \setminus X_t$. As $\hat{B}_t \setminus X_t$ has bounded local treewidth, this implies that G_t^* has treewidth bounded by a function of k. By the result of Demaine and Hajiaghayi [13], this function can be chosen to be linear.

(v) In this step, we first find a tree-decomposition $(\mathcal{T}_\gamma, \{U_p\})$ of G_t^*. Since $A_s \cap G_t^*$ is a clique, it appears in a bag of this tree-decomposition. Let p be the node representing this bag in $\mathcal{T}_\gamma$. We create now a new bag containing the vertices of $A_s \cap G_t^*$, and modify $\mathcal{T}_\gamma$ by adding a leaf connected to p which contains this new bag. With slight abuse of notation, we call this new decomposition $\mathcal{T}_\gamma$ and denote by s this distinguished leaf containing the bag $A_s \cap G_t^*$. We also add all the vertices of A_t to all the bags of this tree-decomposition, increasing the bag size by at most h. Now we apply a dynamic programming algorithm similar to the one we used for the bounded local treewidth case. Remember that for each child s of t, we have a leaf in this (new) decomposition with the bag $A_s \cap G_s^*$. The aim is to find an induced subgraph of minimum size which respects all the choices we have made earlier.

We start from the leaves of $\mathcal{T}_\gamma$ and move towards its root. At this point we have all the values of $a(s, c')$ for all possible colorings c' of A_s, where s is a child of t (because of the first level of dynamic programming). To compute $a(t, c)$ we apply the dynamic programming algorithm of Lemma 3.1 with the restriction that for each *distinguished* leaf s of this decomposition, we already have all the values $a(s, c)$ for all colorings of $A_s \cap G_s^*$ (we extend this coloring to all A_s by giving the zero values to the vertices of $A_s \setminus G_s^*$). Note that the only difference between this dynamic programming and the one of Lemma 3.1 is the way we initialize the leaves of the tree.

(vi) Among all the subgraphs we found in this way, we keep the minimum size of a subgraph with the degree constraint c on A_t. Let $a(t, c)$ be this minimum.

(vii) If for some vertex t and a coloring $c : A_t \to \{0, 3\}$, we have $1 \leq a(t, c) \leq k$, the algorithm return Yes, meaning that the graph contains a subgraph of size at most k and minimum degree at least three. If not, we conclude that such a subgraph does not exist.

This completes the description of the algorithm. Now we discuss the time complexity of this algorithm. Let C_M be the constant determining the linear local treewidth of the surfaces in which M cannot be embedded. For each fixed coloring c, we need time $4^{\mathcal{C}_M k}(C_M k+1)^9 n^{4h+1}$ to obtain $a(t, c)$, where $t \in T$. Since the number of colorings of each A_t is at most 4^h, and the size of the clique decomposition is $\mathcal{O}(n)$, we get the following theorem:

Theorem 3.11 *Let $\mathcal{C}$ be the class of graphs with excluded minor M. Then, for any graph in $\mathcal{C}$, one can find an induced subgraph of size at most k with degree at least 3 in time $\mathcal{O}(4^{\mathcal{O}(k+h)}(\mathcal{O}(k))^9 n^{\mathcal{O}(1)})$, where the constants in the exponents depend only on M.*

Theorem 3.11 can be generalized to larger values of d with slight modifications. We have the following theorem:

Theorem 3.12 *Let $\mathcal{C}$ be a class of graphs with an excluded minor M. Then, for any graph in $\mathcal{C}$, one can find an induced subgraph of size at most k with degree at least d in time $\mathcal{O}((d+1)^{\mathcal{O}(k+h)}(\mathcal{O}(k))^{d^2} n^{\mathcal{O}(1)})$, where the constants in the exponents depend only on M.*

4 Conclusions and Further Research

In this article we studied the parameterized complexity of the following two problems: given two positive integers d and k, finding a d-regular (induced or not) subgraph with at most k vertices, and finding a subgraph with at most k vertices and of minimum degree at least d.

We first showed that these problems are fixed-parameter intractable in general graphs. More precisely, we proved that the two variants of the first problem, namely kdRS (not necessarily induced subgraph) or kdRIS (induced subgraph), are $W[1]$-hard for fixed $d \geq 3$ using a reduction from Multi-Color Clique. The hardness of the second problem, namely kSMDd, followed from an extension of a known result for any fixed $d \geq 4$. We then provided explicit FPT algorithms to solve the second problem in graphs with bounded local treewidth and graphs with excluded minors. The presented algorithms can be modified to deal with the first problem just with technical modifications, but for simplicity we did not include the details here. For instance, in order to deal with the induced version of the first problem, we can apply the dynamic programming techniques of [35]. Our algorithms are faster than those coming from the meta-theorem of Frick and Grohe [23] about problems definable in first-order logic over the so-called "locally tree-decomposable structures".

Note that the parameterized tractability of the kSMDd problem for the case $d = 3$ remains open. We conjecture that:

Conjecture 4.1 kSMD3 *is $W[1]$-hard.*

Finally, it would be interesting to use the approach of this paper to investigate the parameterized complexity of Traffic Grooming in optical networks, a problem which is related to the kSMDd problem (see Section 1.2). Let n be the size of the optical network, and let

C be the number of requests that can share a link on a given wavelength (usually called *grooming factor*). In [10, Proposition 2] it is shown that RING TRAFFIC GROOMING is in P for fixed n. This result only shows that RING TRAFFIC GROOMING is in XP and not necessarily FPT if n is the parameter. According to [10], M. Fellows has shown that if the number of electronic terminations (called *ADMs* in SONET terminology) is taken to be the parameter, then RING TRAFFIC GROOMING is FPT. Unfortunately, the number of ADMs tends to be much larger than the ring size, so it remains an interesting open problem whether RING TRAFFIC GROOMING is FPT if n is the parameter and C is part of the input.

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