The Recognition of Tolerance and Bounded Tolerance Graphs*

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Abstract

Tolerance graphs model interval relations in such a way that intervals can tolerate a certain degree of overlap without being in conflict. This subclass of perfect graphs has been extensively studied, due to both its interesting structure and its numerous applications (in bioinformatics, constrained-based temporal reasoning, resource allocation, and scheduling problems, among others). Several efficient algorithms for optimization problems that are NP-hard in general graphs have been designed for tolerance graphs. In spite of this, the recognition of tolerance graphs – namely, the problem of deciding whether a given graph is a tolerance graph – as well as the recognition of their main subclass of bounded tolerance graphs, have been the most fundamental open problems on this class of graphs (cf. the book on tolerance graphs [15]) since their introduction in 1982 [12]. In this article we prove that both recognition problems are NP-complete, even in the case where the input graph is a trapezoid graph. The presented results are surprising because, on the one hand, most subclasses of perfect graphs admit polynomial recognition algorithms and, on the other hand, bounded tolerance graphs were believed to be efficiently recognizable as they are a natural special case of trapezoid graphs (which can be recognized in polynomial time) and share a very similar structure with them. For our reduction we extend the notion of an *acyclic orientation* of permutation and trapezoid graphs. Our main tool is a new algorithm that uses *vertex splitting* to transform a given trapezoid graph into a permutation graph, while preserving this new acyclic orientation property. This method of vertex splitting is of independent interest; very recently, it has been proved a powerful tool also in the design of efficient recognition algorithms for other classes of graphs [24].

Keywords: Tolerance graphs, bounded tolerance graphs, recognition, vertex splitting, NP-complete, trapezoid graphs, permutation graphs.

1 Introduction

1.1 Tolerance graphs and related graph classes

A simple undirected graph $G = (V, E)$ on n vertices is a *tolerance* graph if there exists a collection $I = \{I_i \mid i = 1, 2, \ldots, n\}$ of closed intervals on the real line and a set $t = \{t_i \mid i = 1, 2, \ldots, n\}$ of positive numbers, such that for any two vertices $v_i, v_j \in V$, $v_i v_j \in E$ if and only if $|I_i \cap I_j| \geq \min\{t_i, t_j\}$. The pair $\langle I, t \rangle$ is called a *tolerance representation* of G. If G has a tolerance representation $\langle I, t \rangle$, such that $t_i \leq |I_i|$ for every $i = 1, 2, \ldots, n$, then G is called a *bounded tolerance* graph and $\langle I, t \rangle$ a *bounded tolerance representation* of G.

Tolerance graphs were introduced in [12], in order to generalize some of the well known applications of interval graphs. The main motivation was in the context of resource allocation and scheduling problems, in which resources, such as rooms and vehicles, can tolerate sharing

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among users [15]. If we replace in the definition of tolerance graphs the operator *min* by the operator *max*, we obtain the class of *max-tolerance* graphs. Both tolerance and max-tolerance graphs find in a natural way applications in biology and bioinformatics, as in the comparison of DNA sequences from different organisms or individuals [19], by making use of a software tool like BLAST [1]. Tolerance graphs find numerous other applications in constrained-based temporal reasoning, data transmission through networks to efficiently scheduling aircraft and crews, as well as contributing to genetic analysis and studies of the brain [14, 15]. This class of graphs has attracted many research efforts [2, 4, 8, 13–15, 17, 20, 25, 27], as it generalizes in a natural way both interval graphs (when all tolerances are equal) and permutation graphs (when $t_i = |I_i|$ for every $i = 1, 2, \ldots, n$) [12]. For a detailed survey on tolerance graphs we refer to [15].

A graph is *perfect* if the chromatic number of every induced subgraph equals the clique number of that subgraph. Several difficult combinatorial problems can be solved efficiently, i.e. in polynomial time, on the class of perfect graphs, such as minimum coloring, maximum clique, and independent set [16]. Thus, since the class of tolerance graphs is a subclass of perfect graphs [13], there exist polynomial algorithms for these problems on tolerance and bounded tolerance graphs as well. In spite of this, faster algorithms have been designed for tolerance and bounded tolerance graphs, which exploit their special structure [14, 15, 25, 27].

A *comparability* graph is a graph which can be transitively oriented. A *co-comparability* graph is a graph whose complement is a comparability graph. A *trapezoid* (resp. *parallelogram* and *permutation*) graph is the intersection graph of trapezoids (resp. parallelograms and line segments) between two parallel lines L_1 and L_2 [10]. Such a representation with trapezoids (resp. parallelograms and line segments) is called a *trapezoid* (resp. *parallelogram* and *permutation*) *representation* of this graph. A graph is bounded tolerance if and only if it is a parallelogram graph [2, 21]. Permutation graphs are a strict subset of parallelogram graphs [3]. Furthermore, parallelogram graphs are a strict subset of trapezoid graphs [29], and both are subsets of co-comparability graphs [10, 15]. On the contrary, tolerance graphs are not even co-comparability graphs [10,15]. Recently, we have presented in [25] a natural intersection model for general tolerance graphs, given by parallelepipeds in the three-dimensional space. This representation generalizes the parallelogram representation of bounded tolerance graphs, and has been used to improve the time complexity of minimum coloring, maximum clique, and weighted independent set algorithms on tolerance graphs [25].

Although tolerance and bounded tolerance graphs have been studied extensively, the recognition problems for both these classes have been the most fundamental open problems since their introduction in 1982 [5, 10, 15]. Therefore, all existing algorithms assume that, along with the input tolerance graph, a tolerance representation of it is given. The only result about the complexity of recognizing tolerance and bounded tolerance graphs is that they have a (non-trivial) polynomial sized tolerance representation, hence the problems of recognizing tolerance and bounded tolerance graphs are in the class NP [17]. Recently, a linear time recognition algorithm for the subclass of *bipartite tolerance* graphs has been presented in [5]. Furthermore, the class of trapezoid graphs (which strictly contains parallelogram, i.e. bounded tolerance, graphs [29]) can be also recognized in polynomial time [22, 24, 31]. On the other hand, the recognition of max-tolerance graphs is known to be NP-hard [19]. Unfortunately, the structure of max-tolerance graphs differs significantly from that of tolerance graphs (max-tolerance graphs are not even perfect, as they can contain induced C_5's [19]), so the technique used in [19] does not carry over to tolerance graphs.

Since very few subclasses of perfect graphs are known to be NP-hard to recognize (for instance, *perfectly orderable* graphs [26], *EPT* graphs [11], and – recently – *triangle* graphs [23]), it was believed that the recognition of tolerance graphs was in P. Furthermore, as bounded tolerance graphs are equivalent to parallelogram graphs [2, 21], which constitute a natural subclass of trapezoid graphs and have a very similar structure, it was plausible that their recognition was also in P.

1.2 Our contribution

In this article, we establish the complexity of recognizing tolerance and bounded tolerance graphs. Namely, we prove that both problems are surprisingly NP-complete, by providing a reduction from the monotone-Not-All-Equal-3-SAT (monotone-NAE-3-SAT) problem. Consider a boolean formula ϕ in conjunctive normal form with three literals in every clause (3-CNF), which is monotone, i.e. no variable is negated. The formula ϕ is called NAE-satisfiable if there exists a truth assignment of the variables of ϕ, such that every clause has at least one true variable and one false variable. Given a monotone 3-CNF formula ϕ, we construct a trapezoid graph H_ϕ, which is parallelogram, i.e. bounded tolerance, if and only if ϕ is NAE-satisfiable. Moreover, we prove that the constructed graph H_ϕ is tolerance if and only if it is bounded tolerance. Thus, since the recognition of tolerance and of bounded tolerance graphs are in the class NP [17], it follows that both problems are NP-complete. Actually, our results imply that the recognition problems remain NP-complete even if the given graph is trapezoid, since the constructed graph H_ϕ is trapezoid.

For our reduction we extend the notion of an *acyclic orientation* of permutation and trapezoid graphs. Our main tool is a new algorithm that transforms a given trapezoid graph into a permutation graph by *splitting* some specific vertices, while preserving this new acyclic orientation property. One of the main advantages of this algorithm is that the constructed permutation graph does not depend on any particular trapezoid representation of the input graph G. Moreover, this approach based on splitting vertices has already been proved useful for the design of polynomial recognition algorithms for other classes of graphs [24].

Organization of the paper. We first present in Section 2 several properties of permutation and trapezoid graphs, as well as the algorithm Split-U, which constructs a permutation graph from a trapezoid graph. In Section 3 we present the reduction of the monotone-NAE-3-SAT problem to the recognition of bounded tolerance graphs. In Section 4 we prove that this reduction can be extended to the recognition of general tolerance graphs. Finally, we discuss the presented results and further research directions in Section 5.

2 Trapezoid graphs and representations

In this section we first introduce (in Section 2.1) the notion of an *acyclic representation* of permutation and of trapezoid graphs. This is followed (in Section 2.2) by some structural properties of trapezoid graphs, which will be used in the sequel for the splitting algorithm Split-U. Given a trapezoid graph G and a vertex subset U of G with certain properties, this algorithm constructs a permutation graph $G^{\#}(U)$ with $2|U|$ vertices, which is independent of any particular trapezoid representation of the input graph G.

Notation. We consider in this article simple undirected and directed graphs with no loops or multiple edges. In an undirected graph G, the edge between vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. If the graph G is directed, we denote by uv the arc from u to v. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the vertices in S, and we use $E[S]$ to denote $E(G[S])$. Whenever we deal with a trapezoid (resp. permutation and bounded tolerance, i.e. parallelogram) graph, we will consider without loss of generality a trapezoid (resp. permutation and parallelogram) representation, in which all endpoints of the trapezoids (resp. line segments and parallelograms) are distinct [9,15,18]. Given a permutation graph P along with a permutation representation R, we may not distinguish in the following between a vertex of P and the corresponding line segment in R, whenever it is clear from the context. Furthermore, with a slight abuse of notation, we will refer to the line segments of a permutation representation just as *lines*.

2.1 Acyclic permutation and trapezoid representations

Let $P = (V, E)$ be a permutation graph and R be a permutation representation of P. For a vertex $u \in V$, denote by $\theta_R(u)$ the angle of the line of u with L_2 in R. The class of permutation graphs is the intersection of comparability and co-comparability graphs [10]. Thus, given a permutation representation R of P, we can define two partial orders $(V, <_R)$ and $(V, \ll_R)$ on the vertices of P [10]. Namely, for two vertices u and v of G, $u <_R v$ if and only if $uv \in E$ and $\theta_R(u) < \theta_R(v)$, while $u \ll_R v$ if and only if $uv \notin E$ and u lies to the left of v in R. The partial order $(V, <_R)$ implies a transitive orientation Φ_R of P, such that $uv \in \Phi_R$ whenever $u <_R v$.

Note that an alternative definition of the transitive orientation Φ_R of P is that $uv \in \Phi_R$ if and only if $u \ll_{R'} v$ in the representation R' obtained by reversing in R the ordering of the points on the top line L_1. However, in the rest of the paper we will use the first definition of Φ_R that involves the angles $\theta_R(u)$ and $\theta_R(v)$ of the lines of u and v in R, respectively. Intuitively, the main reason for using this definition of Φ_R is that, in any parallelogram representation, the two lines of every parallelogram have the same angle (see for example the proof of Lemma 1 below).

Let $G = (V, E)$ be a trapezoid graph, and R be a trapezoid representation of G, where for any vertex $u \in V$, the trapezoid corresponding to u in R is denoted by T_u. Since trapezoid graphs are also co-comparability graphs [10], we can similarly define the partial order $(V, \ll_R)$ on the vertices of G, such that $u \ll_R v$ if and only if $uv \notin E$ and T_u lies completely to the left of T_v in R. In this case, we may denote also $T_u \ll_R T_v$, instead of $u \ll_R v$.

In a given trapezoid representation R of a trapezoid graph G, we denote by $l(T_u)$ and $r(T_u)$ the left and the right line of T_u in R, respectively. Similarly to the case of permutation graphs, we use the relation $\ll_R$ for the lines $l(T_u)$ and $r(T_u)$, e.g. $l(T_u) \ll_R r(T_v)$ means that the line $l(T_u)$ lies to the left of the line $r(T_v)$ in R. Moreover, if the trapezoids of all vertices of a subset $S \subseteq V$ lie completely to the left (resp. right) of the trapezoid T_u in R, we write $R(S) \ll_R T_u$ (resp. $T_u \ll_R R(S)$). Note that there are several trapezoid representations of a particular trapezoid graph G. Given one such representation R, we can obtain another one R' by *vertical axis flipping* of R, i.e. R' is the mirror image of R along an imaginary line perpendicular to L_1 and L_2. Moreover, we can obtain another representation R'' of G by *horizontal axis flipping* of R, i.e. R'' is the mirror image of R along an imaginary line parallel to L_1 and L_2. We will use extensively these two basic operations throughout the article.

In the next two definitions we introduce the notions of acyclic permutation and acyclic trapezoid graphs. These two new notions of acyclicity are essential for proving some basic properties of our Algorithm Split-U (cf. Theorem 1), as well as for proving the correctness of our reduction in Section 3.

Definition 1 *Let P be a permutation graph with $2n$ vertices $\{u_1^1, u_1^2, u_2^1, u_2^2, \ldots, u_n^1, u_n^2\}$. Let R be a permutation representation and Φ_R be the corresponding transitive orientation of P. The simple directed graph F_R is obtained by merging u_i^1 and u_i^2 into a single vertex u_i, for every $i = 1, 2, \ldots, n$, where the arc directions of F_R are implied by the corresponding directions in Φ_R. That is, u_iu_j is an arc in F_R if and only if $u_i^x u_j^y \in E(P)$ and $\theta_R(u_i^x) < \theta_R(u_j^y)$ for some $x, y \in \{1, 2\}$. Then,*

1. *R is an* acyclic permutation representation *with respect to* $\{u_i^1, u_i^2\}_{i=1}^n$*, *if F_R has no directed cycle,*

2. *P is an* acyclic permutation graph *with respect to* $\{u_i^1, u_i^2\}_{i=1}^n$, *if P has an acyclic representation R with respect to $\{u_i^1, u_i^2\}_{i=1}^n$.*

In Figure 1 we show an example of a permutation graph P with six vertices in Figure 1(a), a permutation representation R of P in Figure 1(b), the transitive orientation Φ_R of P in Figure 1(c), and the corresponding simple directed graph F_R in Figure 1(d). In the figure, the pairs $\{u_i^1, u_i^2\}_{i=1}^3$ are grouped inside ellipses. In this example, R is not an acyclic permutation representation with respect to $\{u_i^1, u_i^2\}_{i=1}^3$, since F_R has a directed cycle of length two. However, note that, by exchanging the lines u_1^1 and u_2^1 in R, the resulting permutation representation R' is acyclic with respect to $\{u_i^1, u_i^2\}_{i=1}^3$, and thus P is acyclic with respect to $\{u_i^1, u_i^2\}_{i=1}^3$.

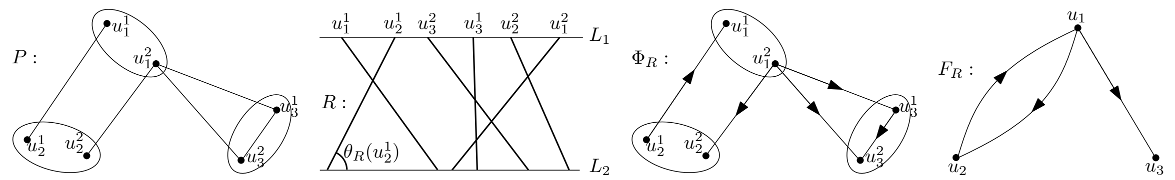


Figure 1: (a) A permutation graph P, (b) a permutation representation R of P, (c) the transitive orientation Φ_R of P, and (d) the corresponding simple directed graph F_R.

Definition 2 *Let G be a trapezoid graph with n vertices and R be a trapezoid representation of G. Let P be the permutation graph with $2n$ vertices corresponding to the left and right lines of the trapezoids in R, R_P be the permutation representation of P induced by R, and $\{u_i^1, u_i^2\}$ be the vertices of P that correspond to the same vertex u_i of G, $i = 1, 2, \ldots, n$. Then,*

1. *R is an* acyclic trapezoid representation*, if R_P is an acyclic permutation representation with respect to $\{u_i^1, u_i^2\}_{i=1}^n$,*
2. *G is an* acyclic trapezoid graph*, if it has an acyclic representation R.*

The following lemma follows easily from Definitions 1 and 2.

Lemma 1 *Any parallelogram graph is an acyclic trapezoid graph.*

Proof. Let G be a parallelogram graph with n vertices $\{u_1, u_2, \ldots, u_n\}$ and R be a parallelogram representation of G. That is, R is a trapezoid representation of G, such that the left and right lines $l(T_{u_i})$ and $r(T_{u_i})$ of the trapezoid T_{u_i}, $i = 1, 2, \ldots, n$, are parallel in R, i.e. $\theta_R(l(T_{u_i})) = \theta_R(r(T_{u_i}))$. Let P be the permutation graph with $2n$ vertices $\{u_1^1, u_1^2, u_2^1, u_2^2, \ldots, u_n^1, u_n^2\}$ corresponding to the left and right lines of the trapezoids of G in R, i.e. the vertices u_i^1 and u_i^2 correspond to $l(T_{u_i})$ and $r(T_{u_i})$, $i = 1, 2, \ldots, n$, respectively. Let R_P be the permutation representation of P induced by R, and Φ_{R_P} be the corresponding transitive orientation of the permutation graph P. Recall that, for two intersecting lines a, b in R_P, it holds $ab \in \Phi_{R_P}$ whenever $\theta_R(a) < \theta_R(b)$. It follows that for any $i = 1, 2, \ldots, n$, the pair $\{u_i^1, u_i^2\}$ of vertices in P has incoming edges from (resp. outgoing edges to) vertices of other pairs $\{u_j^1, u_j^2\}$ in Φ_{R_P}, which have smaller (resp. greater) angle with the line L_2 in R_P. Thus, the simple directed graph F_{R_P} defined in Definition 1 has no directed cycles, and therefore R_P is an acyclic permutation representation with respect to $\{u_i^1, u_i^2\}_{i=1}^n$, i.e. R is an acyclic trapezoid representation of G by Definition 2. ■

*To simplify the presentation, we use throughout the paper $\{u_i^1, u_i^2\}_{i=1}^n$ to denote the set of n unordered pairs $\{u_1^1, u_1^2\}, \{u_2^1, u_2^2\}, \ldots, \{u_n^1, u_n^2\}$.

2.2 Structural properties of trapezoid graphs

In the following, we state some definitions and notions concerning an arbitrary simple undirected graph $G = (V, E)$. These notions are essential in order to present and analyze our Algorithm Split-U (in Section 2.3). Although these definitions apply to any graph, we will use them only for trapezoid graphs. Similar definitions, for the restricted case where the graph G is connected, were studied in [6]. For $u \in V$ and $U \subseteq V$, $N(u) = \{v \in V \mid uv \in E\}$ is the set of adjacent vertices of u in G, $N[u] = N(u) \cup \{u\}$, and $N(U) = \bigcup_{u \in U} N(u) \setminus U$. If $N(U) \subseteq N(W)$ for two vertex subsets U and W, then U is said to be *neighborhood dominated* by W. Clearly, the relationship of neighborhood domination is transitive.

Let $C_1, C_2, \ldots, C_\omega$, $\omega \geq 1$, be the connected components of $G \setminus N[u]$ and $V_i = V(C_i)$, $i = 1, 2, \ldots, \omega$. For simplicity of the presentation, we will identify in the sequel the component C_i and its vertex set V_i, $i = 1, 2, \ldots, \omega$. For $i = 1, 2, \ldots, \omega$, the *neighborhood domination closure* of V_i with respect to u is the set $D_u(V_i) = \{V_p \mid N(V_p) \subseteq N(V_i),\ p = 1, 2, \ldots, \omega\}$ of connected components of $G \setminus N[u]$. The *closure complement* of the neighborhood domination closure $D_u(V_i)$ is the set $D_u^*(V_i) = \{V_1, V_2, \ldots, V_\omega\} \setminus D_u(V_i)$.

For a subset $S \subseteq \{V_1, V_2, \ldots, V_\omega\}$, a component V_i of S is called *maximal*, if there is no component $V_j \in S$, such that $N(V_i) \subsetneqq N(V_j)$. Furthermore, a connected component V_i of $G \setminus N[u]$ is called a *master component* of u, if V_i is a maximal component of $\{V_1, V_2, \ldots, V_\omega\}$.

Intuitively, if G is a trapezoid graph and R is a trapezoid representation of G, one can think of a master component V_i of u as the first connected component of $G \setminus N[u]$ to the right, or to the left of T_u in R. For example, consider the trapezoid graph G with vertex set $\{u, u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$, which is given by the trapezoid representation R of Figure 2. The connected components of $G \setminus N[u] = \{v_1, v_2, v_3, v_4\}$ are $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, and $V_4 = \{v_4\}$. Then, $N(V_1) = \{u_1\}$, $N(V_2) = \{u_1, u_3\}$, $N(V_3) = \{u_2, u_3\}$, and $N(V_4) = \{u_3\}$; thus V_2 and V_3 are the only master components of u. Furthermore, $D_u(V_1) = \{V_1\}$, $D_u(V_2) = \{V_1, V_2, V_4\}$, $D_u(V_3) = \{V_3, V_4\}$, and $D_u(V_4) = \{V_4\}$. Therefore, $D_u^*(V_1) = \{V_2, V_3, V_4\}$, $D_u^*(V_2) = \{V_3\}$, $D_u^*(V_3) = \{V_1, V_2\}$, and $D_u^*(V_4) = \{V_1, V_2, V_3\}$.

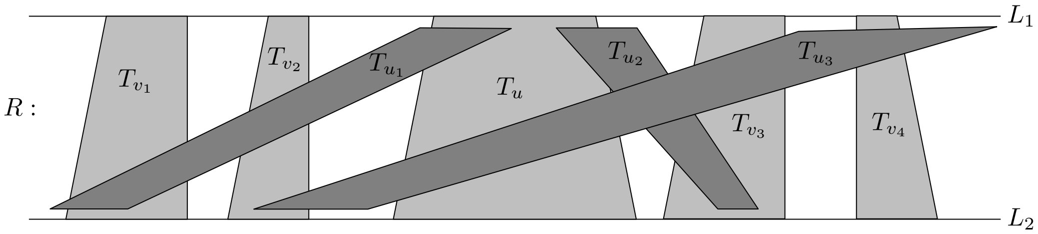


Figure 2: A trapezoid representation R of a trapezoid graph G.

Lemma 2 *Let G be a simple graph, u be a vertex of G, and let $V_1, V_2, \ldots, V_\omega$, $\omega \geq 1$, be the connected components of $G \setminus N[u]$. If V_i is a master component of u, such that $D_u^*(V_i) \neq \emptyset$, then $D_u^*(V_j) \neq \emptyset$ for every component V_j of $G \setminus N[u]$.*

Proof. The proof is done by contradiction. Suppose that there exists a component V_j of $G \setminus N[u]$, such that $D_u^*(V_j) = \emptyset$. That is, $N(V_k) \subseteq N(V_j)$ for every component V_k of $G \setminus N[u]$. Therefore, in particular, $N(V_i) \subseteq N(V_j)$. Suppose first that $N(V_i) = N(V_j)$. Then $N(V_k) \subseteq N(V_i)$ for every component V_k of $G \setminus N[u]$, and thus $D_u^*(V_i) = \emptyset$, which is a contradiction. Suppose now that $N(V_i) \subsetneqq N(V_j)$. Then V_i is not a master component of u, which is again a contradiction. Therefore $D_u^*(V_j) \neq \emptyset$ for every component V_j of $G \setminus N[u]$. ■

In the following we investigate several properties of trapezoid graphs, in order to derive the vertex-splitting algorithm Split-U in Section 2.3.

Remark 1 *Similar properties of trapezoid graphs have been studied in [6], leading to another vertex-splitting algorithm, called Split-All. However, the algorithm proposed in [6] is incorrect, since it is based on an incorrect property*[†]*, as was also verified by [7]. In the sequel of this section, we present new definitions and properties. In the cases where a similarity arises with those of [6], we refer to it specifically.*

The next lemma, which has been stated in Observation 3.1(4) in [6] (without a proof), will be used in our analysis below. For the sake of completeness, we present in the following its proof.

Lemma 3 *Let R be a trapezoid representation of a trapezoid graph G, and V_i be a master component of a vertex u of G, such that $R(V_i) \ll_R T_u$. Then, $T_u \ll_R R(V_j)$ for every component $V_j \in D_u^*(V_i)$.*

Proof. Suppose otherwise that $R(V_j) \ll_R T_u$, for some $V_j \in D_u^*(V_i)$. Consider first the case where $R(V_j) \ll_R R(V_i) \ll_R T_u$. Then, since V_i lies between V_j and T_u in R, all trapezoids that intersect T_u and V_j, must also intersect V_i. Thus, $N(V_j) \subseteq N(V_i)$, i.e. $V_j \in D_u(V_i)$, which is a contradiction, since $V_j \in D_u^*(V_i)$. Consider now the case where $R(V_i) \ll_R R(V_j) \ll_R T_u$. Then, we obtain similarly that $N(V_i) \subseteq N(V_j)$. If $N(V_i) = N(V_j)$, then $V_j \in D_u(V_i)$, which is a contradiction to the assumption, since $V_j \in D_u^*(V_i)$. Otherwise, if $N(V_i) \subsetneqq N(V_j)$, then V_i is not a master component of u, which is again a contradiction to the assumption. Thus, $T_u \ll_R R(V_j)$ for every $V_j \in D_u^*(V_i)$. ■

In the following two definitions, we partition the neighbors $N(u)$ of a vertex u in a trapezoid graph G into four possibly empty sets. In the first definition, these sets depend on the graph G itself and on two particular connected components V_i and V_j of $G \setminus N[u]$, while in the second one, they depend on a particular trapezoid representation R of G.

Definition 3 *Let G be a trapezoid graph, and u be a vertex of G. Let V_i be a master component of u, such that $D_u^*(V_i) \neq \emptyset$, and V_j be a maximal component of $D_u^*(V_i)$. Then, the vertices of $N(u)$ are partitioned into four possibly empty sets:*

1. *$N_0(u, V_i, V_j)$: vertices not adjacent to either V_i or V_j,*
2. *$N_1(u, V_i, V_j)$: vertices adjacent to V_i but not to V_j,*
3. *$N_2(u, V_i, V_j)$: vertices adjacent to V_j but not to V_i,*
4. *$N_{12}(u, V_i, V_j)$: vertices adjacent to both V_i and V_j.*

Definition 4 *Let G be a trapezoid graph, R be a representation of G, and u be a vertex of G. Denote by $D_1(u, R)$ and $D_2(u, R)$ the sets of trapezoids of R that lie completely to the left and to the right of T_u in R, respectively. Then, the vertices of $N(u)$ are partitioned into four possibly empty sets:*

1. *$N_0(u, R)$: vertices not adjacent to either $D_1(u, R)$ or $D_2(u, R)$,*
2. *$N_1(u, R)$: vertices adjacent to $D_1(u, R)$ but not to $D_2(u, R)$,*

[†]In [6], a different definition of a master component has been given. Namely, according to [6], a component V_i is called a *master component* of u if $|D_u(V_i)| \geq |D_u(V_j)|$ for all $j = 1, 2, \ldots, \omega$. In Observation 3.1(5) of [6], it is claimed that for an arbitrary trapezoid representation R of a connected trapezoid graph G, where V_i is a master component of u such that $D_u^*(V_i) \neq \emptyset$ and $R(V_i) \ll_R T_u$, it holds $R(D_u(V_i)) \ll_R T_u \ll_R R(D_u^*(V_i))$. However, the first part of the latter inequality is not true. For instance, in the trapezoid graph G of Figure 2, $V_2 = \{v_2\}$ is a master component of u (according to the definition of [6]), where $D_u^*(V_2) = \{V_3\} = \{\{v_3\}\} \neq \emptyset$ and $R(V_2) \ll_R T_u$. However, $V_4 = \{v_4\} \in D_u(V_2)$ and $T_u \ll_R T_{v_4}$, and thus, $R(D_u(V_2)) \not\ll_R T_u$.

3. $N_2(u,R)$*: vertices adjacent to* $D_2(u,R)$ *but not to* $D_1(u,R)$*,*

4. $N_{12}(u,R)$*: vertices adjacent to both* $D_1(u,R)$ *and* $D_2(u,R)$*.*

The following lemma connects the last two definitions; in particular, it states that, if $R(V_i) \ll_R T_u$, then the partitions of the set $N(u)$ defined in Definitions 3 and 4 coincide. This lemma will enable us to define in the sequel a partition of the set $N(u)$, independently of any trapezoid representation R of G, and regardless of any particular connected components V_i and V_j of $G \setminus N[u]$, cf. Definition 6.

Lemma 4 *Let G be a trapezoid graph, R be a representation of G, and u be a vertex of G. Let V_i be a master component of u, such that $D_u^*(V_i) \neq \emptyset$, and let V_j be a maximal component of $D_u^*(V_i)$. If $R(V_i) \ll_R T_u$, then $N_X(u, V_i, V_j) = N_X(u, R)$ for every $X \in \{0, 1, 2, 12\}$.*

Proof. Since $D_u^*(V_i) \neq \emptyset$ and $R(V_i) \ll_R T_u$, it follows by Lemma 3 that $T_u \ll_R R(V_j)$, i.e. $V_j \subseteq D_2(u,R)$. Suppose that a component $V_\ell \neq V_j$ is the leftmost one of $D_2(u,R)$ in R, i.e. $T_u \ll_R R(V_\ell) \ll_R R(V_j)$. Since V_ℓ lies between T_u and V_j in R, all trapezoids that intersect T_u and V_j, must also intersect V_ℓ, and thus, $N(V_j) \subseteq N(V_\ell)$. It follows that $V_\ell \in D_u^*(V_i)$, i.e. $V_\ell \notin D_u(V_i)$, since otherwise $V_j \in D_u(V_i)$, which is a contradiction. Furthermore, since V_j is a maximal component of $D_u^*(V_i)$, and since $N(V_j) \subseteq N(V_\ell)$, it follows that $N(V_j) = N(V_\ell)$, i.e. $N_X(u, V_i, V_j) = N_X(u, V_i, V_\ell)$ for every $X \in \{0, 1, 2, 12\}$.

Suppose that a component $V_k \neq V_i$ is the rightmost one of $D_1(u,R)$ in R, i.e. $R(V_i) \ll_R R(V_k) \ll_R T_u$. Then, $V_k \in D_u(V_i)$, since otherwise $T_u \ll_R R(V_k)$ by Lemma 3, which is a contradiction. Thus, $N(V_k) \subseteq N(V_i)$. Furthermore, since V_k lies between V_j and T_u in R, all trapezoids that intersect T_u and V_j, must also intersect V_k, and thus, $N(V_i) \subseteq N(V_k)$. Therefore, $N(V_i) = N(V_k)$, i.e. $N_X(u, V_i, V_\ell) = N_X(u, V_k, V_\ell)$ for every $X \in \{0, 1, 2, 12\}$, and thus, $N_X(u, V_i, V_j) = N_X(u, V_k, V_\ell)$ for every $X \in \{0, 1, 2, 12\}$.

Consider now a vertex $v \in N(u)$, and recall that V_k (resp. V_ℓ) is the rightmost (resp. leftmost) component of $D_1(u,R)$ (resp. $D_2(u,R)$) in R. Thus, if T_v intersects at least one component of $D_1(u,R)$ (resp. $D_2(u,R)$), then T_v intersects also with V_k (resp. V_ℓ). On the other hand, if T_v does not intersect any component of $D_1(u,R)$ (resp. $D_2(u,R)$), then T_v clearly does not intersect V_k (resp. V_ℓ), since $V_k \subseteq D_1(u,R)$ (resp. $V_j \subseteq D_2(u,R)$). It follows that $N_X(u, V_k, V_\ell) = N_X(u,R)$, and thus, $N_X(u, V_i, V_j) = N_X(u,R)$ for every $X \in \{0, 1, 2, 12\}$. This proves the lemma. ∎

Note that, given a trapezoid representation R of G, we may assume in Lemma 4 without loss of generality that $R(V_i) \ll_R T_u$, by possibly performing a vertical axis flipping of R. Thus, we can state now the following definition of the sets δ_u and δ_u^*, regardless of the choice the components V_i and V_j of u.

Definition 5 *Let $G = (V,E)$ be a trapezoid graph, u be a vertex of G, and V_i be an arbitrarily chosen master component of u. Then, $\delta_u = V_i$ and*

1. if $D_u^*(V_i) = \emptyset$, then $\delta_u^* = \emptyset$,

2. if $D_u^*(V_i) \neq \emptyset$, then $\delta_u^* = V_j$, for an arbitrarily chosen maximal component $V_j \in D_u^*(V_i)$.

From now on, whenever we speak about δ_u and δ_u^*, we assume that these arbitrary choices of V_i and V_j have been already made. Now, we are ready to define the following partition of the set $N(u)$, which will be used for the vertex splitting in Algorithm Split-U, cf. Definition 7.

Definition 6 *Let G be a trapezoid graph and u be a vertex of G. The vertices of $N(u)$ are partitioned into four possibly empty sets:*

1. $N_0(u)$: *vertices not adjacent to either* δ_u *or* δ_u^*,

2. $N_1(u)$: *vertices adjacent to* δ_u *but not to* δ_u^*,

3. $N_2(u)$: *vertices adjacent to* δ_u^* *but not to* δ_u,

4. $N_{12}(u)$: *vertices adjacent to both* δ_u *and* δ_u^*.

The next corollary follows now from Lemma 4 and Definitions 5 and 6. Intuitively, Corollary 1 states that, by possibly performing a vertical axis flipping of a given trapezoid representation R of G, the components V_i and V_j of Definition 3 can be thought as the rightmost (resp. leftmost) connected component of $G \setminus N[u]$ to the left (resp. to the right) of T_u in R.

Corollary 1 *Let G be a trapezoid graph, R be a representation of G, and u be a vertex of G with $\delta_u^* \neq \emptyset$. Let V_i be the master component of u that corresponds to δ_u. If $R(V_i) \ll_R T_u$, then $N_X(u) = N_X(u, R)$ for every $X \in \{0, 1, 2, 12\}$.*

The next lemma, which connects δ_u^* with the sets $N_1(u, R)$ and $N_2(u, R)$ in an arbitrary trapezoid representation R (see Definition 4), will be used in the proof of Theorem 1.

Lemma 5 *Let G be a trapezoid graph, R be a trapezoid representation of G, and u be a vertex of G. Then, $\delta_u^* \neq \emptyset$ if and only if $N_1(u, R) \neq \emptyset$ and $N_2(u, R) \neq \emptyset$.*

Proof. Recall first by Definition 4 that $D_1(u, R)$ and $D_2(u, R)$ are the sets of trapezoids of R that lie completely to the left and to the right of T_u in R, respectively. Furthermore, recall by Definition 4 that $N_1(u, R)$ are the neighbors of u that are adjacent to $D_1(u, R)$ but not to $D_2(u, R)$, while $N_2(u, R)$ are the neighbors of u that are adjacent to $D_2(u, R)$ but not to $D_1(u, R)$.

Suppose first that $\delta_u^* \neq \emptyset$. Let $\delta_u = V_i$ and $\delta_u^* = V_j$, where V_i is a master component of u and V_j is a maximal component of $D_u^*(V_i)$. By possibly performing a vertical axis flipping of R, we may assume without loss of generality that $R(V_i) \ll_R T_u$, and thus Corollary 1 implies that $N_1(u) = N_1(u, R)$ and $N_2(u) = N_2(u, R)$. Recall by Definition 6 that $N(V_i) = N_1(u) \cup N_{12}(u)$ and that $N(V_j) = N_2(u) \cup N_{12}(u)$. Assume that $N_2(u) = \emptyset$. Then $N(V_j) = N_{12}(u) \subseteq N_1(u) \cup N_{12}(u) = N(V_i)$, i.e. $N(V_j) \subseteq N(V_i)$, and thus $V_j \in D_u(V_i)$, which is a contradiction. Therefore $N_2(u) \neq \emptyset$, and thus also $N_2(u, R) \neq \emptyset$. Assume now that $N_1(u) = \emptyset$. Then $N(V_i) = N_{12}(u) \subseteq N_2(u) \cup N_{12}(u) = N(V_j)$, i.e. $N(V_i) \subseteq N(V_j)$. If $N(V_i) \subsetneqq N(V_j)$, then V_i is not a master component, which is a contradiction. Otherwise, if $N(V_i) = N(V_j)$, then $V_j \in D_u(V_i)$, which is again a contradiction. Therefore $N_1(u) \neq \emptyset$, and thus also $N_1(u, R) \neq \emptyset$. Summarizing, if $\delta_u^* \neq \emptyset$, then $N_1(u, R) \neq \emptyset$ and $N_2(u, R) \neq \emptyset$.

Conversely, suppose that $N_1(u, R) \neq \emptyset$ and $N_2(u, R) \neq \emptyset$. Assume that $\delta_u^* = \emptyset$. Let V_i be the master component of u that corresponds to δ_u. Then, since $\delta_u^* = \emptyset$, it follows that $D_u^*(V_i) = \emptyset$. By possibly performing a vertical axis flipping of R, we may assume without loss of generality that $R(V_i) \ll_R T_u$, and thus Corollary 1 implies that $N_1(u) = N_1(u, R)$. Now, since $R(V_i) \ll_R T_u$ and $N_2(u, R) \neq \emptyset$, there exists by Definition 4 a vertex $v \notin N(u)$ and a vertex $v' \in N(u)$, such that $T_u \ll_R T_v$ and $v' \in N(v) \setminus N(V_i)$. Let V_j be the connected component of $G \setminus N[u]$ that contains vertex v. Then $v' \in N(V_j) \setminus N(V_i)$, and thus $N(V_j) \nsubseteq N(V_i)$, i.e. $V_j \in D_u^*(V_i)$. This is a contradiction, since $D_u^*(V_i) = \emptyset$. Therefore $\delta_u^* \neq \emptyset$. This completes the proof of the lemma. ∎

Algorithm 1 Split-U

Input: A trapezoid graph G and a vertex subset $U = \{u_1, u_2, \ldots, u_k\}$, such that $\delta^*_{u_i} \neq \emptyset$ for all $i = 1, 2, \ldots, k$

Output: The permutation graph $G^{\#}(U)$

$\overline{U} \leftarrow V(G) \setminus U$; $H_0 \leftarrow G$

for $i = 1$ to k **do**

  $H_i \leftarrow H^{\#}_{i-1}(u_i)$ {H_i is obtained by the vertex splitting of u_i in H_{i-1}}

$G^{\#}(U) \leftarrow H_k[V(H_k) \setminus \overline{U}]$ {remove from H_k all unsplitted vertices}

return $G^{\#}(U)$

2.3 A splitting algorithm

We define now the splitting of a vertex u of a trapezoid graph G, where $\delta^*_u \neq \emptyset$. Note that this splitting operation does not depend on any trapezoid representation of G. Intuitively, if the graph G was given along with a specific trapezoid representation R, this would have meant that we replace the trapezoid T_u in R by its two lines $l(T_u)$ and $r(T_u)$.

Definition 7 *Let G be a trapezoid graph and u be a vertex of G, where $\delta^*_u \neq \emptyset$. The graph $G^{\#}(u)$ obtained by the* vertex splitting *of u is defined as follows:*

1. $V(G^{\#}(u)) = V(G) \setminus \{u\} \cup \{u_1, u_2\}$, *where u_1 and u_2 are the two new vertices.*
2. $E(G^{\#}(u)) = E[V(G) \setminus \{u\}] \cup \{u_1 x \mid x \in N_1(u)\} \cup \{u_2 x \mid x \in N_2(u)\} \cup \{u_1 x, u_2 x \mid x \in N_{12}(u)\}$.

The vertices u_1 and u_2 are the derivatives *of vertex u.*

We state now the notion of a standard trapezoid representation with respect to a particular vertex.

Definition 8 *Let G be a trapezoid graph and u be a vertex of G, where $\delta^*_u \neq \emptyset$. A trapezoid representation R of G is* standard with respect to u, *if the following properties are satisfied:*

1. $l(T_u) \ll_R R(N_0(u) \cup N_2(u))$.
2. $R(N_0(u) \cup N_1(u)) \ll_R r(T_u)$.

Now, given a trapezoid graph G and a vertex subset $U = \{u_1, u_2, \ldots, u_k\}$, such that $\delta^*_{u_i} \neq \emptyset$, $N_1(u_i) \setminus U \neq \emptyset$, and $N_2(u_i) \setminus U \neq \emptyset$, for every $i = 1, 2, \ldots, k$, Algorithm Split-U returns a graph $G^{\#}(U)$ by splitting every vertex of U exactly once. At every step, Algorithm Split-U splits a vertex of U, and finally, it removes all vertices of the set $V(G) \setminus U$, which have not been split.

Remark 2 *As mentioned in Remark 1, a similar algorithm, called Split-All, was presented in [6]. We would like to emphasize here the following four differences between the two algorithms. First, that Split-All gets as input a sibling-free graph G (two vertices u, v of a graph G are called* siblings, *if $N[u] = N[v]$; G is called* sibling-free *if G has no pair of sibling vertices), while our Algorithm Split-U gets as an input any graph (though, we will use it only for trapezoid graphs), which may contain pairs of sibling vertices. Second, Split-All splits all the vertices of the input graph, while Split-U splits only a subset of them, which satisfy a special property. Third, the order of vertices that are split by Split-All depends on a certain property (inclusion-minimal neighbor set), while Split-U splits the vertices in an arbitrary order. Last, the main difference between these two algorithms is that they perform a different vertex splitting operation at every step, since Definitions 5 and 6 do not comply with the corresponding Definitions 4.1 and 4.2 of [6].*

Theorem 1 *Let G be a trapezoid graph and $U = \{u_1, u_2, \ldots, u_k\}$ be a vertex subset of G, such that $\delta^*_{u_i} \neq \emptyset$, $N_1(u_i) \setminus U \neq \emptyset$, and $N_2(u_i) \setminus U \neq \emptyset$, for every $i = 1, 2, \ldots, k$. Then, the graph $G^{\#}(U)$ obtained by Algorithm Split-U, is a permutation graph with $2k$ vertices. Furthermore, if G is acyclic, then $G^{\#}(U)$ is acyclic with respect to $\{u^1_i, u^2_i\}^k_{i=1}$, where u^1_i and u^2_i are the derivatives of u_i, $i = 1, 2, \ldots, k$.*

Proof. Let R be a trapezoid representation of G. In order to prove that the graph $G^{\#}(U)$ constructed by Algorithm Split-All is a permutation graph, we will construct from R a permutation representation $R^{\#}(U)$ of $G^{\#}(U)$. To this end, we will construct sequentially, for every $i = 1, 2, \ldots, k$, a standard trapezoid representation of H_{i-1} with respect to u_i, in which all derivatives u^1_j, u^2_j, $1 \leq j \leq i-1$, are represented by trivial trapezoids, i.e. lines.

Let $u = u_1$. If R is not a standard representation with respect to u, we construct first from R a trapezoid representation R' of G that satisfies the first condition of Definition 8. Then, we construct from R' a trapezoid representation R'' of G that satisfies also the second condition of Definition 8, i.e. R'' is a standard trapezoid representation R' of G with respect to u.

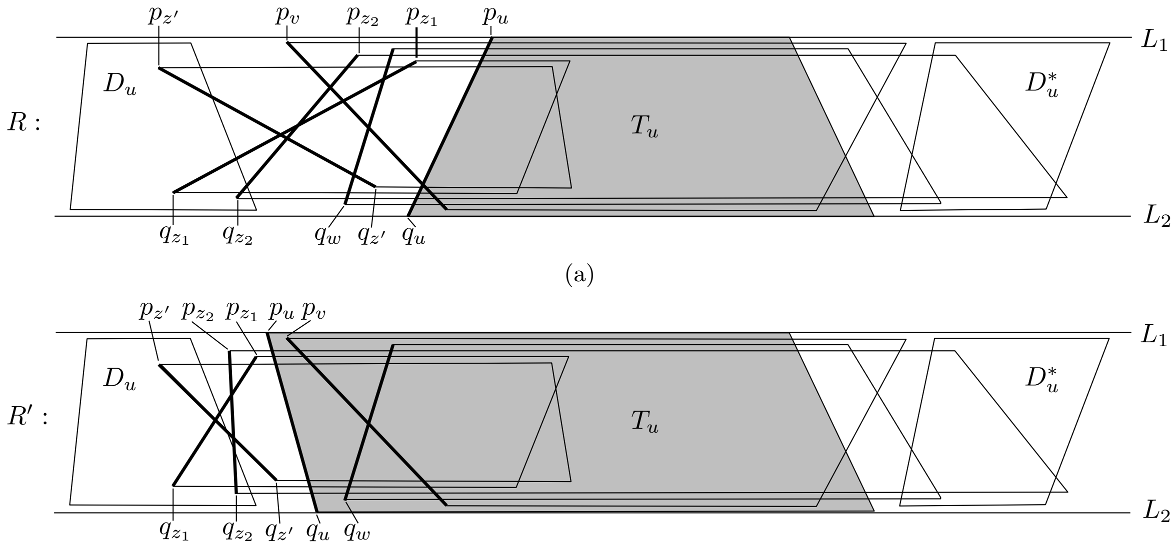
For the sake of presentation, we divide the proof of the theorem into several parts.

Properties of the representation R. Let V_i be the master component of u that corresponds to δ_u. By possibly performing a vertical axis flipping of R, we may assume without loss of generality that $R(V_i) \ll_R T_u$. Furthermore, the sets $N_0(u)$, $N_1(u)$, $N_2(u)$, and $N_{12}(u)$ coincide by Corollary 1 with the sets $N_0(u,R)$, $N_1(u,R)$, $N_2(u,R)$, and $N_{12}(u,R)$, respectively. Recall that, by Definition 4, $D_1(u,R)$ and $D_2(u,R)$ denote the sets of trapezoids of R that lie completely to the left and to the right of T_u in R, respectively.

Let p_x and q_x be the endpoints on L_1 and L_2, respectively, of the left line $l(T_x)$ of an arbitrary trapezoid T_x in R. Suppose that $N_0(u) \cup N_2(u) \neq \emptyset$. Let p_v and q_w be the leftmost endpoints on L_1 and L_2, respectively, of the trapezoids of $N_0(u) \cup N_2(u)$, and suppose that $p_v < p_u$ and $q_w < q_u$, cf. Figure 3(a). Note that, possibly, $v = w$. Then, all vertices x, for which T_x has an endpoint between p_v and p_u on L_1 (resp. between q_w and q_u on L_2) are adjacent to u. Indeed, suppose otherwise that $T_x \cap T_u = \emptyset$, for such a vertex x. Then, $T_x \ll_R T_u$, i.e. $x \in D_1(u,R)$, since T_x has an endpoint to the left of T_u in R. Furthermore, since $T_v \cap T_u \neq \emptyset$ (resp. $T_w \cap T_u \neq \emptyset$), it follows that $T_x \cap T_v \neq \emptyset$ (resp. $T_x \cap T_w \neq \emptyset$). However, since $x \in D_1(u,R)$, it follows that $v \in N_1(u,R) \cup N_{12}(u,R) = N_1(u) \cup N_{12}(u)$ (resp. $w \in N_1(u,R) \cup N_{12}(u,R) = N_1(u) \cup N_{12}(u)$), which is a contradiction.

Consider now a vertex $z \in N_1(u) \cup N_{12}(u)$ with $l(T_z) \ll_R l(T_u)$, where $p_v < p_z < p_u$ (cf. the vertices z_1 and z_2 in Figure 3(a)). Then, $q_z < q_w$. Indeed, suppose otherwise that $q_w < q_z$ (recall that all endpoints are assumed to be distinct). Then, since $z \in N_1(u) \cup N_{12}(u)$, there exists a vertex $x \in D_1(u,R)$, i.e. with $T_x \ll_R T_u$, such that $T_z \cap T_x \neq \emptyset$. Since $v, w \in N_0(u) \cup N_2(u)$, it follows that $T_v \cap T_x = \emptyset$ and $T_w \cap T_x = \emptyset$, and thus, $T_x \ll_R T_v$ and $T_x \ll_R T_w$. Therefore, since $p_v < p_z$ and $q_w < q_z$, we obtain that $T_x \ll_R T_z$, and thus, $T_z \cap T_x = \emptyset$, which is a contradiction. It follows that $q_z < q_w$. Moreover, z is adjacent to all vertices x in G, whose trapezoid T_x has an endpoint on L_1 between p_v and p_z, including p_v. Indeed, otherwise, $T_x \ll_R T_z$, and thus, $T_x \ll_R T_u$, since $l(T_z) \ll_R l(T_u)$. This is however a contradiction, since $x \in N(u)$, as we have proved above. Similarly, if $q_w < q_z < q_u$, then $p_z < p_v$ and z is adjacent to all vertices x in G, whose trapezoid T_x has an endpoint on L_2 between q_w and q_z, including q_w (cf. vertex z' in Figure 3(a)).

Construction of the representation R'. We construct now from R a new trapezoid representation R' of G as follows. First, for all vertices $z \in N_1(u) \cup N_{12}(u)$ with $l(T_z) \ll_R l(T_u)$, for which $p_v < p_z < p_u$ (and thus $q_z < q_w$), we move the endpoint p_z of $l(T_z)$ directly before p_v on L_1 (cf. the vertices z_1 and z_2 in Figures 3(a) and 3(b)). Then, for all vertices $z' \in N_1(u) \cup N_{12}(u)$ with $l(T_{z'}) \ll_R l(T_u)$, for which $q_w < q_{z'} < q_u$ (and thus $p_z < p_v$), we



(a)

(b)

Figure 3: The movement of the left line $l(T_u)$ of the trapezoid T_u, in order to construct a standard trapezoid representation with respect to u.

move the endpoint $q_{z'}$ of $l(T_{z'})$ directly before q_w on L_2 (cf. vertex z' in Figures 3(a) and 3(b)). During the movement of all these lines $l(T_z)$ (resp. $l(T_{z'})$), we keep the same relative positions of their endpoints p_z on L_1 (resp. $q_{z'}$ on L_2) as in R, and thus we introduce no new line intersection among the lines of the trapezoids of G. Since all these vertices z (resp. z') are adjacent to all vertices x of G, whose trapezoid T_x has an endpoint on L_1 (resp. L_2) between p_v and p_z, including p_v (resp. between q_w and q_z, including q_w), these movements do not remove any adjacency from, and do not add any new adjacency to G.

Finally, we move both endpoints p_u and q_u of $l(T_u)$ directly before p_v and q_w on L_1 and L_2, respectively. Since u is adjacent to all vertices x, for which T_x has an endpoint between p_v and p_u on L_1, or between q_w and q_u on L_2 in R, the resulting representation R' is a trapezoid representation of G, in which the first condition of Definition 8 is satisfied. Since we moved all lines $l(T_z)$ and $l(T_{z'})$ to the left of T_v and T_w, R' has no additional line intersections than R. Moreover, note that for any line intersection of two lines a and b in R', the relative position of the endpoints of a and b on L_1 and L_2 remains the same as in R. In the case where $p_v > p_u$ (resp. $q_w > q_u$) we replace in the above construction p_v by p_u (resp. q_w by q_u), while in the case where $N_0(u) \cup N_2(u) = \emptyset$, we define $R' = R$. An example of the construction of R' is given in Figure 3. In this example, $v \in N_0(u)$, $w \in N_2(u)$, $z_1, z' \in N_1(u)$ and $z_2 \in N_{12}(u)$.

Construction of the representation R''. If R' is not a standard trapezoid representation with respect to u, then we move $r(T_u)$ to the right (similarly to the above), obtaining thus a trapezoid representation R'' of G, in which the second condition of Definition 8 is satisfied. Since during the construction of R'' from R' only the line $r(T_u)$, and other lines that lie completely to the right of $r(T_u)$, are moved to the right, the first condition of Definition 8 is satisfied for R'' as well. Thus, R'' is a standard representation of G with respect to u. Similarly to R', R'' has no additional line intersections than R. Moreover, for any line intersection of two lines a and b in R'', the relative position of the endpoints of a and b on L_1 and L_2 remains the same as in R.

Splitting of vertex u. Since R'' is standard with respect to u, the left line $l(T_u)$ of T_u in R'' intersects exactly with those trapezoids T_z, for which $z \in N_1(u) \cup N_{12}(u)$. On the other hand, the right line $r(T_u)$ of T_u in R'' intersects exactly with those trapezoids T_z, for which

$z \in N_2(u) \cup N_{12}(u)$. Thus, if we replace in R'' the trapezoid T_u by the two trivial trapezoids (lines) $l(T_u)$ and $r(T_u)$, we obtain a trapezoid representation $R^{\#}(u)$ of the graph $G^{\#}(u)$ defined in Definition 7.

Consider now a vertex $v \in \{u_2, u_3, \ldots, u_k\}$. Recall by the assumption in the statement of the theorem that $\delta_v^* \neq \emptyset$, $N_1(v) \setminus U \neq \emptyset$, and $N_2(v) \setminus U \neq \emptyset$ in G (before the splitting of vertex u). We now prove in the next claim that the same conditions on v remain true also in the trapezoid graph $G^{\#}(u)$ (after the splitting of vertex u), and thus the above construction can be iteratively applied to eventually split all vertices of U.

Claim 1 *Let $v \in \{u_2, u_3, \ldots, u_k\}$. Then, in $G^{\#}(u)$ (i.e. after the splitting of $u = u_1$), it remains $\delta_v^* \neq \emptyset$, $N_1(v) \setminus U \neq \emptyset$, and $N_2(v) \setminus U \neq \emptyset$.*

Proof of Claim 1. Let V_i and V_j be the components that correspond to δ_v and δ_v^*, respectively (before the vertex splitting of u). By possibly performing a vertical axis flipping of R'', we may assume without loss of generality that $R''(V_i) \ll_{R''} T_v$, and thus Corollary 1 implies that $N_1(v) = N_1(v, R'')$ and $N_2(v) = N_2(v, R'')$. Since by assumption $N_1(v) \setminus U \neq \emptyset$ and $N_2(v) \setminus U \neq \emptyset$ before the splitting of u, there exist vertices $x_v \in N_1(v) = N_1(v, R'')$ and $y_v \in N_2(v) = N_2(v, R'')$, such that $x_v, y_v \notin U$. That is, the trapezoid T_{x_v} is adjacent to the trapezoids to the left (but not to the right) of T_v in R'', and the trapezoid T_{y_v} is adjacent to the trapezoids to the right (but not to the left) of T_v in R''. Furthermore, since $x_v, y_v \notin U$, the trapezoids T_{x_v} and T_{y_v} are never split during the execution of Algorithm Split-U. Thus, in particular, T_{x_v} and T_{y_v} remain unchanged in both R'' and $R^{\#}(u)$, i.e. both before and after the splitting of vertex u.

Let now u_l and u_r be the two derivatives of vertex u, which correspond to the lines $l(T_u)$ and $r(T_u)$ of T_u, respectively. Suppose first that $v \in N(u)$ (before the splitting of u). Then, in $R^{\#}(u)$, each of the lines of u_l and u_r, either intersects T_v, or lies to the left/right of T_v. In both cases, the trapezoid T_{x_v} remains adjacent to the trapezoids to the left (but not to the right) of T_v in $R^{\#}(u)$, and the trapezoid T_{y_v} remains adjacent to the trapezoids to the right (but not to the left) of T_v in $R^{\#}(u)$. Suppose now that $u \notin N(u)$ (before the splitting of u), i.e. either $T_u \ll_{R''} T_v$ or $T_v \ll_{R''} T_u$. Since the two cases are exactly symmetrical, it suffices to consider only the case where $T_u \ll_{R''} T_v$. In this case, $u \in N(x_v)$ before the splitting of u if and only if $u_r \in N(x_v)$ after the splitting of u. Furthermore, since $y_v \in N_2(v, R'')$, it follows that $u \notin N(y_v)$ before the splitting of u and also that $u_l, u_r \notin N(y_v)$ after the splitting of u. Thus, the trapezoid T_{x_v} remains adjacent to the trapezoids to the left (but not to the right) of T_v in $R^{\#}(u)$, and the trapezoid T_{y_v} remains adjacent to the trapezoids to the right (but not to the left) of T_v in $R^{\#}(u)$.

Summarizing, in both cases where $v \in N(u)$ and $v \notin N(u)$ before the splitting of u, it follows that $x_v \in N_1(v, R^{\#}(u))$ and $y_v \in N_2(v, R^{\#}(u))$ after the splitting of u. Therefore, since $x_v, y_v \notin U$, it follows that $N_1(v, R^{\#}(u)) \setminus U \neq \emptyset$ and $N_2(v, R^{\#}(u)) \setminus U \neq \emptyset$ after the splitting of u. Furthermore, since $N_1(v, R^{\#}(u)) \neq \emptyset$ and $N_2(v, R^{\#}(u)) \neq \emptyset$, Lemma 5 implies that $\delta_u^* \neq \emptyset$ after the splitting of u. Therefore, Corollary 1 implies that the sets $N_1(v)$ and $N_2(v)$ are the same as the sets $N_1(v, R^{\#}(u))$ and $N_2(v, R^{\#}(u))$, and thus also $N_1(v) \setminus U \neq \emptyset$ and $N_2(v) \setminus U \neq \emptyset$ after the splitting of u. Summarizing, after the splitting of $u = u_1$, we have that $\delta_v^* \neq \emptyset$, $N_1(v) \setminus U \neq \emptyset$, and $N_2(v) \setminus U \neq \emptyset$, for every $v \in \{u_2, u_3, \ldots, u_k\}$. □

Iterative splitting of all the vertices of the set U. Due to Claim 1, we can iteratively apply the above construction for all $u = u_i$, where $i = 2, 3, \ldots, k$, i.e. we can split sequentially all vertices of U exactly once. Then, after k vertex splittings, and after removing from the resulting graph the vertices of $\overline{U} = V(G) \setminus U$, we obtain a trapezoid representation $R^{\#}(U)$ of the graph $G^{\#}(U)$ returned by Algorithm Split-U. Since every trapezoid T_u, $u \in U$, has been replaced by two trivial trapezoids (i.e. lines) in $R^{\#}(U)$, it follows that $G^{\#}(U)$ is a permutation graph with $2k$ vertices, and $R^{\#}(U)$ is a permutation representation of $G^{\#}(U)$.

Acyclicity of the permutation graph $G^{\#}(U)$. Finally, suppose that R is an acyclic trapezoid representation of G. According to Definition 2, let P be the permutation graph with $2n$ vertices corresponding to the left and right lines of the trapezoids in R, R_P be the permutation representation of P induced by R, and $\{u_i^1, u_i^2\}$ be the vertices of P that correspond to the same vertex u_i of G, $i = 1, 2, \ldots, n$. Since R is an acyclic trapezoid representation of G, it follows by Definition 2 that R_P is an acyclic permutation representation with respect to $\{u_i^1, u_i^2\}_{i=1}^n$. That is, the simple directed graph F_{R_P} obtained (according to Definition 1) by merging u_i^1 and u_i^2 in P into a single vertex u_i, for every $i = 1, 2, \ldots, n$, has no directed cycle.

Since, during the construction of $R^{\#}(U)$, the trapezoid representation obtained after every vertex splitting has no additional line intersections than the previous one, it follows that $R^{\#}(U)$ has no additional line intersections than R. Moreover, for any line intersection of two lines a and b in $R^{\#}(U)$, the relative position of the endpoints of a and b on L_1 and L_2 remains the same as in R. Thus, the simple directed graph $F_{R\#(U)}$ obtained (according to Definition 1) by merging u_i^1 and u_i^2 in $G^{\#}(U)$ into a single vertex u_i, for every $i = 1, 2, \ldots, k$, is a subdigraph of F_{R_P}. Therefore, since F_{R_P} has no directed cycle, $F_{R\#(U)}$ has no directed cycle as well, i.e. $G^{\#}(U)$ is an acyclic permutation graph with respect to $\{u_i^1, u_i^2\}_{i=1}^k$. This completes the proof of the theorem. ■

3 The recognition of bounded tolerance graphs

In this section we provide a reduction from the *monotone-Not-All-Equal-3-SAT (monotone-NAE-3-SAT)* problem to the problem of recognizing whether a given graph is a bounded tolerance graph. A boolean formula ϕ is called *monotone* if no variable in ϕ is negated. Given a monotone boolean formula ϕ in conjunctive normal form with three literals in each clause (3-CNF), ϕ is *NAE-satisfiable* if there is a truth assignment of ϕ, such that every clause contains at least one true literal and at least one false one. The problem of deciding whether a given 3-CNF formula ϕ is NAE-satisfiable is known to be NP-complete [30]. In the next lemma we provide a reduction of the NAE-3-SAT problem to the *monotone*-NAE-3-SAT problem, which proves that monotone-NAE-3-SAT is NP-complete.

Lemma 6 *Monotone-NAE-3-SAT problem is NP-complete.*

Proof. To reduce NAE-3-SAT to monotone-NAE-3-SAT, consider first a 3-CNF formula ϕ (the input of NAE-3-SAT). We construct from ϕ a monotone 3-CNF formula ϕ' as follows. Replace each appearance of a variable x in ϕ with two variables x_0 and x_1 (depending on whether x appears negated or not), add variables x_2, x_3, x_4, and add the clauses $(x_0 \vee x_1 \vee x_2)$, $(x_0 \vee x_1 \vee x_3)$, $(x_0 \vee x_1 \vee x_4)$, and $(x_2 \vee x_3 \vee x_4)$. Then, it is easy to check that the constructed 3-CNF formula ϕ' is monotone (i.e. no variable appears negated in ϕ') and that ϕ' is NAE-satisfiable if and only if ϕ is NAE-satisfiable. ■

We can assume in the following without loss of generality that each clause has three distinct literals, i.e. variables. Given a monotone 3-CNF formula ϕ, we construct in polynomial time a trapezoid graph H_ϕ, such that H_ϕ is a bounded tolerance graph if and only if ϕ is NAE-satisfiable. To this end, we construct first a permutation graph P_ϕ and a trapezoid graph G_ϕ.

3.1 The permutation graph P_ϕ

Consider a monotone 3-CNF formula $\phi = \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k$ with k clauses and n boolean variables $x_1, x_2, \ldots, x_n$, such that $\alpha_i = (x_{r_{i,1}} \vee x_{r_{i,2}} \vee x_{r_{i,3}})$ for $i = 1, 2, \ldots, k$, where $1 \leq r_{i,1} < r_{i,2} < r_{i,3} \leq n$. We construct the permutation graph P_ϕ, along with a permutation representation R_P of P_ϕ, as follows. Let L_1 and L_2 be two parallel lines and let $\theta(\ell)$ denote the angle of the line ℓ with L_2 in R_P. For every clause α_i, $i = 1, 2, \ldots, k$, we correspond to each of the literals, i.e. variables, $x_{r_{i,1}}$, $x_{r_{i,2}}$, and $x_{r_{i,3}}$ a pair of intersecting lines with endpoints on L_1 and L_2. Namely, we correspond to the variable $x_{r_{i,1}}$ the pair $\{a_i, c_i\}$, to $x_{r_{i,2}}$ the pair $\{e_i, b_i\}$ and to $x_{r_{i,3}}$ the pair $\{d_i, f_i\}$, respectively, such that $\theta(a_i) > \theta(c_i)$, $\theta(e_i) > \theta(b_i)$, $\theta(d_i) > \theta(f_i)$, and such that the lines a_i, c_i lie completely to the left of e_i, b_i in R_P, and e_i, b_i lie completely to the left of d_i, f_i in R_P, as it is illustrated in Figure 4. Denote the lines that correspond to the variable $x_{r_{i,j}}$, $j = 1, 2, 3$, by $\ell^1_{i,j}$ and $\ell^2_{i,j}$, respectively, such that $\theta(\ell^1_{i,j}) > \theta(\ell^2_{i,j})$. That is, $(\ell^1_{i,1}, \ell^2_{i,1}) = (a_i, c_i)$, $(\ell^1_{i,2}, \ell^2_{i,2}) = (e_i, b_i)$, and $(\ell^1_{i,3}, \ell^2_{i,3}) = (d_i, f_i)$. Note that no line of a pair $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ intersects with a line of another pair $\{\ell^1_{i',j'}, \ell^2_{i',j'}\}$.

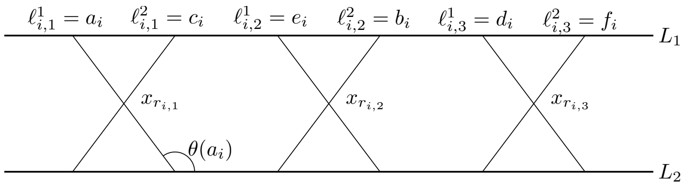


Figure 4: The six lines of the permutation graph P_ϕ, which correspond to the clause $\alpha_i = (x_{r_{i,1}} \vee x_{r_{i,2}} \vee x_{r_{i,3}})$ of the boolean formula ϕ.

Denote by S_p, $p = 1, 2, \ldots, n$, the set of pairs $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ that correspond to the variable x_p, i.e. $r_{i,j} = p$. We order the pairs $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ such that any pair of S_{p_1} lies completely to the left of any pair of S_{p_2}, whenever $p_1 < p_2$, while the pairs that belong to the same set S_p are ordered arbitrarily. For two consecutive pairs $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ and $\{\ell^1_{i',j'}, \ell^2_{i',j'}\}$ in S_p, where $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ lies to the left of $\{\ell^1_{i',j'}, \ell^2_{i',j'}\}$, we add a pair $\{u^{i',j'}_{i,j}, v^{i',j'}_{i,j}\}$ of parallel lines that intersect both $\ell^1_{i,j}$ and $\ell^1_{i',j'}$, but no other line. Note that $\theta(\ell^1_{i,j}) > \theta(u^{i',j'}_{i,j})$ and $\theta(\ell^1_{i',j'}) > \theta(u^{i',j'}_{i,j})$, while $\theta(u^{i',j'}_{i,j}) = \theta(v^{i',j'}_{i,j})$. This completes the construction. Denote the resulting permutation graph by P_ϕ, and the corresponding permutation representation of P_ϕ by R_P. Observe that P_ϕ has n connected components, which are called *blocks*, one for each variable $x_1, x_2, \ldots, x_n$.

An example of the construction of P_ϕ and R_P from ϕ with $k = 3$ clauses and $n = 4$ variables is illustrated in Figure 5. In this figure, the lines $u^{i',j'}_{i,j}$ and $v^{i',j'}_{i,j}$ are drawn in bold.

The formula ϕ has $3k$ literals, and thus the permutation graph P_ϕ has $6k$ lines $\ell^1_{i,j}, \ell^2_{i,j}$ in R_P, one pair for each literal. Furthermore, two lines $u^{i',j'}_{i,j}, v^{i',j'}_{i,j}$ correspond to each pair of consecutive pairs $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ and $\{\ell^1_{i',j'}, \ell^2_{i',j'}\}$ in R_P, except for the case where these pairs of lines belong to different variables, i.e. when $r_{i,j} \neq r_{i',j'}$. Therefore, since ϕ has n variables, there are $2(3k - n) = 6k - 2n$ lines $u^{i',j'}_{i,j}, v^{i',j'}_{i,j}$ in R_P. Thus, R_P has in total $12k - 2n$ lines, i.e. P_ϕ has $12k - 2n$ vertices. In the example of Figure 5, $k = 3$, $n = 4$, and thus, P_ϕ has 28 vertices.

Let $m = 6k - n$, where $2m$ is the number of vertices in P_ϕ. We group the lines of R_P, i.e. the vertices of P_ϕ, into pairs $\{u^1_i, u^2_i\}^m_{i=1}$, as follows. For every clause α_i, $i = 1, 2, \ldots, k$, we group the lines $a_i, b_i, c_i, d_i, e_i, f_i$ into the three pairs $\{a_i, b_i\}$, $\{c_i, d_i\}$, and $\{e_i, f_i\}$. The remaining lines are grouped naturally according to the construction; namely, every two lines $\{u^{i',j'}_{i,j}, v^{i',j'}_{i,j}\}$ constitute a pair.

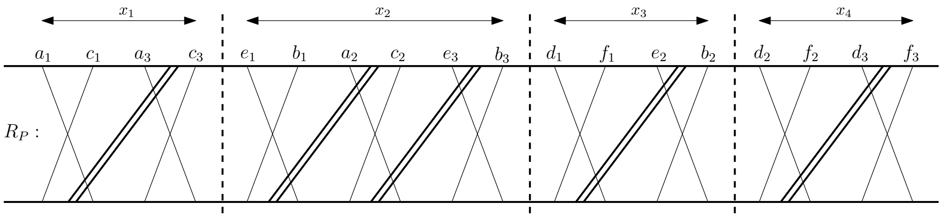


Figure 5: The permutation representation R_P of the permutation graph P_ϕ for $\phi = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_4)$.

Lemma 7 *If the permutation graph P_ϕ is acyclic with respect to $\{u_i^1, u_i^2\}_{i=1}^m$ then the formula ϕ is NAE-satisfiable.*

Proof. Suppose that P_ϕ is acyclic with respect to $\{u_i^1, u_i^2\}_{i=1}^m$, and let R_0 be an acyclic permutation representation of P_ϕ with respect to $\{u_i^1, u_i^2\}_{i=1}^m$. Then, in particular, R_0 is acyclic with respect to $\{a_i, b_i\}, \{c_i, d_i\}, \{e_i, f_i\}$, for every $i = 1, 2, \ldots, k$. We will construct a truth assignment of the variables $x_1, x_2, \ldots, x_n$ that NAE-satisfies ϕ, as follows. For every $i = 1, 2, \ldots, k$, we define $x_{r_{i,1}} = 1$ if and only if $\theta(c_i) < \theta(a_i)$ in R_0, $x_{r_{i,2}} = 1$ if and only if $\theta(b_i) < \theta(e_i)$ in R_0, and $x_{r_{i,3}} = 1$ if and only if $\theta(f_i) < \theta(d_i)$ in R_0.

Note that this assignment is consistent; that is, all variables $x_{r_{i,j}}$ that correspond to the same x_k are assigned the same value. Indeed, every block (i.e. connected component) of the permutation graph P_ϕ is a very particular graph, namely an odd path with pendent vertices on alternating vertices and duplicating the other vertices. It is easy to see that each such connected component of P_ϕ has exactly two permutation representations (related by the horizontal axis flipping), where these representations correspond to the values 0 and 1 of x_k in the assignment, respectively. In other words, the existence of the lines $u_{i,j}^{i',j'}, v_{i,j}^{i',j'}$ (cf. the bold lines in Figure 6(a)) forces all pairs of crossing lines $\{\ell_{i,j}^1, \ell_{i,j}^2\}$ in the same connected component to correspond to either 0 or 1 in the assignment.

Now, we show that in each clause α_i, $i = 1, 2, \ldots, k$, there exists at least one true and at least one false variable. For an arbitrary index $i \in \{1, 2, \ldots, k\}$, let P_i be the subgraph induced by the vertices $a_i, b_i, c_i, d_i, e_i, f_i$ in P_ϕ, and R_i be the permutation representation of P_i, which is induced by R_0. According to Definition 1, we construct the simple directed graph F_{R_i} by merging into a single vertex each of the pairs $\{a_i, b_i\}$, $\{c_i, d_i\}$ and $\{e_i, f_i\}$ of vertices of P_i. The arc directions of F_{R_i} are implied by the corresponding directions in Φ_{R_i} (or equivalently, in Φ_{R_0}). Then, since R_0 is acyclic with respect to $\{a_i, b_i\} \cup \{c_i, d_i\} \cup \{e_i, f_i\}$, so is R_i. Thus, it follows by Definition 1 that F_{R_i} has no directed cycle. Therefore, it does not hold simultaneously $c_i a_i, b_i e_i, f_i d_i \in \Phi_{R_0}$, or $a_i c_i, e_i b_i, d_i f_i \in \Phi_{R_0}$. That is, it does not hold simultaneously $\theta(c_i) < \theta(a_i)$, $\theta(b_i) < \theta(e_i)$, and $\theta(f_i) < \theta(d_i)$, or $\theta(a_i) < \theta(c_i)$, $\theta(e_i) < \theta(b_i)$, and $\theta(d_i) < \theta(f_i)$ in R_0, respectively. Then, by the definition of the above truth assignment, it follows that it does not hold simultaneously $x_{r_{i,1}} = x_{r_{i,2}} = x_{r_{i,3}} = 1$, or $x_{r_{i,1}} = x_{r_{i,2}} = x_{r_{i,3}} = 0$, and therefore, the clause $\alpha_i = (x_{r_{i,1}} \vee x_{r_{i,2}} \vee x_{r_{i,3}})$ is NAE-satisfied. Finally, since this holds for every $i = 1, 2, \ldots, k$, ϕ is NAE-satisfiable. ■

Note here that the converse of Lemma 7 is also true, i.e. if the formula ϕ is NAE-satisfiable, then the permutation graph P_ϕ is acyclic with respect to $\{u_i^1, u_i^2\}_{i=1}^m$ (this can be easily proved, similarly to the necessity part of the proof of Theorem 2 below). That is, the permutation graph P_ϕ is acyclic with respect to $\{u_i^1, u_i^2\}_{i=1}^m$ if and only if the monotone formula ϕ is NAE-satisfiable. Therefore, since monotone-NAE-3-SAT problem is NP-complete by Lemma 6, it follows that, given a permutation graph P with vertices $\{u_1^1, u_1^2, \ldots, u_m^1, u_m^2\}$, it is NP-hard to decide whether P is acyclic with respect to $\{u_i^1, u_i^2\}_{i=1}^m$.

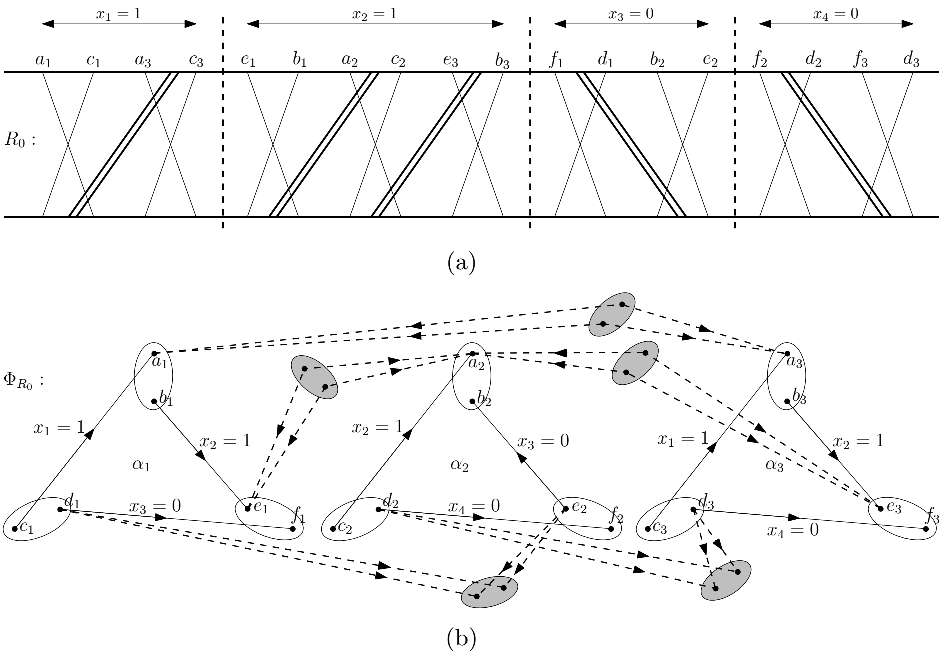


Figure 6: The NAE-satisfying truth assignment $(x_1, x_2, x_3, x_4) = (1, 1, 0, 0)$ of the formula ϕ of Figure 5: (a) an acyclic permutation representation R_0 of P_ϕ and (b) the corresponding transitive orientation Φ_{R_0} of P_ϕ.

For the formula ϕ of Figure 5, an example of an acyclic permutation representation R_0 of P_ϕ with respect to $\{u_i^1, u_i^2\}_{i=1}^m$, along with the corresponding transitive orientation Φ_{R_0} of P_ϕ, is illustrated in Figure 6. This transitive orientation corresponds to the NAE-satisfying truth assignment $(x_1, x_2, x_3, x_4) = (1, 1, 0, 0)$ of ϕ. Similarly to Figure 5, the lines $u_{i,j}^{i',j'}$ and $v_{i,j}^{i',j'}$ are drawn in bold in Figure 6(a). Furthermore, for better visibility, the vertices that correspond to these lines are grouped in shadowed ellipses in Figure 6(b), while the arcs incident to them are drawn dashed.

3.2 The trapezoid graphs G_ϕ and H_ϕ

Let $\{u_i^1, u_i^2\}_{i=1}^m$ be the pairs of vertices in the constructed permutation graph P_ϕ and R_P be its permutation representation. We construct now from P_ϕ the trapezoid graph G_ϕ with m vertices $\{u_1, u_2, \ldots, u_m\}$, as follows. We replace in the permutation representation R_P for every $i = 1, 2, \ldots, m$ the lines u_i^1 and u_i^2 by the trapezoid T_{u_i}, which has u_i^1 and u_i^2 as its left and right lines, respectively. Let R_G be the resulting trapezoid representation of G_ϕ.

Finally, we construct from G_ϕ the trapezoid graph H_ϕ with $7m$ vertices, by adding to every trapezoid T_{u_i}, $i = 1, 2, \ldots, m$, six parallelograms $T_{u_{i,1}}, T_{u_{i,2}}, \ldots, T_{u_{i,6}}$ in the trapezoid representation R_G, as follows. Let ε be the smallest distance in R_G between two different endpoints on L_1, or on L_2. The right (resp. left) line of $T_{u_{1,1}}$ lies to the right (resp. left) of u_1^1, and it is parallel to it at distance $\frac{\varepsilon}{2}$. The right (resp. left) line of $T_{u_{1,2}}$ lies to the left of u_1^1, and it is parallel to it at distance $\frac{\varepsilon}{4}$ (resp. $\frac{3\varepsilon}{4}$). Moreover, the right (resp. left) line of $T_{u_{1,3}}$ lies to the left of u_1^1, and it is parallel to it at distance $\frac{3\varepsilon}{8}$ (resp. $\frac{7\varepsilon}{8}$). Similarly, the left (resp. right) line of $T_{u_{1,4}}$ lies to the left (resp. right) of u_1^2, and it is parallel to it at distance $\frac{\varepsilon}{2}$. The left (resp. right) line of $T_{u_{1,5}}$ lies to the right of u_1^2, and it is parallel to it at distance $\frac{\varepsilon}{4}$ (resp. $\frac{3\varepsilon}{4}$). Finally, the right (resp. left) line of $T_{u_{1,6}}$ lies to the right of u_1^2, and it is parallel to it at distance $\frac{3\varepsilon}{8}$ (resp. $\frac{7\varepsilon}{8}$), as illustrated in Figure 7.

After adding the parallelograms $T_{u_{1,1}}, T_{u_{1,2}}, \ldots, T_{u_{1,6}}$ to a trapezoid T_{u_1}, we update the

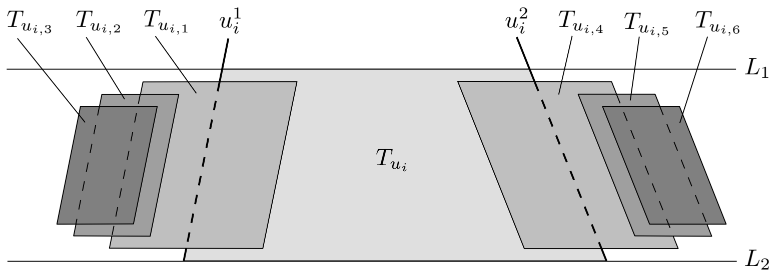


Figure 7: The addition of the six parallelograms $T_{u_{i,1}}, T_{u_{i,2}}, \ldots, T_{u_{i,6}}$ to the trapezoid T_{u_i}, $i = 1, 2, \ldots, m$, in the construction of the trapezoid graph H_ϕ from G_ϕ.

smallest distance ε between two different endpoints on L_1, or on L_2 in the resulting representation, and we continue the construction iteratively for all $i = 2, \ldots, m$. Denote by H_ϕ the resulting trapezoid graph with $7m$ vertices, and by R_H the corresponding trapezoid representation. Note that in R_H, between the endpoints of the parallelograms $T_{u_{i,1}}$, $T_{u_{i,2}}$, and $T_{u_{i,3}}$ (resp. $T_{u_{i,4}}$, $T_{u_{i,5}}$, and $T_{u_{i,6}}$) on L_1 and L_2, there are no other endpoints of H_ϕ, except those of u_i^1 (resp. u_i^2), for every $i = 1, 2, \ldots, m$. The next lemma is crucial in the proof of Theorem 2.

Lemma 8 *In the trapezoid graph H_ϕ, let $U = \{u_1, u_2, \ldots, u_m\}$. Then $\delta^*_{u_i} \neq \emptyset$, $N_1(u_i) \backslash U \neq \emptyset$, and $N_2(u_i) \setminus U \neq \emptyset$, for every $i = 1, 2, \ldots, m$.*

Proof. Consider the trapezoid representation R_H of H_ϕ. Let $i \in \{1, 2, \ldots, m\}$. Recall by Definition 4 that $D_1(u_i, R_H)$ (resp. $D_2(u_i, R_H)$) denotes the set of trapezoids of H_ϕ that lie completely to the left (resp. to the right) of T_{u_i} in R_H. In particular, $T_{u_{i,2}}, T_{u_{i,3}} \in D_1(u_i, R_H)$ and $T_{u_{i,5}}, T_{u_{i,6}} \in D_2(u_i, R_H)$. Furthermore, recall by Definition 4 that $N_1(u_i, R_H)$ are the neighbors of u_i that are adjacent to $D_1(u_i, R_H)$ but not to $D_2(u_i, R_H)$, while $N_2(u_i, R_H)$ are the neighbors of u_i that are adjacent to $D_2(u_i, R_H)$ but not to $D_1(u_i, R_H)$. In particular, $u_{i,1} \in N_1(u_i, R_H)$ and $u_{i,4} \in N_2(u_i, R_H)$. Therefore, since $u_{i,1}, u_{i,4} \notin U$, it follows that $N_1(u_i, R_H) \setminus U \neq \emptyset$ and $N_2(u_i, R_H) \setminus U \neq \emptyset$. Furthermore, since $N_1(u_i, R_H) \neq \emptyset$ and $N_2(u_i, R_H) \neq \emptyset$, Lemma 5 implies that $\delta^*_{u_i} \neq \emptyset$.

By the construction of R_H, note that $T_{u_{i,2}} \cup T_{u_{i,3}}$ (resp. $T_{u_{i,5}} \cup T_{u_{i,6}}$) is the rightmost (resp. leftmost) connected component of $D_1(u_i, R_H)$ (resp. $D_2(u_i, R_H)$). Therefore $N(V_k) \subseteq N(\{u_{i,2}, u_{i,3}\})$ (resp. $N(V_\ell) \subseteq N(\{u_{i,5}, u_{i,6}\})$), for every connected component V_k (resp. V_ℓ) of $D_1(u_i, R_H)$ (resp. $D_2(u_i, R_H)$). Let V_p be the master component of u_i that corresponds to δ_{u_i}. Then, either $V_p = \{u_{i,2}, u_{i,3}\}$, or $V_p = \{u_{i,5}, u_{i,6}\}$. In the case where $V_p = \{u_{i,2}, u_{i,3}\}$, Corollary 1 implies that $N_1(u_i) = N_1(u_i, R_H)$ and $N_2(u_i) = N_2(u_i, R_H)$. Thus, since $N_1(u_i, R_H) \setminus U \neq \emptyset$ and $N_2(u_i, R_H) \setminus U \neq \emptyset$ by the previous paragraph, it follows that $N_1(u_i) \setminus U \neq \emptyset$ and $N_2(u_i) \setminus U \neq \emptyset$. Similarly, in the case where $V_p = \{u_{i,5}, u_{i,6}\}$, Corollary 1 implies (by performing a vertical axis flipping of R_H) that $N_1(u_i) = N_2(u_i, R_H)$ and $N_2(u_i) = N_1(u_i, R_H)$. Thus, since $N_2(u_i, R_H) \setminus U \neq \emptyset$ and $N_1(u_i, R_H) \setminus U \neq \emptyset$ by the previous paragraph, it follows that $N_1(u_i) \setminus U \neq \emptyset$ and $N_2(u_i) \setminus U \neq \emptyset$. Summarizing, $\delta^*_{u_i} \neq \emptyset$, $N_1(u_i) \backslash U \neq \emptyset$, and $N_2(u_i) \backslash U \neq \emptyset$, for every $i = 1, 2, \ldots, m$. This completes the proof of the lemma. ■

Let $i \in \{1, 2, \ldots, m\}$. Note that, by the construction of R_H, the left line $l(T_{u_i})$ (resp. the right line $r(T_{u_i})$) of T_{u_i} intersects in R_H exactly with the trapezoids that intersect $T_{u_{i,2}} \cup T_{u_{i,3}}$ (resp. $T_{u_{i,5}} \cup T_{u_{i,6}}$). That is, the left line $l(T_{u_i})$ intersects exactly with the trapezoids of $N_1(u_i, R_H) \cup N_{12}(u_i, R_H)$, while the right line $r(T_{u_i})$ intersects exactly with the trapezoids of $N_2(u_i, R_H) \cup N_{12}(u_i, R_H)$. Let now V_p be the master component of u_i that corresponds to δ_{u_i} in H_ϕ. Recall by the proof of Lemma 8 that either $V_p = \{u_{i,2}, u_{i,3}\}$ or $V_p = \{u_{i,5}, u_{i,6}\}$, since $\{u_{i,2}, u_{i,3}\}$ and $\{u_{i,5}, u_{i,6}\}$ are the two master components of u_i (i.e. the two maximal connected components of $H_\phi \setminus N[u_i]$). However, since $\delta_{u_i} = V_p$ is an arbitrarily chosen master component of u_i by Definition 5, we can choose $V_p = \{u_{i,2}, u_{i,3}\}$, i.e. $R_H(V_P) \ll_{R_H} T_{u_i}$. Furthermore, since $\delta^*_{u_i} \neq \emptyset$ by Lemma 8, it follows by Corollary 1 that $N_1(u_i) \cup N_{12}(u_i) = N_1(u_i, R_H) \cup N_{12}(u_i, R_H)$ and that $N_2(u_i) \cup N_{12}(u_i) = N_2(u_i, R_H) \cup N_{12}(u_i, R_H)$. Therefore, the left line $l(T_{u_i})$ of T_{u_i} intersects in R_H exactly with the trapezoids of $N_1(u_i) \cup N_{12}(u_i)$, while the right line $r(T_{u_i})$ intersects exactly with the trapezoids of $N_2(u_i, R_H) \cup N_{12}(u_i, R_H)$. Thus, by Definition 8, R_H is a standard trapezoid representation with respect to u_i.

Theorem 2 *The formula ϕ is NAE-satisfiable if and only if the trapezoid graph H_ϕ is a bounded tolerance graph.*

Proof. Since a graph is a bounded tolerance graph if and only if it is a parallelogram graph [2,21], it suffices to prove that ϕ is NAE-satisfiable if and only if the trapezoid graph H_ϕ is a parallelogram graph.

($\Leftarrow$) Suppose that H_ϕ is a parallelogram graph, and let $U = \{u_1, u_2, \ldots, u_m\}$. Then, H_ϕ is an acyclic trapezoid graph by Lemma 1. Consider the permutation graph $H^{\#}_\phi(U)$ with $2m$ vertices, which is obtained by Algorithm Split-U on H_ϕ. Starting with the trapezoid representation R_H of H_ϕ, we obtain by the construction of Theorem 1 a permutation representation $R^{\#}_H(U)$ of $H^{\#}_\phi(U)$. Note that, since R_H is a standard trapezoid representation of H_ϕ with respect to every u_i, $i = 1, 2, \ldots, m$, the line u^1_i (resp. u^2_i) of T_{u_i} is not moved during the construction of $R^{\#}_H(U)$ from R_H, for every $i = 1, 2, \ldots, m$. Therefore, $H^{\#}_\phi(U) = P_\phi$. On the other hand, since by Lemma 8 $\delta^*_{u_i} \neq \emptyset$, $N_1(u_i) \setminus U \neq \emptyset$, and $N_2(u_i) \setminus U \neq \emptyset$ for every vertex $u_i \in U$, and since H_ϕ is an acyclic trapezoid graph, Theorem 1 implies that $H^{\#}_\phi(U) = P_\phi$ is an acyclic permutation graph with respect to $\{u^1_i, u^2_i\}^m_{i=1}$. Thus, ϕ is NAE-satisfiable by Lemma 7.

($\Rightarrow$) Conversely, suppose that ϕ has a NAE-satisfying truth assignment τ. We will construct first a permutation representation R_0 of P_ϕ, and then two trapezoid representations R'_0 and R''_0 of G_ϕ and H_ϕ, respectively, as follows. Similarly to the representation R_P, the representation R_0 has n blocks, i.e. connected components, one for each variable $x_1, x_2, \ldots, x_n$. R_0 is obtained from R_P by performing a horizontal axis flipping of every block, which corresponds to a variable $x_p = 0$ in the truth assignment τ. Every other block, which corresponds to a variable $x_p = 1$ in the assignment τ, remains the same in R_0, as in R_P. Thus, $\theta(\ell^1_{i,j}) > \theta(\ell^2_{i,j})$ if $x_{r_{i,j}} = 1$ in τ, and $\theta(\ell^1_{i,j}) < \theta(\ell^2_{i,j})$ if $x_{r_{i,j}} = 0$ in τ, for every pair $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ of lines in R_0 (which correspond to the literal $x_{r_{i,j}}$ of the clause α_i in ϕ). An example of the construction of this representation R_0 of P_ϕ for the truth assignment $\tau = (1, 1, 0, 0)$ is illustrated in Figure 6(a).

Since τ is a NAE-satisfying truth assignment of ϕ, at least one literal is true and at least one is false in τ in every clause α_i, $i = 1, 2, \ldots, k$. Thus, there are six possible truth assignments for every clause, namely $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(0, 0, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$. For the first three ones, we can assign appropriate angles to the lines a_i, b_i, c_i, d_i, e_i, and f_i in the representation R_0, such that the relative positions of all endpoints in L_1 and L_2 remain unchanged, and such that a_i is parallel to b_i, c_i is parallel to d_i, and e_i is parallel to f_i, as illustrated in Figure 8. The last three truth assignments of α_i are the complement of the first three ones. Thus, by performing a horizontal axis flipping of the blocks in Figure 8, to which the lines a_i, b_i, c_i, d_i, e_i, and f_i belong, it is easy to see that for these assignments, we can also assign appropriate angles to these lines in the representation R_0, such that the relative positions of all endpoints in L_1 and L_2 remain unchanged, and such that a_i is parallel to b_i, c_i is parallel to d_i, and e_i is parallel to f_i.

Recall that for every two consecutive pairs $\{\ell^1_{i,j}, \ell^2_{i,j}\}$ and $\{\ell^1_{i',j'}, \ell^2_{i',j'}\}$ of lines in R_P (resp. R_0), which belong to the same block, i.e. where $r_{i,j} = r_{i',j'}$, there are two parallel lines

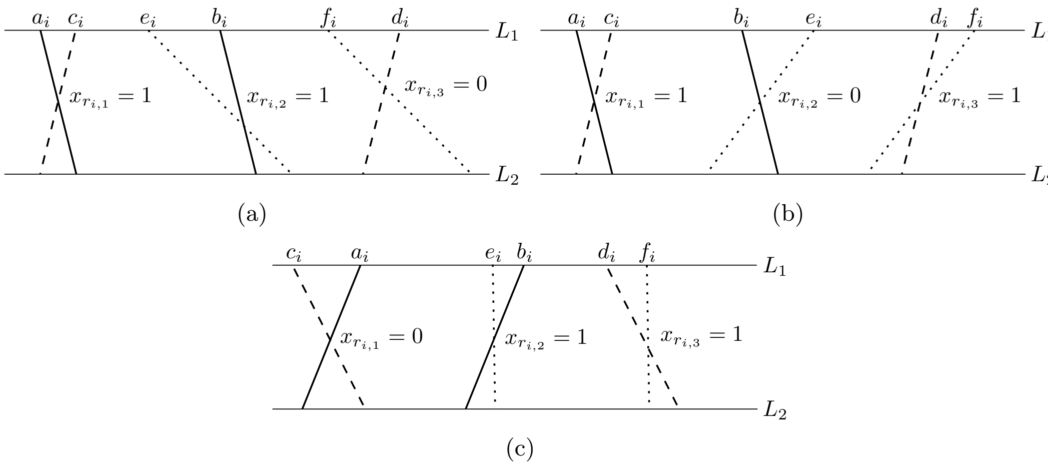


Figure 8: The relative positions of the lines a_i, b_i, c_i, d_i, e_i, and f_i for the truth assignments (a) $(1, 1, 0)$, (b) $(1, 0, 1)$, and (c) $(0, 1, 1)$ of the clause α_i.

$u_{i,j}^{i',j'}, v_{i,j}^{i',j'}$ that intersect both $\ell_{i,j}^1$ and $\ell_{i',j'}^1$. Thus, after assigning the appropriate angles to the lines $\{\ell_{i,j}^1, \ell_{i,j}^2\}$, $i = 1, 2, \ldots, k$, $j = 1, 2, 3$, we can clearly assign the appropriate angles to the lines $u_{i,j}^{i',j'}, v_{i,j}^{i',j'}$, such that the relative positions of all endpoints in L_1 and L_2 remain unchanged, and such that $u_{i,j}^{i',j'}$ remains parallel to $v_{i,j}^{i',j'}$. Summarizing, the lines u_i^1 and u_i^2 are parallel in R_0, for every $i = 1, 2, \ldots, m$.

We construct now the trapezoid representation R_0' of G_ϕ from the permutation representation R_0, by replacing for every $i = 1, 2, \ldots, m$ the lines u_i^1 and u_i^2 by the trapezoid T_{u_i}, which has u_i^1 and u_i^2 as its left and right lines, respectively. Since R_0 is obtained by performing horizontal axis flipping of some blocks of R_P, and then changing the angles of the lines, while respecting the relative positions of the endpoints, R_0' is indeed another trapezoid representation of G_ϕ than R_G. Since u_i^1 is now parallel to u_i^2 for every $i = 1, 2, \ldots, m$, it follows clearly that R_0' is a parallelogram representation, and thus, G_ϕ is a parallelogram graph.

Finally, we construct the trapezoid representation R_0'' of H_ϕ from R_0', similarly to the construction of R_H from R_G. Namely, we add for every trapezoid T_{u_i}, $i = 1, 2, \ldots, m$, six parallelograms $T_{u_{i,1}}, T_{u_{i,2}}, \ldots, T_{u_{i,6}}$, resulting in a trapezoid graph with $7m$ vertices. Since in R_0'' the parallelograms $T_{u_{i,1}}$, $T_{u_{i,2}}$, and $T_{u_{i,3}}$ (resp. $T_{u_{i,4}}$, $T_{u_{i,5}}$, and $T_{u_{i,6}}$) are sufficiently close to the left line u_i^1 (resp. right line u_i^2) of T_{u_i}, $i = 1, 2, \ldots, m$, and since between the endpoints of the parallelograms $T_{u_{i,1}}$, $T_{u_{i,2}}$, and $T_{u_{i,3}}$ (resp. $T_{u_{i,4}}$, $T_{u_{i,5}}$, and $T_{u_{i,6}}$) on L_1 and L_2, there are no other endpoints, it follows that R_0'' is indeed another trapezoid representation of H_ϕ than R_H. Finally, since R_0' is a parallelogram representation, and since $T_{u_{i,1}}, T_{u_{i,2}}, \ldots, T_{u_{i,6}}$, $i = 1, 2, \ldots, m$, are all parallelograms, R_0'' is also a parallelogram representation, and thus, H_ϕ is a parallelogram graph. ■

Therefore, since monotone-NAE-3-SAT is NP-complete, the problem of recognizing bounded tolerance graphs is NP-hard by Theorem 2. Moreover, since the recognition of bounded tolerance graphs lies in NP [17], we can summarize our results as follows.

Theorem 3 *It is NP-complete to decide whether a given graph G is a bounded tolerance graph.*

4 The recognition of tolerance graphs

In this section we show that the reduction from the monotone-NAE-3-SAT problem to the problem of recognizing bounded tolerance graphs presented in Section 3, can be extended to the problem of recognizing general tolerance graphs. In particular, we prove that a given monotone 3-CNF formula ϕ is NAE-satisfiable if and only if the graph H_ϕ constructed in Section 3.2 is a tolerance graph.

4.1 Structural properties of tolerance graphs

In the following we assume without loss of generality that any tolerance graph has a tolerance representation, in which all tolerances are distinct and no two different intervals share an endpoint [13, 14]. We state now similarly to [14, 15] some definitions and lemmas concerning tolerance graphs. In a certain tolerance representation $\langle I, t \rangle$ of a tolerance graph $G = (V, E)$, a vertex v is called *bounded* if $t_v \leq |I_v|$; otherwise, v is called *unbounded*. An unbounded vertex v of G is called *inevitable* (for a certain tolerance representation), if v is not an isolated vertex, and if setting $t_v = |I_v|$ creates a new edge in the representation, that is, the representation is no longer a tolerance representation of G. A tolerance representation of G is called *inevitable unbounded*, if every unbounded vertex in this representation is inevitable. For an inevitable unbounded vertex v of G (for a certain tolerance representation), a vertex u is called a *hovering vertex* of v, if $uv \notin E$ and $I_v \subseteq I_u$. The next lemma follows easily from the above definitions.

Lemma 9 *There exists a hovering vertex u for every inevitable unbounded vertex v of the tolerance graph G (for a certain tolerance representation).*

Proof. Since v is an inevitable unbounded vertex, setting $t_v = |I_v|$ creates a new edge in G; let uv be such an edge. Then, clearly $I_u \cap I_v \neq \emptyset$. Since initially $uv \notin E$, it follows that $|I_u \cap I_v| < \min\{t_u, t_v\} \leq t_u$. Furthermore, since setting $t_v = |I_v|$ creates a new edge in G, we obtain that $\min\{t_u, |I_v|\} \leq |I_u \cap I_v| < t_u$, and thus, $|I_u \cap I_v| = |I_v|$, i.e. $I_v \subseteq I_u$. Therefore, since $uv \notin E$ and $I_v \subseteq I_u$, it follows that u is a hovering vertex of v. ∎

Lemma 10 ([25]) *Every tolerance representation can be transformed into an inevitable one in $O(n \log n)$ time.*

Lemma 11 *Let v be an inevitable unbounded vertex of a tolerance graph G and u be a hovering vertex of v, in a certain tolerance representation of G. Then, $N(v) \subseteq N(u)$ in G.*

Proof. Since v is an inevitable unbounded vertex, $N(v) \neq \emptyset$. Let $w \in N(v)$ be a neighbor of v in G. Since u is a hovering vertex of v, it follows that $uv \notin E$, and thus, $w \neq u$. Furthermore, since $vw \in E$, and since v is unbounded, we obtain that $\min\{t_v, t_w\} \leq |I_v \cap I_w| \leq |I_v| < t_v$, and thus, $t_w \leq |I_v \cap I_w|$. Then, since $I_v \subseteq I_u$, it follows that $|I_v \cap I_w| \leq |I_u \cap I_w|$, and thus, $t_w \leq |I_u \cap I_w|$, i.e. $w \in N(u)$. Therefore, $N(v) \subseteq N(u)$ in G. ∎

4.2 The reduction

Consider now a monotone 3-CNF formula ϕ and the trapezoid graph H_ϕ constructed from ϕ in Section 3.2.

Lemma 12 *In the trapezoid graph H_ϕ, there are no two vertices u and v, such that $uv \notin E(H_\phi)$ and $N(v) \subseteq N(u)$ in H_ϕ.*

Proof. The proof is done by investigating all cases for a pair of non-adjacent vertices u, v. First, observe that, by the construction of H_ϕ from G_ϕ, we have $N[u_{i,2}] = N[u_{i,3}]$, $N[u_{i,1}] = N[u_{i,2}] \cup \{u_i\}$, $N[u_{i,5}] = N[u_{i,6}]$, and $N[u_{i,4}] = N[u_{i,5}] \cup \{u_i\}$.

Consider first two vertices u_i and u_k in H_ϕ, for some $i, k = 1, 2, \ldots, m$ and $i \neq k$. Then, by the construction of H_ϕ from G_ϕ, and since u_i and u_k are non-adjacent, $u_{i,1} \in N(u_i) \setminus N(u_k)$ and $u_{k,1} \in N(u_k) \setminus N(u_i)$. Consider next the vertices u_i and $u_{k,j}$, for some $i, k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 6$. If $i = k$, then $j \in \{2, 3, 5, 6\}$, since $u_{i,1}, u_{i,4} \in N(u_i)$. In the case where $j \in \{2, 3\}$, we have $u_{i,4} \in N(u_i) \setminus N(u_{k,j})$ and $u_{k,5-j} \in N(u_{k,j}) \setminus N(u_i)$, while in the case where $j \in \{5, 6\}$, we have $u_{i,1} \in N(u_i) \setminus N(u_{k,j})$ and $u_{k,11-j} \in N(u_{k,j}) \setminus N(u_i)$. Suppose that $i \neq k$. Then, it follows by the construction of H_ϕ from G_ϕ that $u_{i,1} \in N(u_i) \setminus N(u_{k,j})$. Furthermore, if $j \in \{1, 2, 3\}$ (resp. $j \in \{4, 5, 6\}$), then $u_{k,j'} \in N(u_{k,j}) \setminus N(u_i)$ for any index $j' \in \{1, 2, 3\} \setminus \{j\}$ (resp. $j' \in \{4, 5, 6\} \setminus \{j\}$).

Consider finally the vertices $u_{i,\ell}$ and $u_{k,j}$, for some $i, k = 1, 2, \ldots, m$ and $\ell, j = 1, 2, \ldots, 6$. If $i = k$, then without loss of generality $\ell \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, since $u_{i,\ell}$ and $u_{k,j}$ are non-adjacent. In this case, $u_{i,\ell'} \in N(u_{i,\ell}) \setminus N(u_{k,j})$ and $u_{k,j'} \in N(u_{k,j}) \setminus N(u_{i,\ell})$, for all indices $\ell' \in \{1, 2, 3\} \setminus \{\ell\}$ and $j' \in \{4, 5, 6\} \setminus \{j\}$. Suppose that $i \neq k$. If $j \in \{1, 2, 3\}$ (resp. $j \in \{4, 5, 6\}$), let j' be any index of $\{1, 2, 3\} \setminus \{j\}$ (resp. $\{4, 5, 6\} \setminus \{j\}$). Similarly, if $\ell \in \{1, 2, 3\}$ (resp. $\ell \in \{4, 5, 6\}$), let ℓ' be any index of $\{1, 2, 3\} \setminus \{\ell\}$ (resp. $\{4, 5, 6\} \setminus \{\ell\}$). Then, it follows by the construction of H_ϕ from G_ϕ that $u_{i,\ell'} \in N(u_{i,\ell}) \setminus N(u_{k,j})$ and $u_{k,j'} \in N(u_{k,j}) \setminus N(u_{i,\ell})$.

Therefore, for all possible choices of non-adjacent vertices u, v in the trapezoid graph H_ϕ, we have $N(u) \setminus N(v) \neq \emptyset$ and $N(v) \setminus N(u) \neq \emptyset$, which proves the lemma. ∎

Lemma 13 *If H_ϕ is a tolerance graph then it is a bounded tolerance graph.*

Proof. Suppose that H_ϕ is a tolerance graph, and consider a tolerance representation R of H_ϕ. Due to Lemma 10, we may assume without loss of generality that R is an inevitable unbounded representation. If R has no unbounded vertices, then we are done. Otherwise, there exists at least one inevitable unbounded vertex v in R, which has a hovering vertex u by Lemma 9, where $uv \notin E(H_\phi)$. Then, $N(v) \subseteq N(u)$ in H_ϕ by Lemma 11, which contradicts Lemma 12. Thus, there exists no unbounded vertex in R, i.e. H_ϕ is a bounded tolerance graph. ∎

Theorem 4 *The formula ϕ is NAE-satisfiable if and only if H_ϕ is a tolerance graph.*

Proof. Suppose that ϕ is NAE-satisfiable. Then, by Theorem 2, H_ϕ is a bounded tolerance graph, and thus, H_ϕ is a tolerance graph. Suppose conversely that H_ϕ is a tolerance graph. Then, by Lemma 13, H_ϕ is a bounded tolerance graph. Thus, ϕ is NAE-satisfiable by Theorem 2. ∎

Therefore, since monotone-NAE-3-SAT is NP-complete, the problem of recognizing tolerance graphs is NP-hard by Theorem 4. Moreover, since the recognition of tolerance graphs lies in NP [17], and since H_ϕ is a trapezoid graph, we obtain the following theorem.

Theorem 5 *It is NP-complete to decide whether a given graph G is a tolerance graph, even if G is a trapezoid graph.*

5 Concluding remarks

In this article we proved that both tolerance and bounded tolerance graph recognition problems are NP-complete, by providing a reduction from the monotone-NAE-3-SAT problem, thus answering a longstanding open question. Furthermore, our reduction implies that, given a trapezoid graph, it is NP-complete to decide whether this graph is a tolerance or a bounded tolerance (i.e. parallelogram) graph. A *unit* interval representation is an interval representation in which all intervals have the same length. A *proper* interval representation is one in which no interval is properly contained in another. These terms can apply to both interval graphs and tolerance graphs. It is known that the subclasses of unit and proper interval graphs are equal [28], but the corresponding tolerance subclasses are different [2]. The recognition of unit and of proper tolerance graphs, as well as of any other subclass of tolerance graphs, except bounded tolerance and bipartite tolerance graphs [5], remain interesting open problems [15].

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