On Colored Edge Cuts in Graphs

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Abstract. *In this work we present some results on the classical and parameterized complexity of finding cuts in edge-colored graphs. In general, we are interested in problems of finding cuts $\{A, B\}$ which minimize or maximize the number of colors occurring in the edges with exactly one endpoint in A.*

1. Introduction

A *cut* in a graph G is a partition of $V(G)$ into two disjoint non-empty sets A and B. The *cut-set* of a cut $\{A, B\}$, denoted by $e(A, B)$, is defined as the set of edges of G with exactly one endpoint in A. If s and t are two distinct vertices of a graph G, an (s, t)-*cut* of G is a cut $\{A, B\}$ of G such that $s \in A$ and $t \in B$.

Motivation. MINIMUM CUT and MAXIMUM CUT are two of most popular graph problems and they have several applications. An important application of graph edge cuts is *image segmentation*, in the field of *computer vision*. Image segmentation can be defined as the task of distinguishing objects from background in images, more precisely, image segmentation is the process of partitioning a digital image into multiple segments (sets of pixels, also known as superpixels). Classical methods to obtain image segmentations are based on finding edge cuts in uncolored-edge graphs [Shi and Malik 1997, Felzenszwalb and Huttenlocher 2004]. However, as images can be easily seen as graphs, where pixels' colors can be interpreted as edge's colors, the study of combinatorial problems related to cuts in colored-edge graphs seems to be useful to construct new techniques for image segmentation and for other applications as well.

Let $c : E(G) \to \mathbb{N}$ be a (not necessarily proper) coloring function defined on the edges of a graph G and let $\{A, B\}$ be a cut in G. We denote by $c(A, B)$ the set of colors that appear in the edges of $e(A, B)$, that is,

$$c(A, B) = \{i \in \mathbb{N} : \text{there exists } e \in e(A, B) \text{ with } c(e) = i\}.$$

Let $\mathrm{im}(c)$ be the image of the coloring function $c : E(G) \to \mathbb{N}$.

In this paper we consider the following combinatorial problems related to cuts in edge-colored graphs:

MINIMUM COLORED CUT **Input:** A connected graph G and a coloring function $c : E(G) \to \mathbb{N}$. **Output:** A cut $\{A, B\}$ of G that minimizes $\|c(A, B)\|$.

MINIMUM COLORED (s, t)-CUT **Input:** A connected graph G, two distinct vertices $s, t \in V(G)$, and a coloring function $c : E(G) \to \mathbb{N}$. **Output:** An (s, t)-cut $\{A, B\}$ of G that minimizes $\|c(A, B)\|$.

MAXIMUM COLORED CUT **Input:** A graph G and a coloring function $c : E(G) \to \mathbb{N}$. **Output:** A cut $\{A, B\}$ of G that maximizes $\|c(A, B)\|$.

COLORFUL CUT **Input:** A graph G and a coloring function $c : E(G) \to \mathbb{N}$. **Output:** A cut $\{A, B\}$ of G such that $c(A, B) = \text{im}(c)$, if it exists.

As we can observe, COLORFUL CUT is a special case of MAXIMUM COLORED CUT, and MINIMUM COLORED CUT are easily Turing-reducible to MINIMUM COLORED (s, t)-CUT.

In this article we focus on the computational complexity of these problems, with special emphasis on their parameterized complexity for several choices of the parameters. For an introduction to the field of Parameterized Complexity, see [Flum and Grohe 2006, Niedermeier 2006, Downey and Fellows 2013, Cygan et al. 2015]. We use standard graph-theoretic notation [Diestel 2010]. Throughout the article, we denote by n the number of vertices of the input graph of the problem under consideration.

2. Minimum colored cut and colored (s, t)-cuts

To the best of our knowledge, the MINIMUM COLORED CUT and MINIMUM COLORED (s, t)-CUT problems were first introduced by Coudert *et al.* [Coudert et al. 2007], using different terminology.

Note that if the coloring function c is injective, that is, if all edges get different colors, then the MINIMUM COLORED CUT and MINIMUM COLORED (s, t)-CUT problems correspond exactly to the MINIMUM CUT and MINIMUM (s, t)-CUT problems, respectively, hence they can be both solved in polynomial time by a classical MAXIMUM FLOW algorithm [Diestel 2010].

Theorem 1.

- MINIMUM COLORED (s, t)-CUT *cannot be approximated within a factor of* $(1 - \epsilon)\ln(c)$ *for any constant* $\epsilon > 0$ *unless* $\mathrm{P} = \mathrm{NP}$*, even if the input graph is bipartite planar or complete;*
- MINIMUM COLORED (s, t)-CUT *is* W[2]*-hard parameterized by the cost of the solution, even if the input graph is bipartite planar.*

Proof. Reduction from SET COVER which is known to be W[2]-hard when parameterized by the size of the solution [Flum and Grohe 2006] and $(1 - \epsilon)\ln(c)$ inapproximable (for any constant $\epsilon > 0$ unless $\mathrm{P} = \mathrm{NP}$) [Dinur and Steurer 2014]. □

Theorem 2. MINIMUM COLORED (s, t)-CUT *on planar bipartite graphs G remains* NP*-complete even when:*

- *either, each color of G occurs at most three times, and every (s, t)-path in G has length two;*
- *or, each color of G occurs at most twice, and every (s, t)-path in G has length at most three.*

Proof. Reduction from VERTEX COVER on cubic graphs [Blin et al. 2014]. □

Theorem 3. MINIMUM COLORED (s, t)-CUT *can be solved in polynomial time when every color appears in at most two (s, t)-paths of G.*
Proof. We can show this using the polynomial algorithm for MONOTONE WEIGHTED SAT where each variable occurs at most twice [Porschen and Speckenmeyer 2007]. □

Corollary 4. MINIMUM COLORED (s, t)-CUT *can be solved in polynomial time when each color occurs at most twice and each (s, t)-path has length two.*
Theorem 5. MINIMUM COLORED CUT *and* MINIMUM COLORED (s, t)-CUT *are* FPT *when parameterized by the number of colors.*

Note that MINIMUM COLORED (s, t)-CUT can be solved in polynomial time when $c = O(\log n)$, and can be solved in pseudo-polynomial time when G has an uncolored (s, t)-cut of size $O(\log n)$. The following result complements Theorem 5 above.
Lemma 6. *The* MINIMUM COLORED (s, t)-CUT *problem does not admit polynomial kernels when parameterized by the number of colors, unless* $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.
Proof. The proof uses OR-composition. □

Lemma 7. MINIMUM COLORED (s, t)-CUT *are* FPT *when parameterized by the number of (s, t)-paths and the size of the solution.*
Lemma 8. *The* MINIMUM COLORED (s, t)-CUT *problem does not admit polynomial kernels when parameterized by the number of (s, t)-paths and the size of the solution, unless* $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.
Lemma 9. MINIMUM COLORED CUT *can be solved in polynomial time when G has an uncolored (s, t)-cut of constant size, and it can be solved in $O(n^5)$ time on planar graphs.*
Theorem 10. MINIMUM COLORED CUT *for directed graphs is* NP*-hard.*

We denote by *span* of a color c_i the number of connected components in the graph induced by the set of edges colored with c_i.
Lemma 11. *Given an edge-colored graph (G, c), let c_2 denote the number of colors with span at least two. The* MINIMUM COLORED CUT *problem can be solved in time $3^{c_2} \cdot n^{O(1)}$.*
Lemma 12. *Given an edge-colored graph (G, c) and a positive integer p, let c_p denote the number of colors with span at least p. The* MINIMUM COLORED CUT *problem can be solved by a randomized algorithm in time $3^{c_p} \cdot n^{O(1)}$, where the degree of the polynomial depends on p.*

3. Maximum colored cut and colorful cut

In this section we list our results on COLORFUL CUT and MAXIMUM COLORED CUT.
Theorem 13. MAX COLORED CUT *remains* NP*-hard even when restricted to complete graphs.*
Theorem 14. MAX COLORED CUT *admits a polynomial 2-approximation algorithm.*
Theorem 15. MAX COLORED CUT *can be solved in polynomial time on graphs G colored with a constant number of colors.*
Theorem 16. COLORFUL CUT *is* NP*-complete even when each color class induces a clique.*

Theorem 17. COLORFUL CUT *is* NP*-complete even on planar graphs where each color occurs at most twice and each vertex has degree at most 4.*

Corollary 18. *The* COLORFUL CUT *problem is* NP*-complete, even when restricted to planar graphs with odd cycle transversal number at most 1.*

Theorem 19. COLORFUL CUT *can be solved in polynomial time on planar graphs if any color class induces a clique.*

Lemma 20. *The* COLORFUL CUT *problem is* NP*-complete, even when restricted to graphs with vertex cover number at most 4.*

Lemma 21. *The* COLORFUL CUT *problem is* NP*-complete, even when restricted to planar graphs with feedback vertex set number at most 2.*

Lemma 22. *The* MAXIMUM COLORED CUT *problem admits a cubic kernel when parameterized by the number of colors.*

Lemma 23. *The* MAXIMUM COLORED CUT *problem admits a cubic kernel when parameterized by the cost of the solution.*

References

Blin, G., Bonizzoni, P., Dondi, R., Rizzi, R., and Sikora, F. (2014). Complexity insights of the minimum duplication problem. *Theoretical Computer Science*, 530:66 – 79.

Coudert, D., Datta, P., Perennes, S., Rivano, H., and Voge, M. (2007). Shared risk resource group complexity and approximability issues. *Parallel Process. Lett.*, 17(2):169–184.

Cygan, M., Fomin, F. V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., and Saurabh, S. (2015). *Parameterized Algorithms*. Springer.

Diestel, R. (2010). *Graph Theory*, volume 173. Springer-Verlag, 4th edition.

Dinur, I. and Steurer, D. (2014). Analytical approach to parallel repetition. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, STOC '14, pages 624–633, New York, NY, USA. ACM.

Downey, R. G. and Fellows, M. R. (2013). *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer.

Felzenszwalb, P. F. and Huttenlocher, D. P. (2004). Efficient graph-based image segmentation. *International Journal of Computer Vision*, 59(2):167–181.

Flum, J. and Grohe, M. (2006). *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. Springer.

Niedermeier, R. (2006). *Invitation to Fixed-Parameter Algorithms*, volume 31. Oxford University Press.

Porschen, S. and Speckenmeyer, E. (2007). Algorithms for variable-weighted 2-sat and dual problems. In *Theory and Applications of Satisfiability Testing–SAT 2007*, pages 173–186. Springer.

Shi, J. and Malik, J. (1997). Normalized cuts and image segmentation. In *Computer Vision and Pattern Recognition, 1997. Proceedings., 1997 IEEE Computer Society Conference on*, pages 731–737. IEEE.