On Colored Edge Cuts in Graphs

Luerbio Faria\(^2\), Sulamita Klein\(^1\), Ignasi Sau\(^3\), Uéverton S. Souza\(^4\), Rubens Sucupira\(^{1,2}\)

\(^1\)Universidade Federal do Rio de Janeiro, PESC/COPPE, Brazil
\(^2\)Universidade do Estado do Rio de Janeiro, Brazil
\(^3\)CNRS, LIRMM, Montpellier, France
\(^4\)Universidade Federal Fluminense, Brazil

{luerbio,sula}@cos.ufrj.br, \{ignasi.sau, rasucupira\}@gmail.com, uevertonssouza@yahoo.com.br

Abstract. In this work we present some results on the classical and parameterized complexity of finding cuts in edge-colored graphs. In general, we are interested in problems of finding cuts \(\{A, B\}\) which minimize or maximize the number of colors occurring in the edges with exactly one endpoint in \(A\).

1. Introduction

A cut in a graph \(G\) is a partition of \(V(G)\) into two disjoint non-empty sets \(A\) and \(B\). The cut-set of a cut \(\{A, B\}\), denoted by \(e(A, B)\), is defined as the set of edges of \(G\) with exactly one endpoint in \(A\). If \(s\) and \(t\) are two distinct vertices of a graph \(G\), an \((s, t)\)-cut of \(G\) is a cut \(\{A, B\}\) of \(G\) such that \(s \in A\) and \(t \in B\).

Motivation. \textsc{Minimum Cut} and \textsc{Maximum Cut} are two of most popular graph problems and they have several applications. An important application of graph edge cuts is image segmentation, in the field of computer vision. Image segmentation can be defined as the task of distinguishing objects from background in images, more precisely, image segmentation is the process of partitioning a digital image into multiple segments (sets of pixels, also known as superpixels). Classical methods to obtain image segmentations are based on finding edge cuts in uncolored-edge graphs [Shi and Malik 1997, Felzenszwalb and Huttenlocher 2004]. However, as images can be easily seen as graphs, where pixels’ colors can be interpreted as edge’s colors, the study of combinatorial problems related to cuts in colored-edge graphs seems to be useful to construct new techniques for image segmentation and for other applications as well.

Let \(c : E(G) \to \mathbb{N}\) be a (not necessarily proper) coloring function defined on the edges of a graph \(G\) and let \(\{A, B\}\) be a cut in \(G\). We denote by \(c(A, B)\) the set of colors that appear in the edges of \(e(A, B)\), that is,

\[c(A, B) = \{i \in \mathbb{N} : \text{there exists } e \in e(A, B) \text{ with } c(e) = i\}\.

Let \(\text{im}(c)\) be the image of the coloring function \(c : E(G) \to \mathbb{N}\).

In this paper we consider the following combinatorial problems related to cuts in edge-colored graphs:
**Minimum Colored Cut**  
**Input:** A connected graph $G$ and a coloring function $c : E(G) \to \mathbb{N}$.  
**Output:** A cut $\{A, B\}$ of $G$ that minimizes $|c(A, B)|$.

**Minimum Colored $(s, t)$-Cut**  
**Input:** A connected graph $G$, two distinct vertices $s, t \in V(G)$, and a coloring function $c : E(G) \to \mathbb{N}$.  
**Output:** An $(s, t)$-cut $\{A, B\}$ of $G$ that minimizes $|c(A, B)|$.

**Maximum Colored Cut**  
**Input:** A graph $G$ and a coloring function $c : E(G) \to \mathbb{N}$.  
**Output:** A cut $\{A, B\}$ of $G$ that maximizes $|c(A, B)|$.

**Colorful Cut**  
**Input:** A graph $G$ and a coloring function $c : E(G) \to \mathbb{N}$.  
**Output:** A cut $\{A, B\}$ of $G$ such that $c(A, B) = \text{im}(c)$, if it exists.

As we can observe, **Colorful Cut** is a special case of **Maximum Colored Cut**, and **Minimum Colored Cut** are easily Turing-reducible to **Minimum Colored $(s, t)$-Cut**.

In this article we focus on the computational complexity of these problems, with special emphasis on their parameterized complexity for several choices of the parameters. For an introduction to the field of Parameterized Complexity, see [Flum and Grohe 2006, Niedermeier 2006, Downey and Fellows 2013, Cygan et al. 2015]. We use standard graph-theoretic notation [Diestel 2010]. Throughout the article, we denote by $n$ the number of vertices of the input graph of the problem under consideration.

### 2. Minimum colored cut and colored $(s, t)$-cuts

To the best of our knowledge, the **Minimum Colored Cut** and **Minimum Colored $(s, t)$-Cut** problems were first introduced by Coudert et al. [Coudert et al. 2007], using different terminology.

Note that if the coloring function $c$ is injective, that is, if all edges get different colors, then the **Minimum Colored Cut** and **Minimum Colored $(s, t)$-Cut** problems correspond exactly to the **Minimum Cut** and **Minimum $(s, t)$-Cut** problems, respectively, hence they can be both solved in polynomial time by a classical **Maximum Flow** algorithm [Diestel 2010].

**Theorem 1.**  
- **Minimum Colored $(s, t)$-Cut** cannot be approximated within a factor of $(1 - \epsilon) \ln(c)$ for any constant $\epsilon > 0$ unless P = NP, even if the input graph is bipartite planar or complete;  
- **Minimum Colored $(s, t)$-Cut** is **W[2]**-hard parameterized by the cost of the solution, even if the input graph is bipartite planar.

**Proof.** Reduction from **Set Cover** which is known to be **W[2]**-hard when parameterized by the size of the solution [Flum and Grohe 2006] and $(1 - \epsilon) \ln(c)$ inapproximable (for any constant $\epsilon > 0$ unless P = NP) [Dinur and Steurer 2014].

**Theorem 2.** **Minimum Colored $(s, t)$-Cut** on planar bipartite graphs $G$ remains **NP**-complete even when:
• either, each color of \( G \) occurs at most three times, and every \((s, t)\)-path in \( G \) has length two;

• or, each color of \( G \) occurs at most twice, and every \((s, t)\)-path in \( G \) has length at most three.

Proof. Reduction from VERTEX COVER on cubic graphs [Blin et al. 2014].

**Theorem 3.** **MINIMUM COLORED \((s, t)\)-Cut** can be solved in polynomial time when every color appears in at most two \((s, t)\)-paths of \( G \).

Proof. We can show this using the polynomial algorithm for MONOTONE WEIGHTED SAT where each variable occurs at most twice [Porschen and Speckenmeyer 2007].

**Corollary 4.** **MINIMUM COLORED \((s, t)\)-Cut** can be solved in polynomial time when each color occurs at most twice and each \((s, t)\)-path has length two.

**Theorem 5.** **MINIMUM COLORED \((s, t)\)-Cut and MINIMUM COLORED \((s, t)\)-Cut are FPT when parameterized by the number of colors.**

Note that **MINIMUM COLORED \((s, t)\)-Cut** can be solved in polynomial time when \( c = O(\log n) \), and can be solved in pseudo-polynomial time when \( G \) has an uncolored \((s, t)\)-cut of size \( O(\log n) \). The following result complements Theorem 5 above.

**Lemma 6.** The **MINIMUM COLORED \((s, t)\)-Cut problem does not admit polynomial kernels when parameterized by the number of colors, unless NP \( \subseteq \text{coNP/poly} \).**

Proof. The proof uses OR-composition.

**Lemma 7.** **MINIMUM COLORED \((s, t)\)-Cut are FPT when parameterized by the number of \((s, t)\)-paths and the size of the solution.**

**Lemma 8.** The **MINIMUM COLORED \((s, t)\)-Cut problem does not admit polynomial kernels when parameterized by the number of \((s, t)\)-paths and the size of the solution, unless NP \( \subseteq \text{coNP/poly} \).**

**Lemma 9.** **MINIMUM COLORED Cut can be solved in polynomial time when \( G \) has an uncolored \((s, t)\)-cut of constant size, and it can be solved in \( O(n^5) \) time on planar graphs.**

**Theorem 10.** **MINIMUM COLORED Cut for directed graphs is \( \text{NP-hard} \).**

We denote by span of a color \( c_i \) the number of connected components in the graph induced by the set of edges colored with \( c_i \).

**Lemma 11.** Given an edge-colored graph \((G, c)\), let \( c_2 \) denote the number of colors with span at least two. The **MINIMUM COLORED CUT problem** can be solved in time \( 3^{c_2} \cdot n^{O(1)} \).

**Lemma 12.** Given an edge-colored graph \((G, c)\) and a positive integer \( p \), let \( c_p \) denote the number of colors with span at least \( p \). The **MINIMUM COLORED CUT problem** can be solved by a randomized algorithm in time \( 3^{c_p} \cdot n^{O(1)} \), where the degree of the polynomial depends on \( p \).

### 3. Maximum colored cut and colorful cut

In this section we list our results on **COLORFUL CUT** and **MAXIMUM COLORED CUT**.

**Theorem 13.** **MAX COLORED CUT remains \( \text{NP-hard} \) even when restricted to complete graphs.**

**Theorem 14.** **MAX COLORED CUT admits a polynomial 2-approximation algorithm.**

**Theorem 15.** **MAX COLORED CUT can be solved in polynomial time on graphs \( G \) colored with a constant number of colors.**

**Theorem 16.** **COLORFUL CUT is \( \text{NP-complete} \) even when each color class induces a clique.**
Theorem 17. **Colorful Cut** is NP-complete even on planar graphs where each color occurs at most twice and each vertex has degree at most 4.

Corollary 18. The **Colorful Cut** problem is NP-complete, even when restricted to planar graphs with odd cycle transversal number at most 1.

Theorem 19. **Colorful Cut** can be solved in polynomial time on planar graphs if any color class induces a clique.

Lemma 20. The **Colorful Cut** problem is NP-complete, even when restricted to graphs with vertex cover number at most 4.

Lemma 21. The **Colorful Cut** problem is NP-complete, even when restricted to planar graphs with feedback vertex set number at most 2.

Lemma 22. The **Maximum Colored Cut** problem admits a cubic kernel when parameterized by the number of colors.

Lemma 23. The **Maximum Colored Cut** problem admits a cubic kernel when parameterized by the cost of the solution.

References


