Packet Routing Problems on Plane Grids

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Joint work with Omid Amini, Florian Huc and Janez Žerovnik

AEOLUS Workshop on Scheduling

Outline

- Introduction
  - Statement of the problem
  - Preliminaries
  - Example
- Permutation routing algorithm for triangular grids
  - Description
  - Correctness
  - Optimality
- Permutation routing algorithm for hexagonal grids
- (ℓ, k)-routing algorithms
- Conclusions

(ℓ, k)-routing

- The (ℓ, k)**-routing** problem is a **packet routing** problem.
- Each processor is the **origin of at most** ℓ **packets** and the **destination of no more than** k **packets**.
- The goal is to **minimize the number of time steps** required to route all packets to their respective destinations.
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Permutation Routing

Statement of the problem

- **Input:**
  - a directed graph $G = (V, E)$ (the *host* graph),
  - a subset $S \subseteq V$ of nodes,
  - and a permutation $\pi : S \to S$.
    Each node $u \in S$ wants to send a packet to $\pi(u)$.
- **Output:** Find for each pair $(u, \pi(u))$, a path form u to $\pi(u)$ in G.
- **Constraints:**
  - At each step, a packet can either move or stay at a node.
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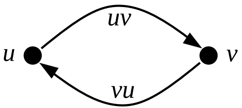
- We consider the **store-and-forward** and **Δ-port** model.
- **Full duplex link**: packets can be sent in the two directions of the link **simultaneously**.



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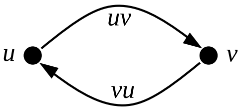

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Previous work

-The permutation routing problem has been studied in:

- Mobile Ad Hoc Networks
- Cube-Connected Cycle Networks
- Wireless and Radio Networks
- All-Optical Networks
- Reconfigurable Meshes...

-But, optimal algorithms (in the worst case):

- 2-circulant graphs, square grids.
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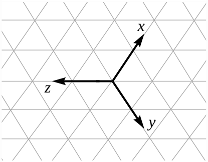
Permutation Routing on Triangular Grids

Notation and preliminary results

Nocetti, Stojmenović and Zhang [IEEE TPDS'02]:

Representation of the relative address of the nodes on a generating system ***i***, ***j***, ***k*** on the directions of the three axis x, y, z.



- This address **is not unique**, but we have that, being (a, b, c) and (a', b', c') the addresses of two $D - S$ pairs,

$$(a, b, c) = (a', b', c') \quad \Leftrightarrow \quad \exists \text{ an integer } d \text{ such that}$$

$$a' = a + d,$$

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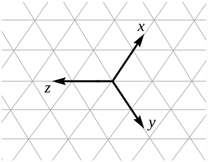

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Notation and preliminary results (2)

- A relative address $D - S = (a, b, c)$ is of the ***shortest path form*** if
  - there is a path C from S to D, C=a***i***+b***j***+c***k***,
  - and C has the shortest length over all paths going from S to D.

Theorem (*NSZ'02*)

*An address (a, b, c) is of the **shortest path form** if and only if*

i) ***at least one component is zero*** *(that is, $abc = 0$),*

ii) *and **any two components do not have the same sign** (that is, $ab \leq 0,\ ac \leq 0,$ and $bc \leq 0$).*

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Corollary (*NSZ'02*)

Any address has a ***unique*** *shortest path form.*

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If $D - S = (a, b, c)$, then the shortest path form is one of those:

$$(0, b-a, c-a),$$

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and thus:

$$|D - S| = \min(|b-a| + |c-a|, |a-b| + |c-b|, |a-c| + |b-c|).$$

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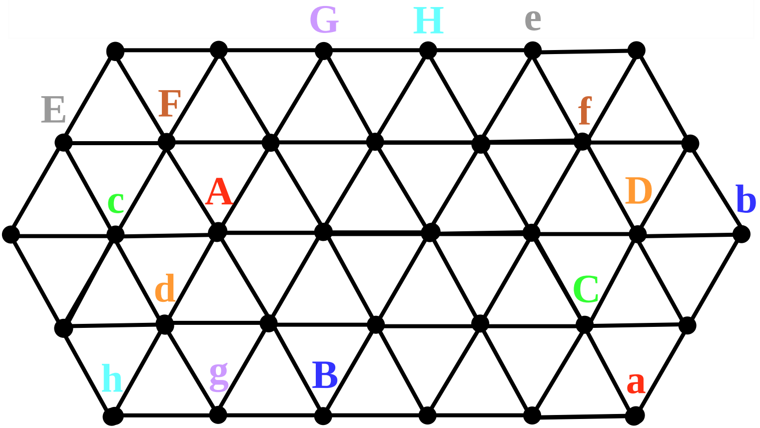
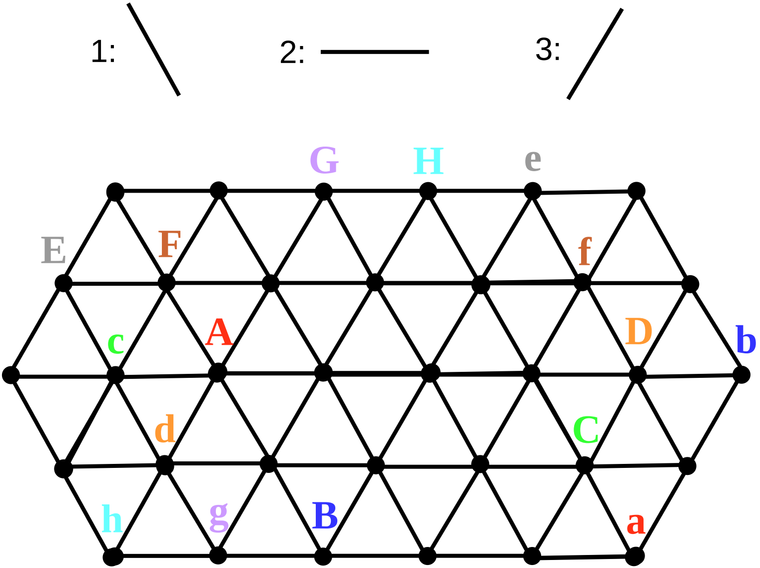
Example of an instance



A non-optimal intuitive algorithm



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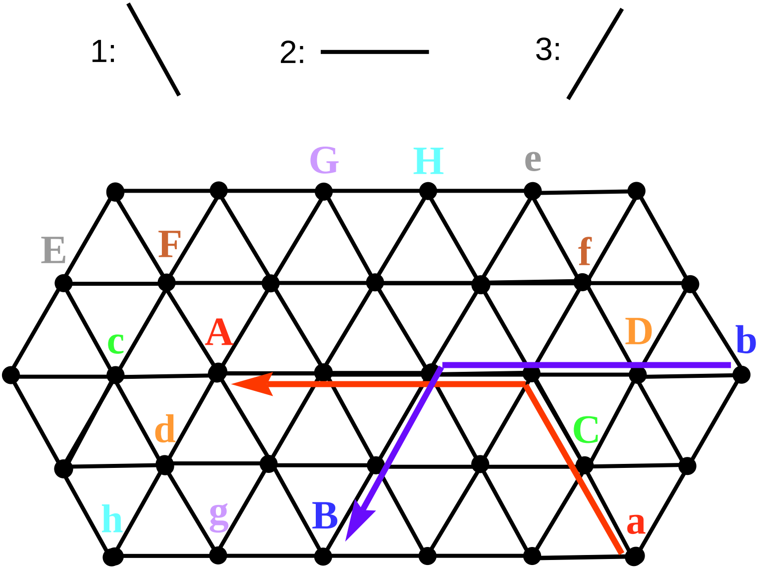

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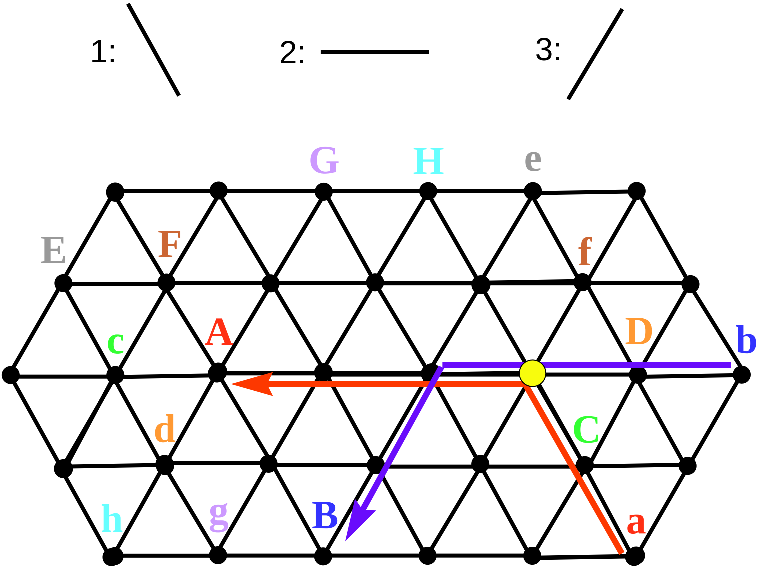

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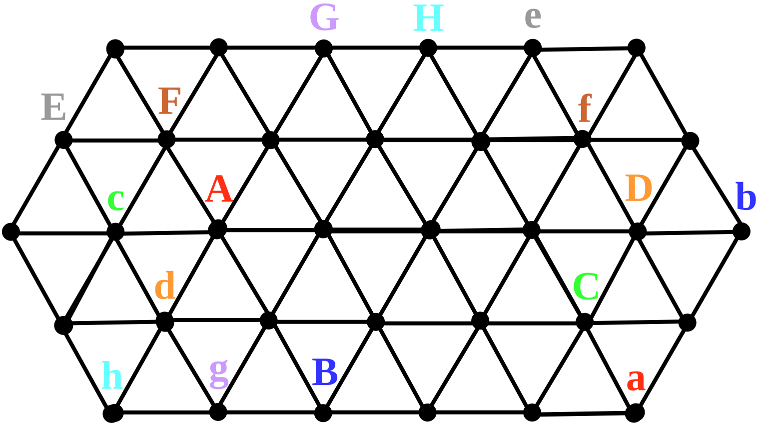
1: 2: 3:



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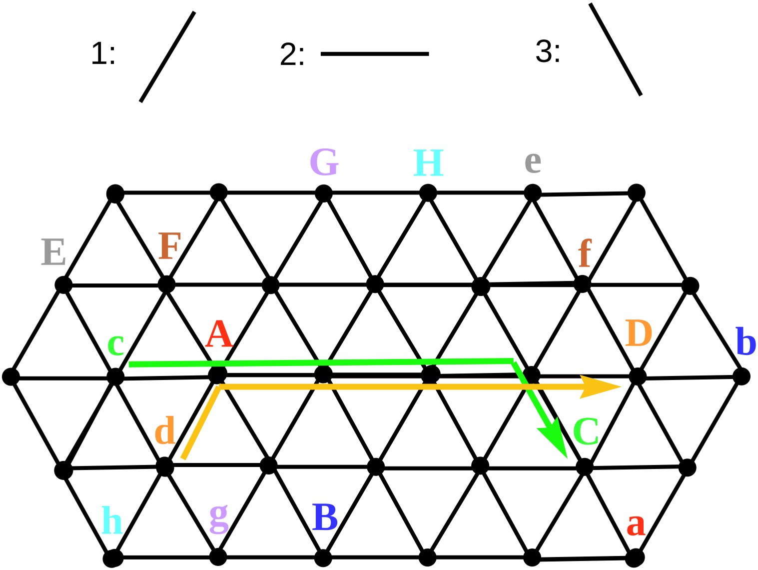

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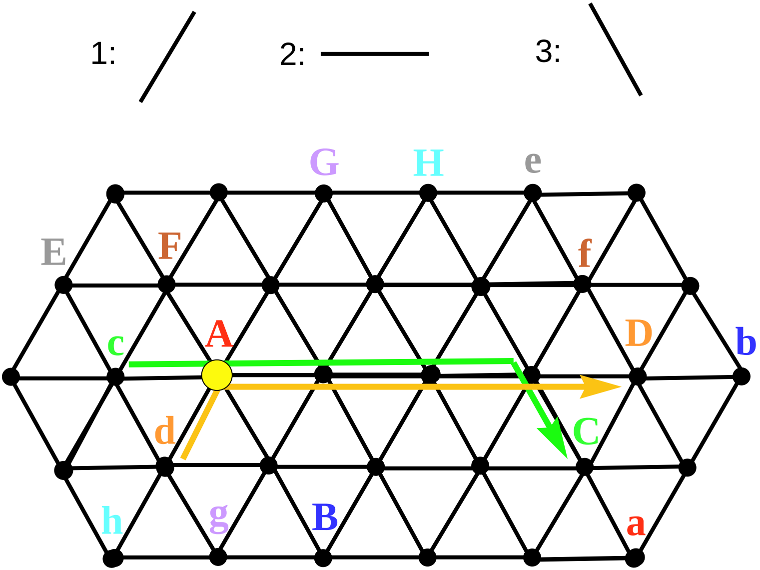

Description of Algorithm $\mathcal{A}$

At each node u of the network:

- ***Preprocessing*:** Initially, if there is a packet at u, compute the relative address $D - S$ of the message in the shortest path form, and add this information to the message.
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- Algorithm $\mathcal{A}$ defines for each packet **two directions of movement** (except if a packet has only one non-zero component)
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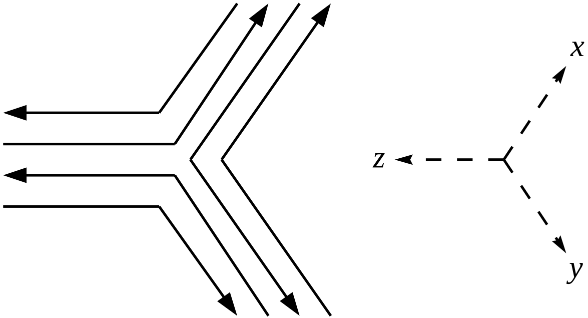
Routing the packets (2)

In this figure all the routing rules are summarized:



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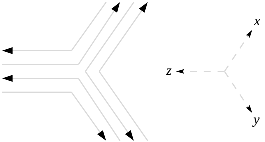

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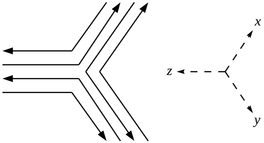

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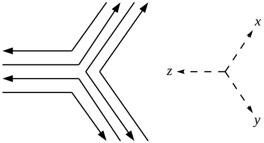

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Correctness (4)

- All the packets in in **b)** are moving along their **last direction**
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Optimality

- Using this algorithm, at each step **all the packets with maximum remaining distance move**

  $\longrightarrow$ every step the maximum remaining distance over all packets decreases by one

  $\longrightarrow$ the total running time is at most ℓ_{max}, **meeting the lower bound**.

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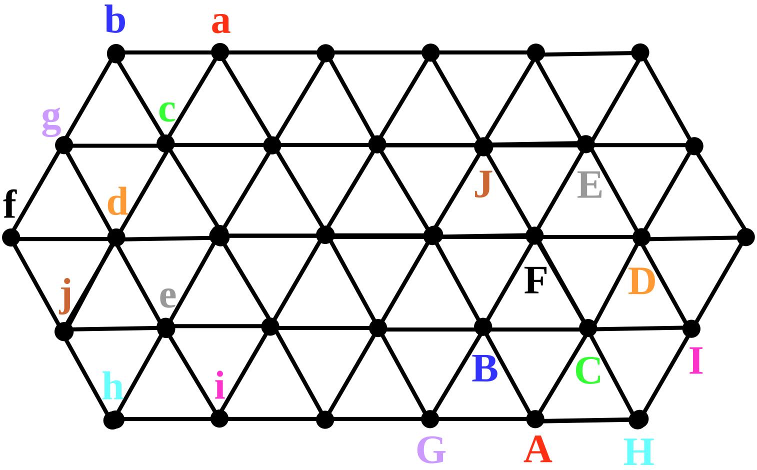

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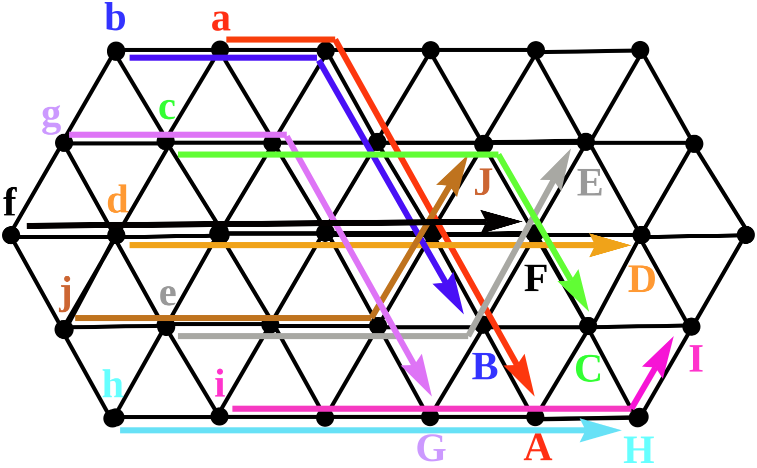
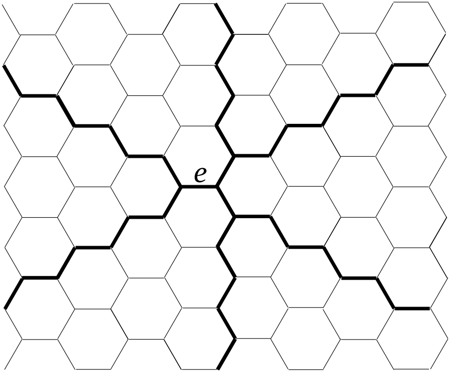


Permutation Routing on Hexagonal Grids

Hexagonal grid

- One can define 3 types of *zigzag chains*:



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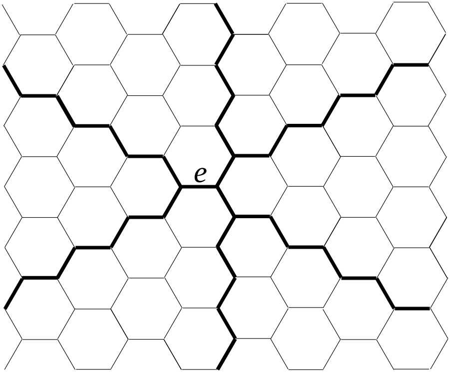

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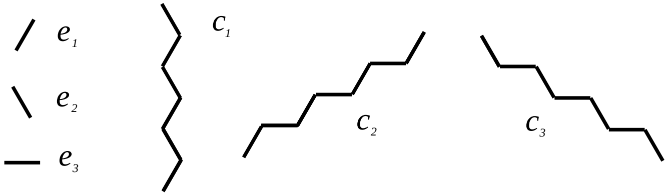
- There are 3 types of edges and 3 types of chains:



- Each edge belongs to exactly 2 different chains, and conversely each chain is made of 2 types of edges.
- Any 2 chains of different type intersect exactly on one edge.
- We can define 2 phases in such a way that at each phase, each type of chain uses only one type of edge.

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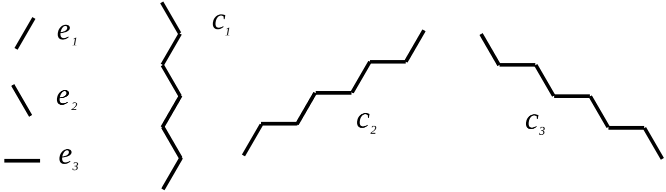

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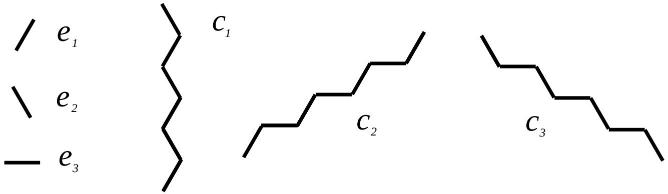

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Optimal algorithm

At each node of the network:

1) During the **first step**, move all packets along the direction of their **negative component**. If a packet's address has only a positive component, move it along this direction.
2) From now on, **change alternatively** between Phase 1 and Phase 2.
3) At each step (the same for both phases):
   a) If there are packets with negative components, send them immediately along the direction of this component.
   b) If not, for each outgoing edge order the packets according to decreasing number of remaining steps, and send the first packet of each queue.

Running time

- Every 2 steps (one of Phase 1 and one of Phase 2) the maximum remaining distance over all packets decreases by one.
- During the first step all packets decrease their remaining distance by one.
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(ℓ, k)-Routing

Algorithm (in any grid)

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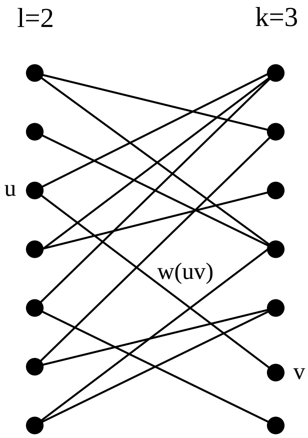

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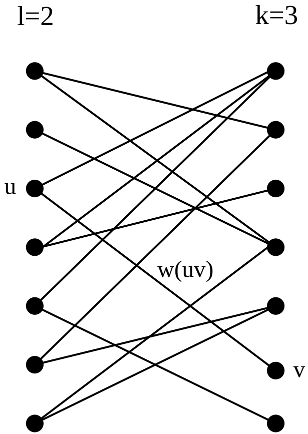

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- Let $c(M_i) := \max\{w(e) | e \in M_i)\}, i = 1, \ldots, m$
- **Objective function**:

$$\min \sum_{i=1}^{m} c(M_i)$$

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Thanks!