Linear kernels on graphs excluding a topological minor

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Outline of the talk

1. Motivation and our result
2. Idea of proof
3. Further research

Next section is...

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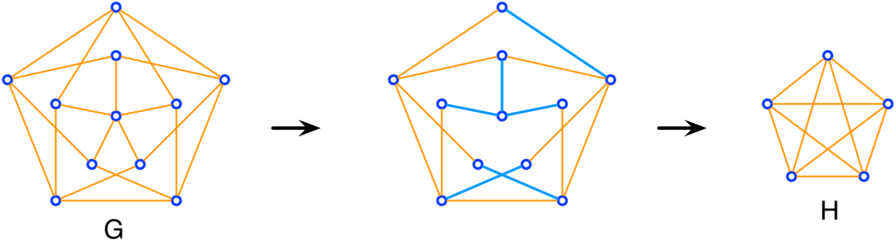
Minors and topological minors



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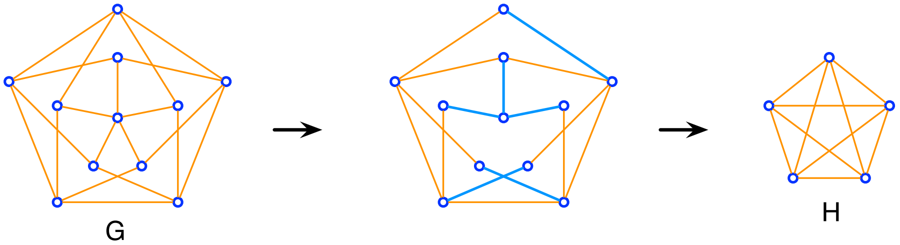
Minors and topological minors



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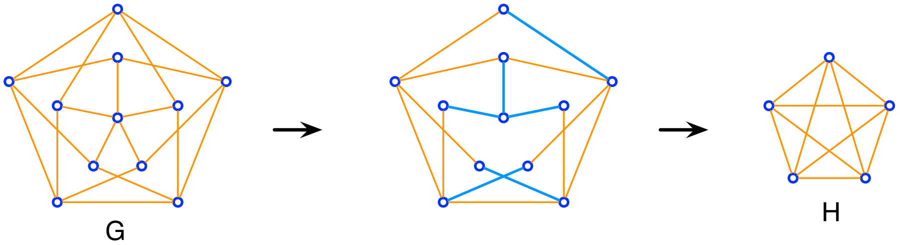

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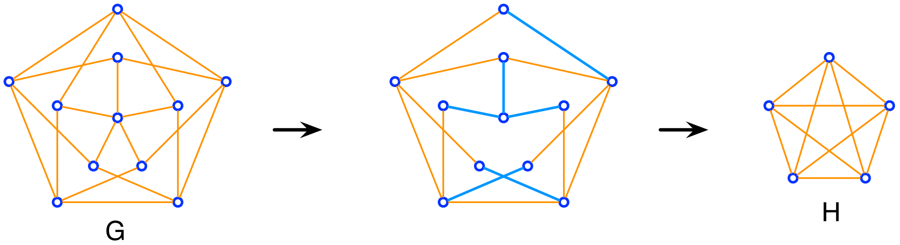

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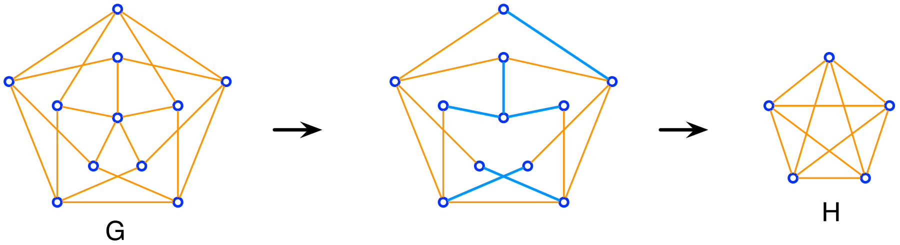

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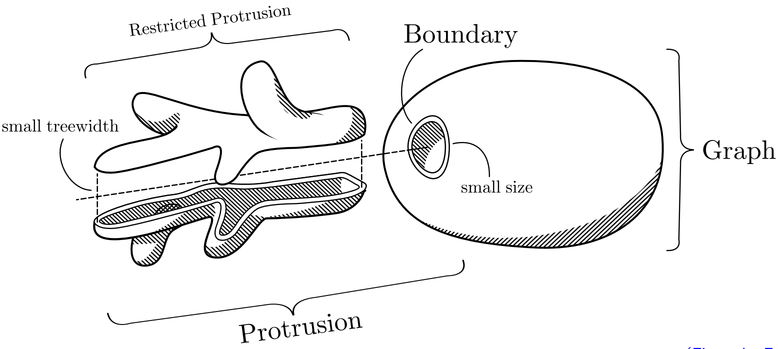
- Fixed H: $\boxed{H\text{-minor-free graphs} \subseteq H\text{-topological-minor-free graphs}}$.

Protrusions

[Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh, Thilikos '09]

- Given a graph G, a set $W \subseteq V(G)$ is a t-protrusion of G if
$$|\partial_G(W)| \leqslant t \quad \text{and} \quad \mathrm{tw}(G[W]) \leqslant t.$$
- The vertex set $W' = W \setminus \partial_G(W)$ is the restricted protrusion of W.
- We call $\partial_G(W)$ the boundary and $|W|$ the size of W.



(Figure by Felix Reidl)

Linear kernels on sparse graphs – an overview

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- Meta-result for H-topological-minor-free graphs. [Our result]

Our result

Theorem

Fix a graph H. Let Π be a parameterized graph problem on the class of H-topological-minor-free graphs that is treewidth-bounding and has finite integer index (FII). Then Π admits a linear kernel.

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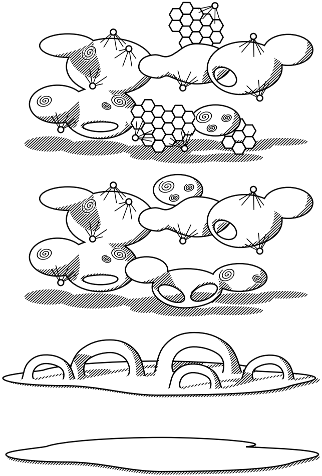
Problems affected by our result:

TREEWIDTH-t VERTEX DELETION, CHORDAL VERTEX DELETION, INTERVAL VERTEX DELETION, EDGE DOMINATING SET, FEEDBACK VERTEX SET, CONNECTED VERTEX COVER, ...

Linear kernels on sparse graphs – the conditions



H-topological-minor-free	treewidth-bounding
$\cup$	
H-minor-free	bidimensional, separation property
$\cup$	
bounded genus	quasi-compact
$\cup$	
planar	"distance-property"

(Figure by Felix Reidl)

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- Thus, our results imply the linear kernels of [Fomin, Lokshtanov, Saurabh, Thilikos '10]

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  1. $G_1 \oplus H \in \mathcal{G}$ iff $G_2 \oplus H \in \mathcal{G}$;
  2. $(G_1 \oplus H, k) \in \Pi$ iff $(G_2 \oplus H, k + \Delta_{\Pi,t}(G_1, G_2)) \in \Pi$.

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- Problem Π has FII in the class $\mathcal{G}$ if for every integer t, the equivalence relation $\equiv_{\Pi,t}$ has a finite number of equivalence classes.

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Disconnected PLANAR-$\mathcal{F}$-DELETION has not FII

- We prove: if $\mathcal{F}$ is a family of graphs containing some disconnected graph H, then PLANAR-$\mathcal{F}$-DELETION has not FII (in general).

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where $\pi(G)$ denotes the optimal value of problem o-Π on graph G.

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- By construction, if $i, j \geqslant 1$, it holds $\pi(G_i \oplus H_j) = \min\{i, j\}$.
- Then, if we take $1 \leqslant n < m$,

$$\begin{aligned} \pi(G_n \oplus H_{n-1}) - \pi(G_m \oplus H_{n-1}) &= (n-1) - (n-1) = 0, \\ \pi(G_n \oplus H_m) - \pi(G_m \oplus H_m) &= n - m < 0. \end{aligned}$$

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- Thus, $G_n, G_m \notin$ same equiv. class of $\sim_{\Pi,1}$ whenever $1 \leqslant n < m$.

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Lemma (The parameter does not increase)

$\forall$ *fixed* t, $\exists$ *finite set* $\mathcal{R}_t$ *of* t*-boundaried graphs s.t. for each* t*-boundaried graph* $G \in \mathcal{G}_t$ $\exists$ $G' \in \mathcal{R}_t$ *s.t.* $G \equiv_{\Pi,t} G'$ *and* $\Delta_{\Pi,t}(G, G') \geqslant 0$.

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Let t be a constant. Given an n-vertex graph G, a t-protrusion of G with the maximum number of vertices can be found in time $O(n^{t+1})$.

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Lemma (Replacing protrusions of constant size)

For $t \in \mathbb{N}$, suppose that the set $\mathcal{R}_t$ of representatives of $\equiv_{\Pi,t}$ is given. If W is a t-protrusion of size at most a fixed constant c, then one can decide in constant time which $G' \in \mathcal{R}_t$ satisfies $G' \equiv_{\Pi,t} G[W]$.

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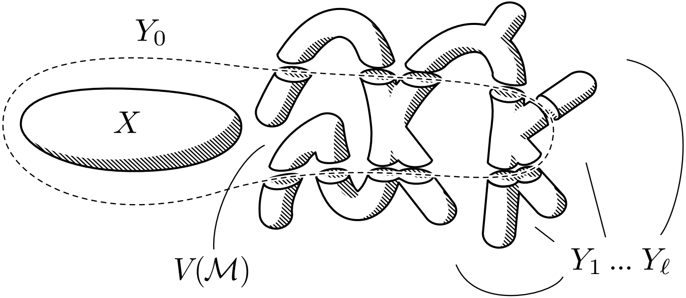
It runs in polynomial time ... given the sets of representatives!

Protrusion decompositions

An (α, t)-protrusion decomposition of a graph G is a partition $\mathcal{P} = Y_0 \uplus Y_1 \uplus \cdots \uplus Y_\ell$ of $V(G)$ such that:

- for every $1 \leqslant i \leqslant \ell$, $N(Y_i) \subseteq Y_0$;
- for every $1 \leqslant i \leqslant \ell$, $Y_i \cup N_{Y_0}(Y_i)$ is a t-protrusion of G;
- $\max\{\ell, |Y_0|\} \leqslant \alpha$.



(Figure by Felix Reidl)

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   Using what Christophe explained yesterday, we can easily prove that:

   Let Π be a parameterized graph problem that has FII and is t-treewidth-bounding, both on the class of H-topological-minor-free graphs. Then any reduced YES-instance (G, k) has a protrusion decomposition $V(G) = Y_0 \uplus Y_1 \uplus \cdots \uplus Y_\ell$ s.t.:

   1. $|Y_0| = O(k)$;
   2. $|Y_i| \leq \rho'_\Pi(2t + \omega_{\mathcal{H}})$ for $1 \leq i \leq \ell$; and
   3. $\ell = O(k)$.

Next section is...

1. Motivation and our result
2. Idea of proof
3. Further research

Limits of our approach and further research

- For which notions of sparseness (beyond H-topological-minor-free graphs) can we use our technique to obtain polynomial kernels?

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- Explicit constants? Lower bounds on their size?

Gràcies!!