Computing Distances on Graph Associahedra is Fixed-parameter Tractable

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Available at **arXiv:2504.18338** (ICALP 2025)

Elimination trees

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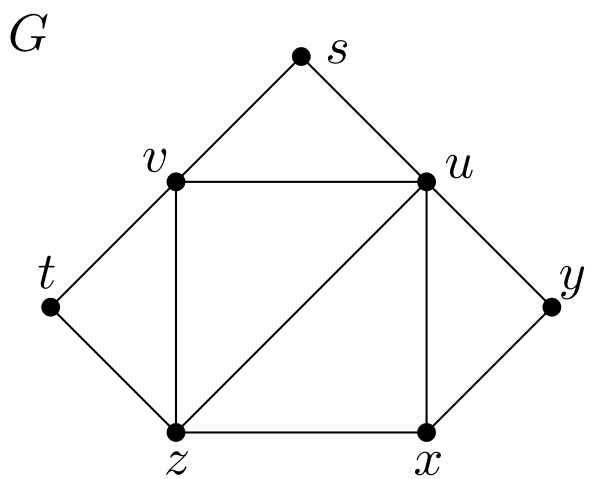

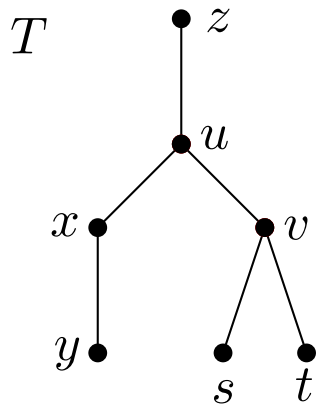

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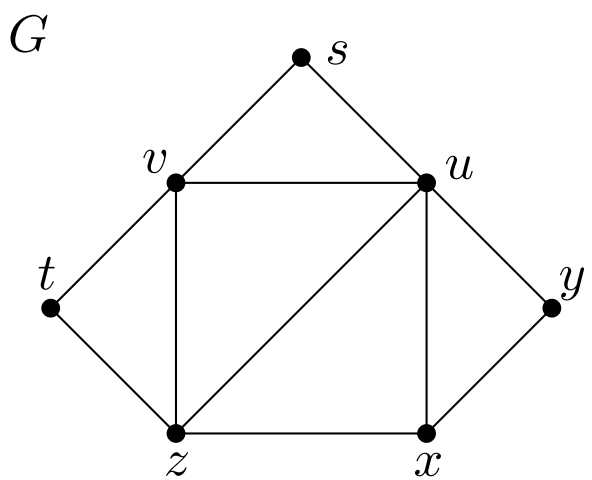

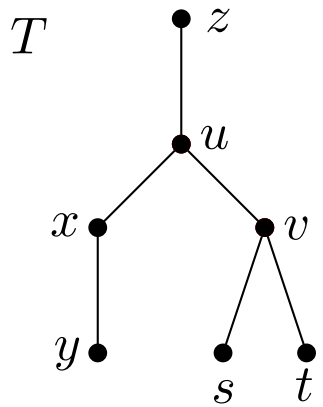

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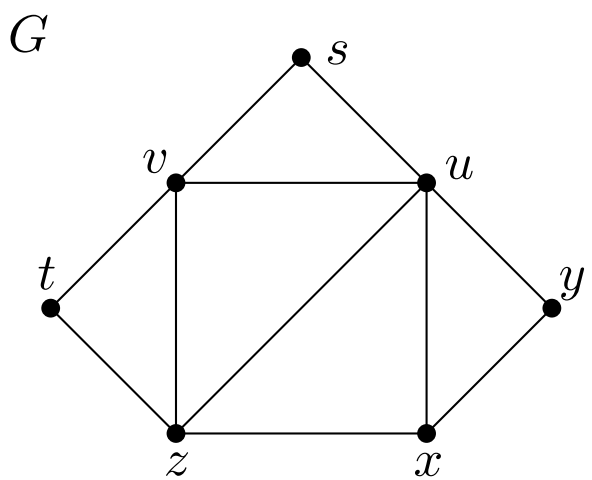

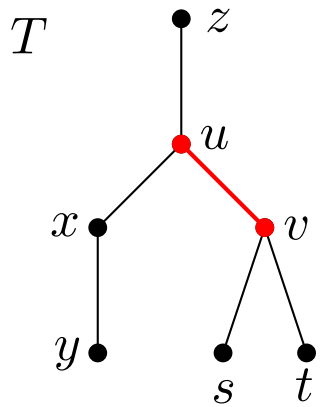

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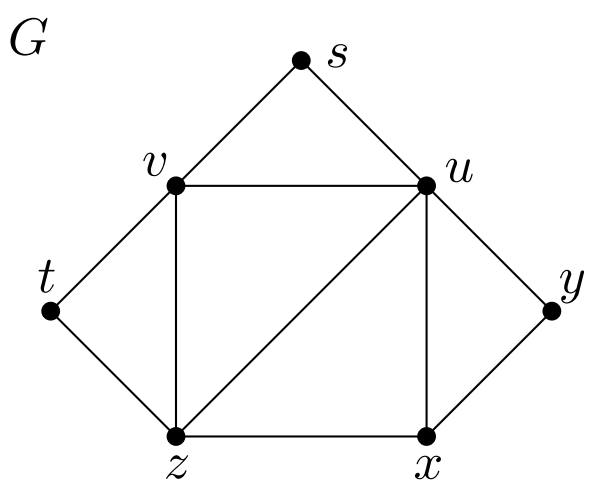

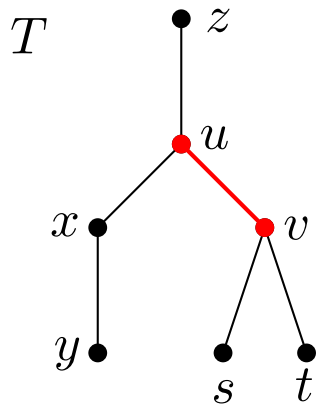

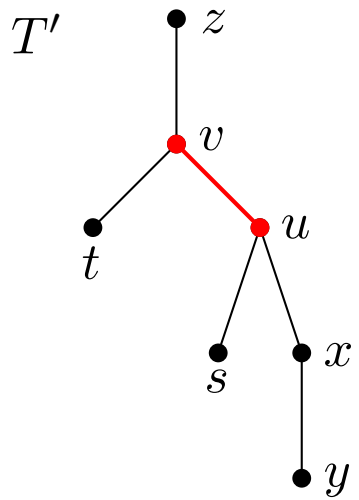

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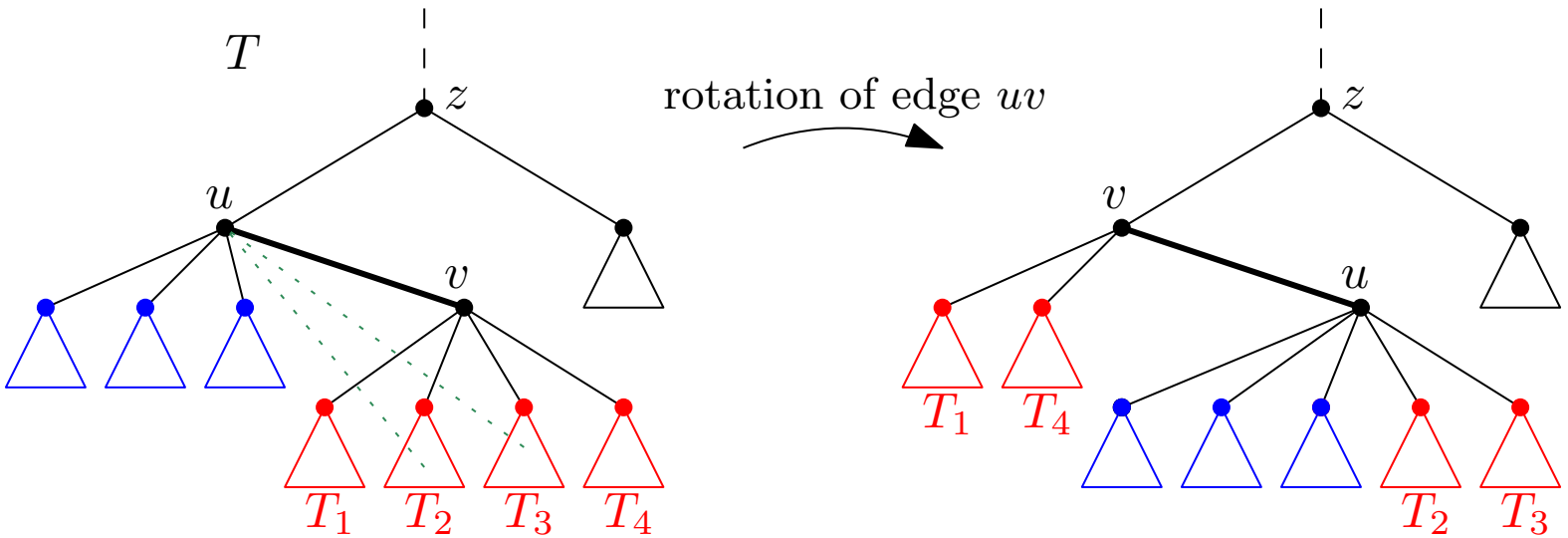
Rotation distance between elimination trees



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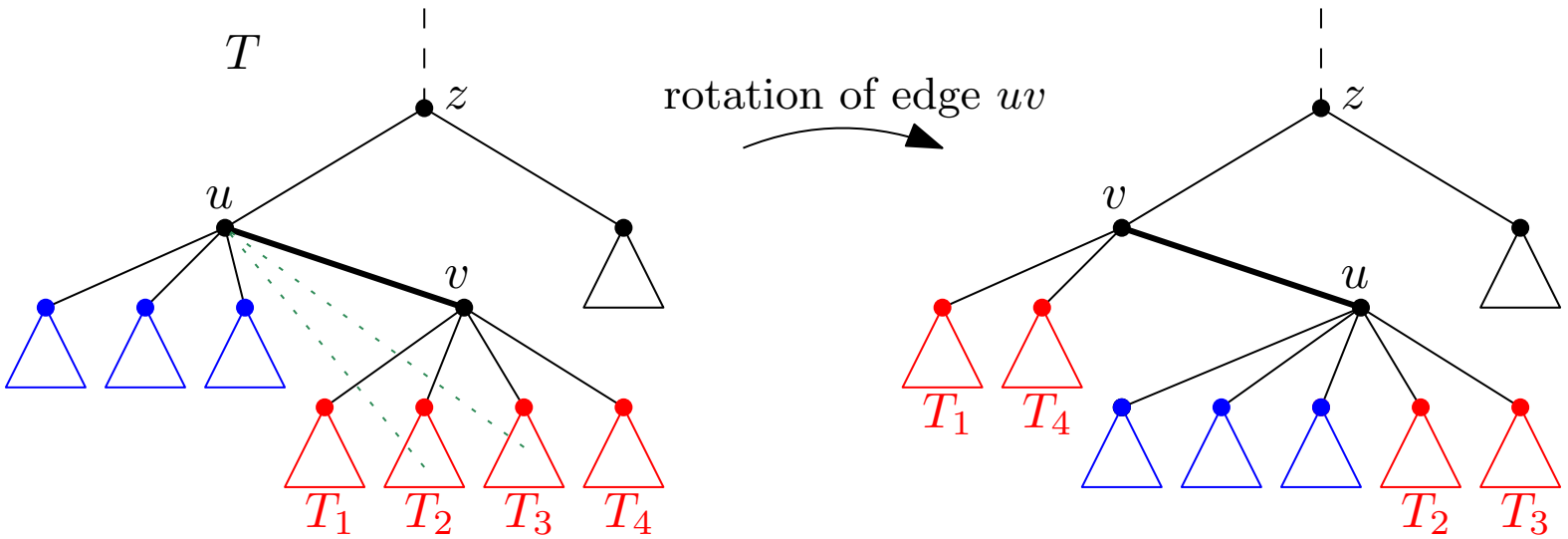

The rotation distance between two elimination trees (forests) T, T' of a graph G, denoted by $\mathrm{dist}(T, T')$, is the minimum number of rotations it takes to transform T into T'.

Graph associahedra

For any graph G, the flip graph of elimination forests of G under edge rotations is the skeleton of a polytope: graph associahedron $\mathcal{A}(G)$.

Object introduced by [Carr, Devadoss, Postnikov. 2006-2009]

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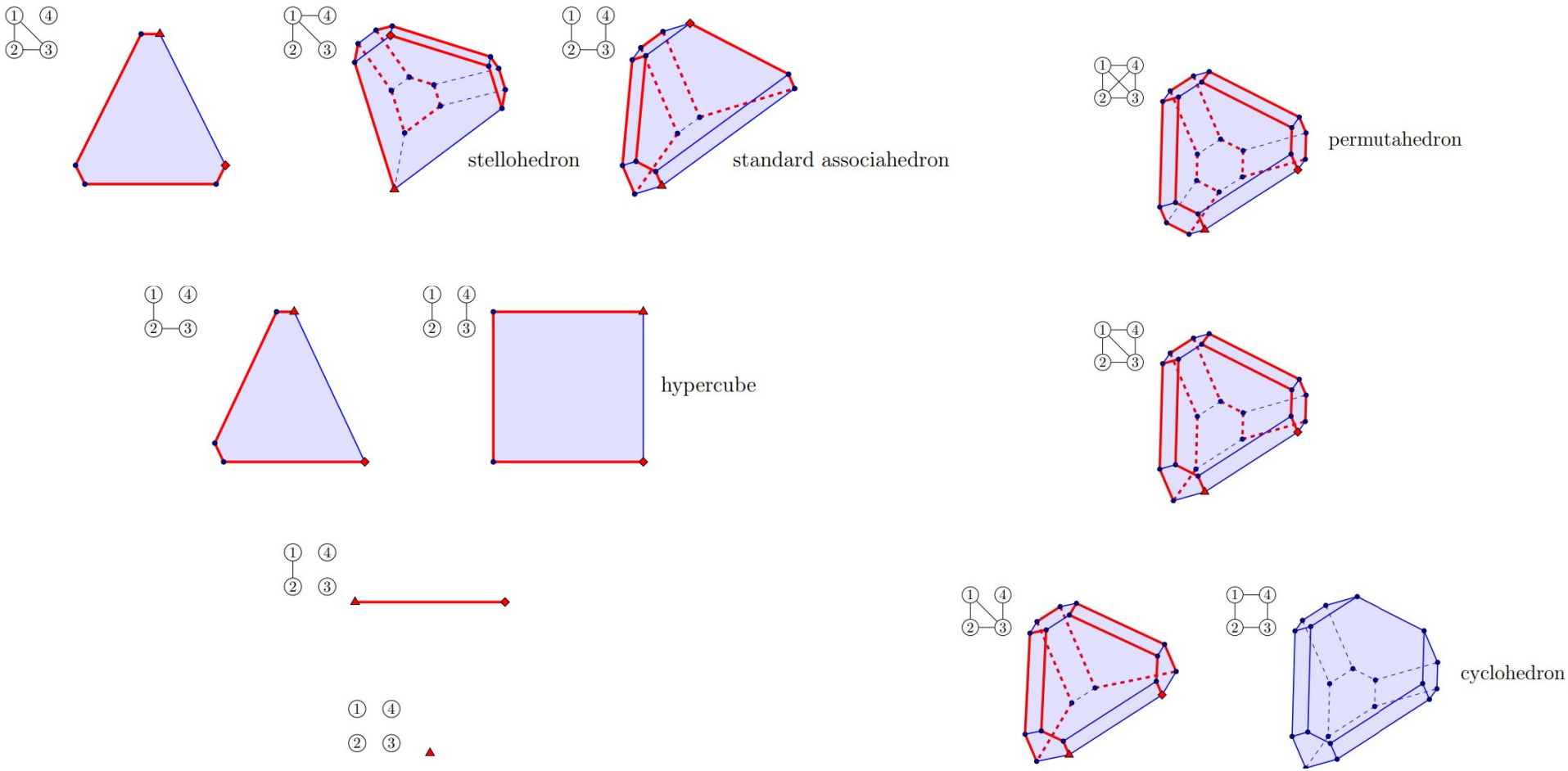
Famous particular cases of $\mathcal{A}(G)$ depending on the underlying graph G:

G	$\mathcal{A}(G)$
path	(standard) associahedron
complete graph	permutahedron
cycle	cyclohedron
star	stellohedron
matching	hypercube

Illustration of some famous examples



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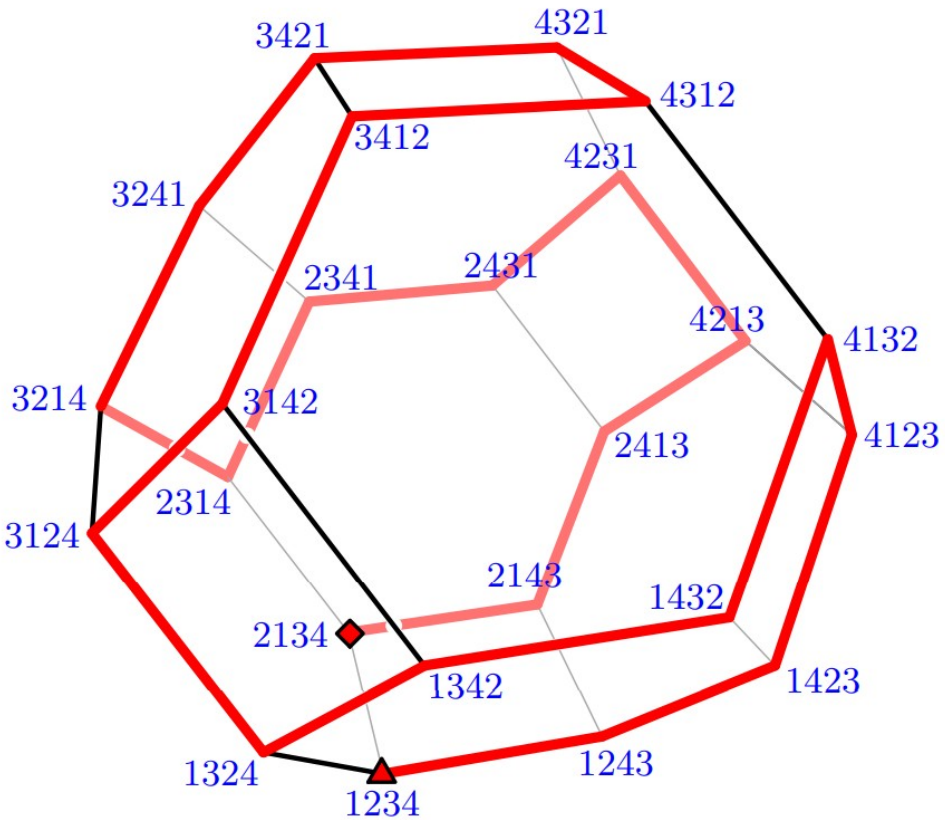
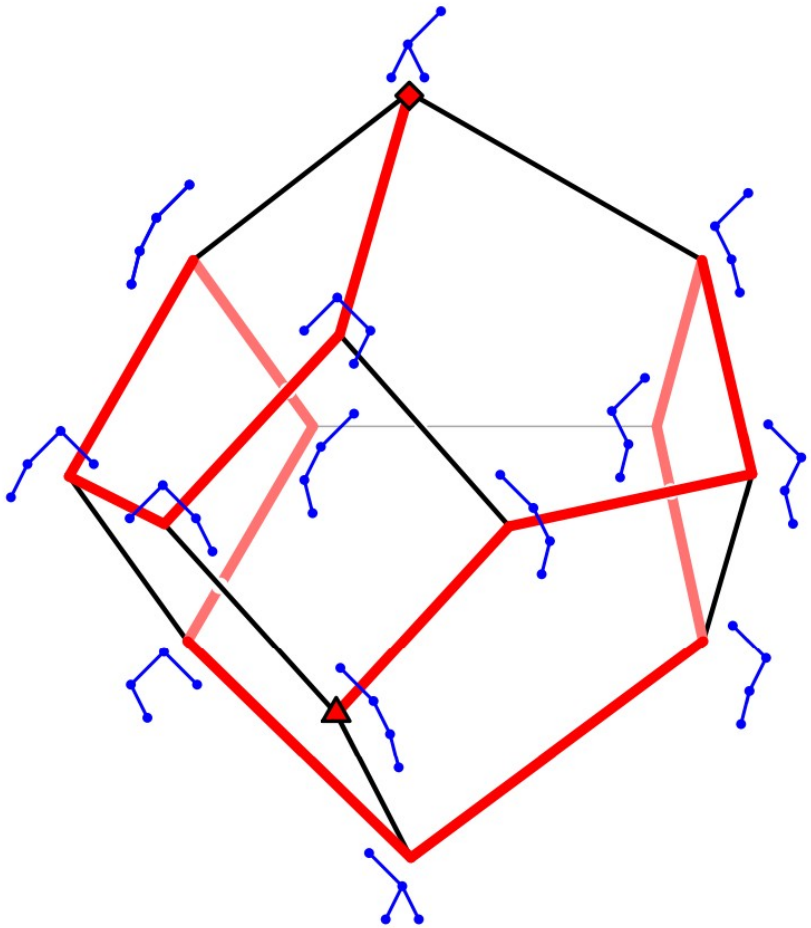
Zooming in: permutahedron and (standard) associahedron





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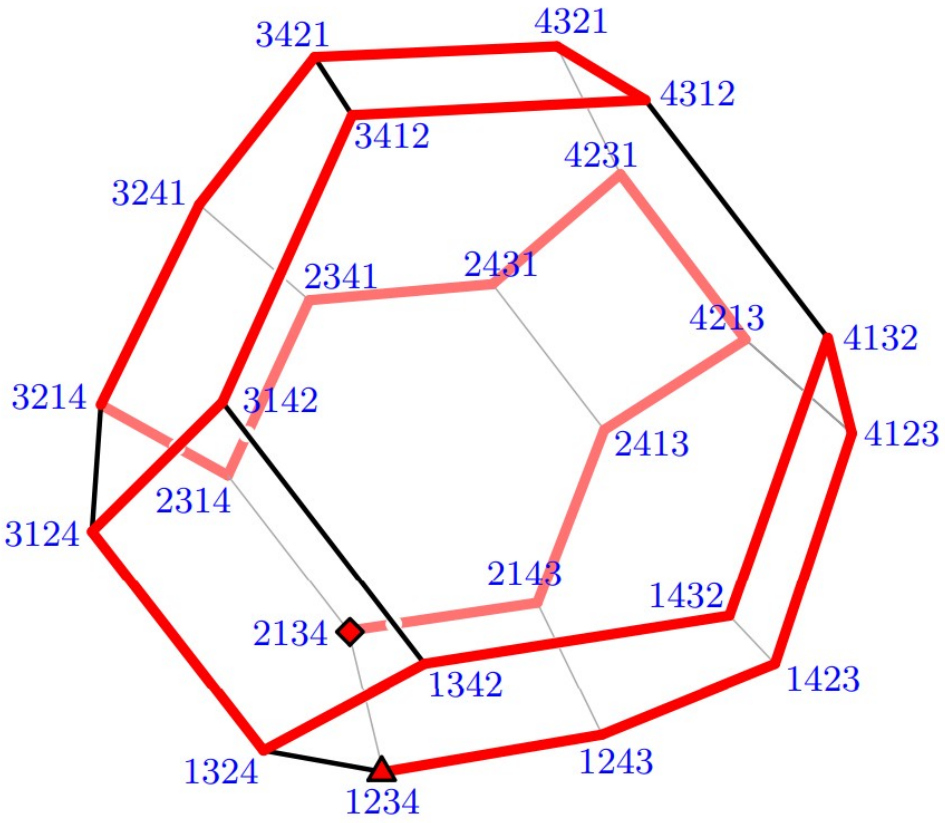

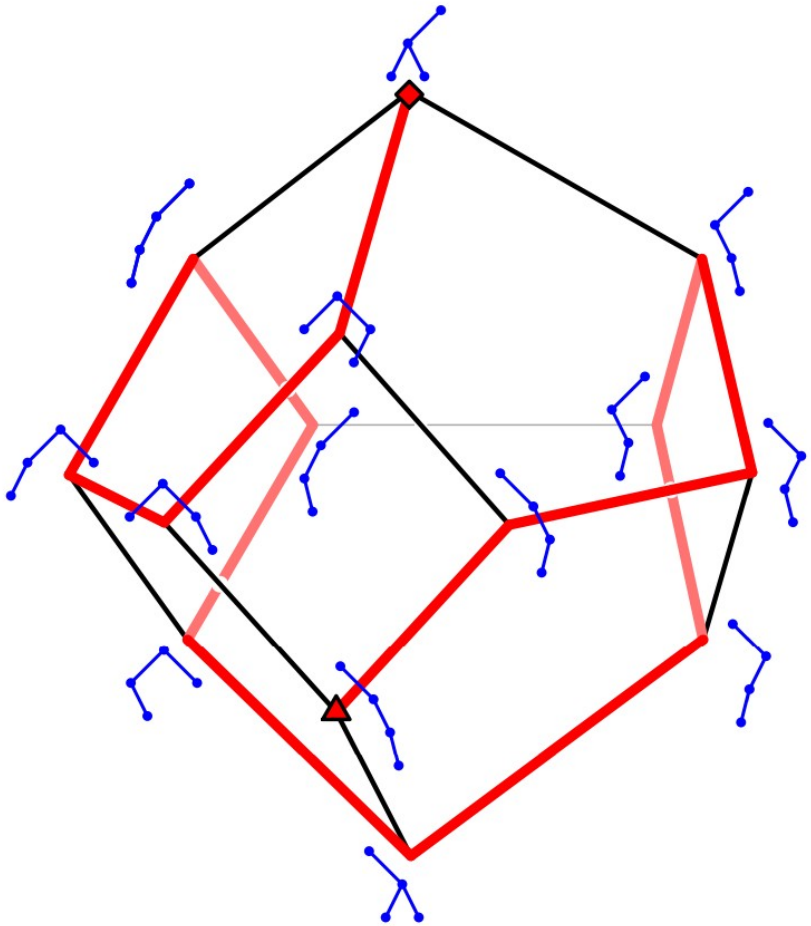

The (standard) associahedron has a rich history and literature, connecting computer science, combinatorics, algebra, and topology.

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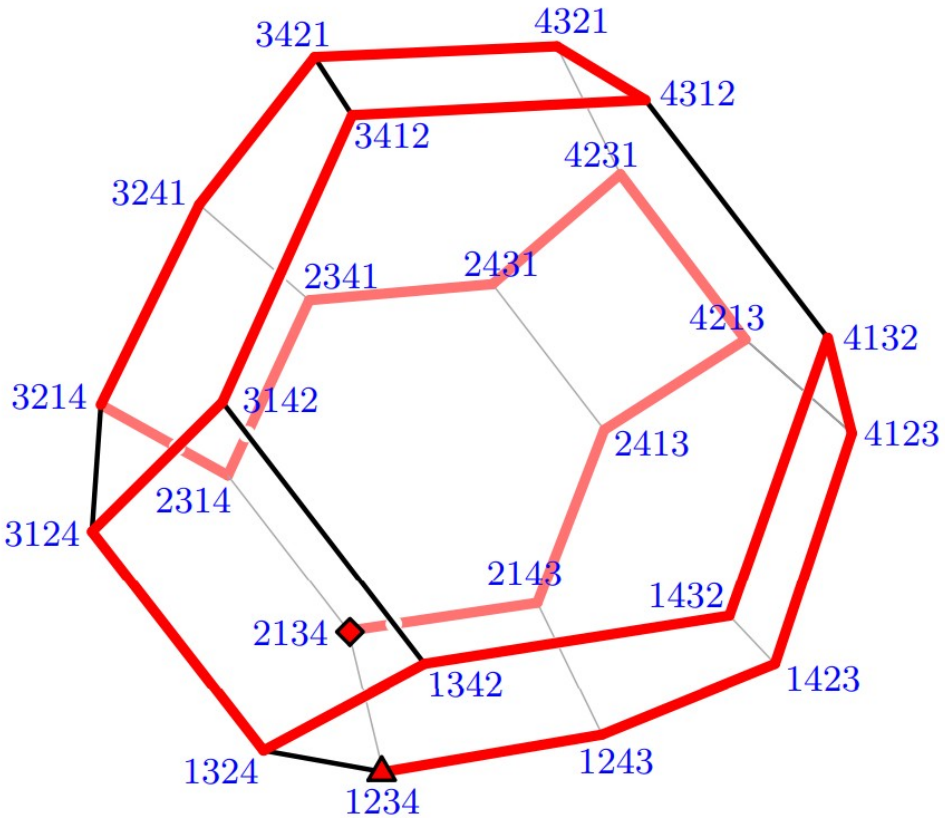

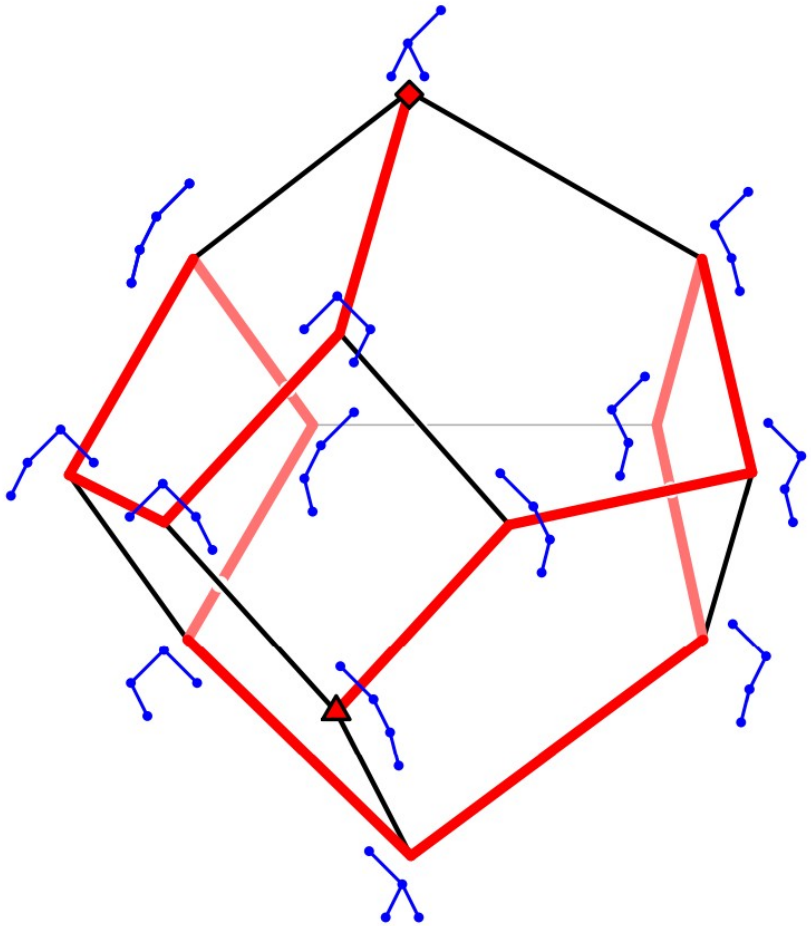

Binary trees are in bijection with many other Catalan objects:
triangulations of a convex polygon, well-formed parenthesis, Dyck paths, ...

Intensively studied: diameter of graph associahedra

Determining the diameter exactly, or upper/lower bounds, or estimates:

- If G is a path: [Sleator, Tarjan, Thurston. 1998] [Pournin. 2014]
- If G is a star: [Manneville, Pilaud. 2010]
- If G is a cycle: [Pournin. 2017]
- If G is a tree: [Manneville, Pilaud. 2010] [Cardinal, Langerman, Pérez-Lantero. 2018]
- If G is a complete bipartite or trivially perfect graph: [Cardinal, Pournin, Valencia-Pabon. 2022]
- If G is a caterpillar: [Berendsohn. 2022]
- If G has bounded treedepth or treewidth: [Cardinal, Pournin, Valencia-Pabon. 2022]

Our focus: computing distances on graph associahedra

Suppose for simplicity that the considered graph G is connected.

Rotation Distance

Instance: A graph G, two elimination trees T and T' of G, and a positive integer k.

Question: Is the rotation distance between T and T' at most k?

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This is not the problem we solve!

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This motivates the study of the parameterized complexity of the problem.

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- para-NP-hard problem: NP-hard for a fixed value of the parameter.

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Prior to our work, only the case where G is a path was known to be FPT.

[Cleary, St. John. 2009] [Lucas. 2010] [Kanj, Sedgwick, Xia. 2017] [Li, Xia. 2023]

Main ideas of the FPT algorithm

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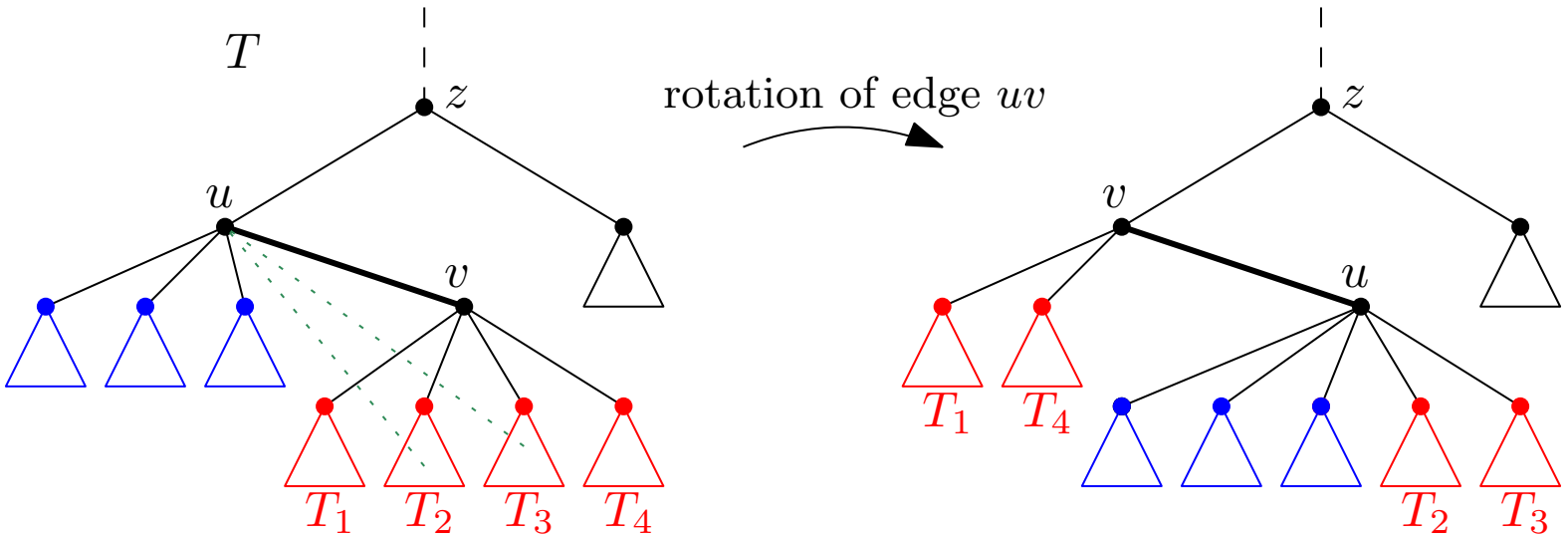
Let us see how we find such a "small" set $M \subseteq V(T)$ of marked vertices...

There are few children-bad vertices



Observation: a rotation may change the set of children of at most three vertices (but the parent of arbitrarily many vertices).

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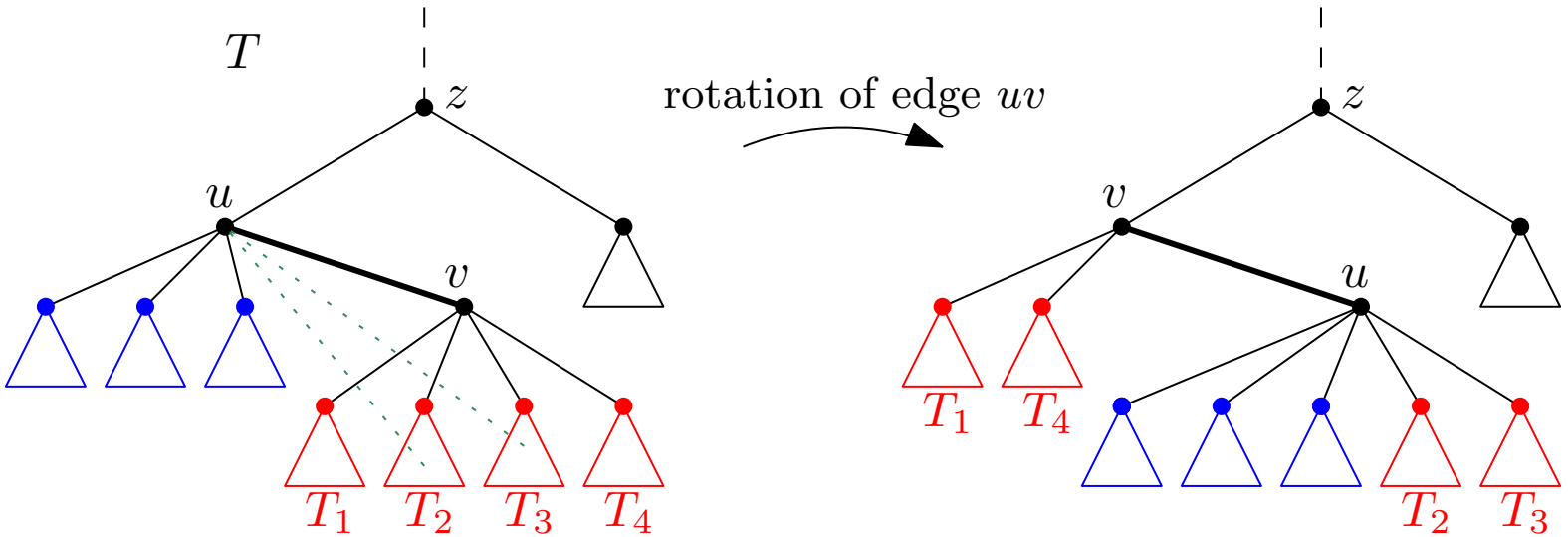

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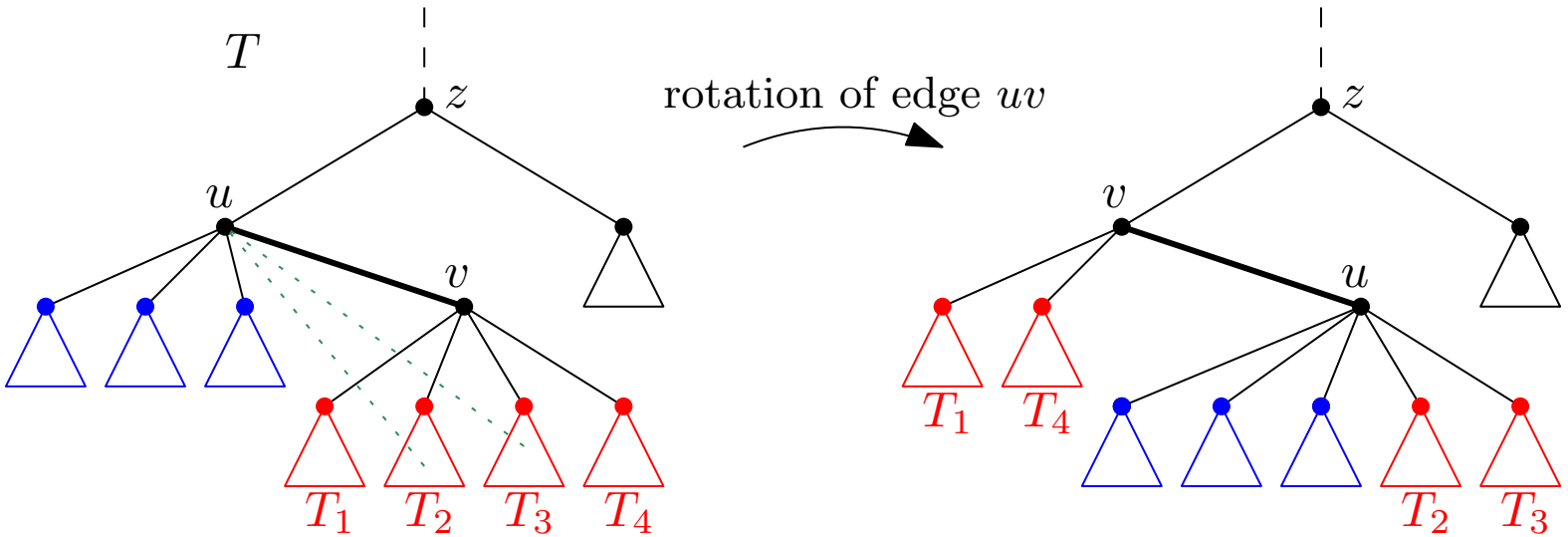

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We may assume that there are at most $3k$ (T, T')-children-bad vertices.

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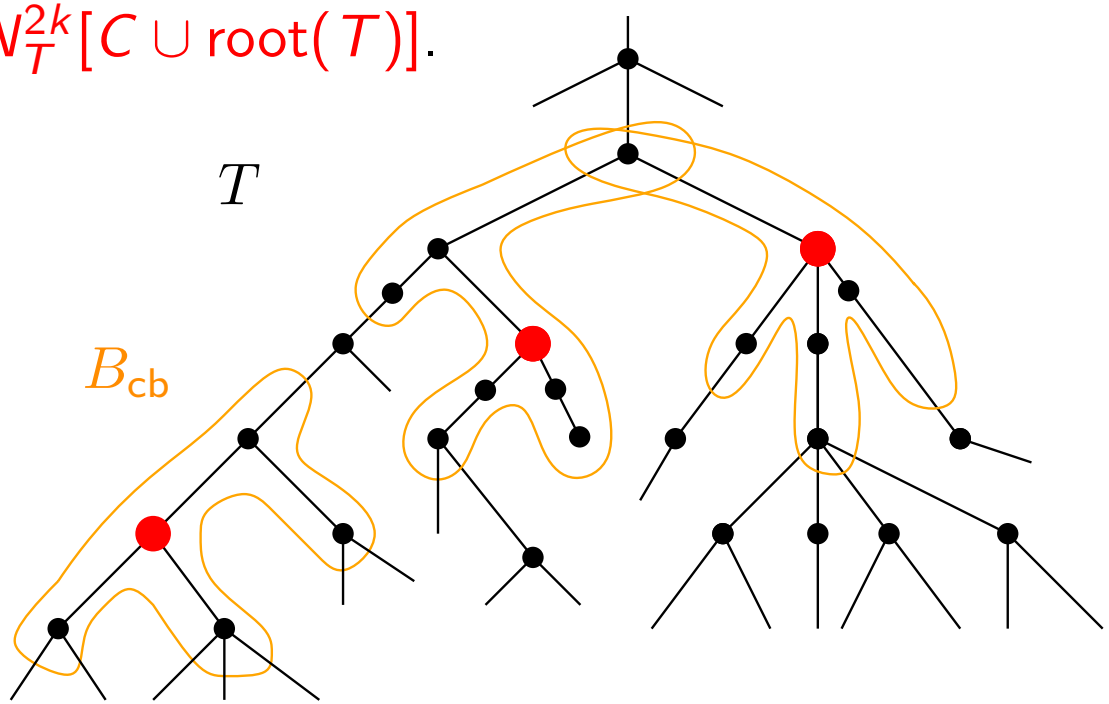
We define $B_{\mathsf{cb}} = N_T^{2k}[C \cup \mathsf{root}(T)]$.



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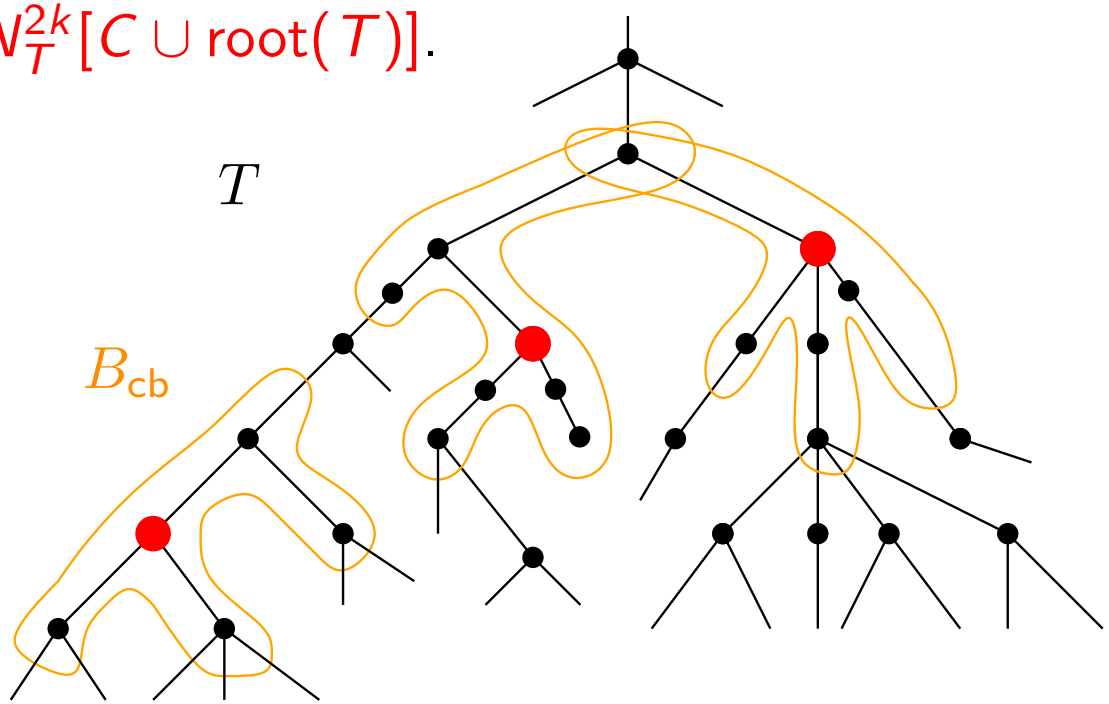

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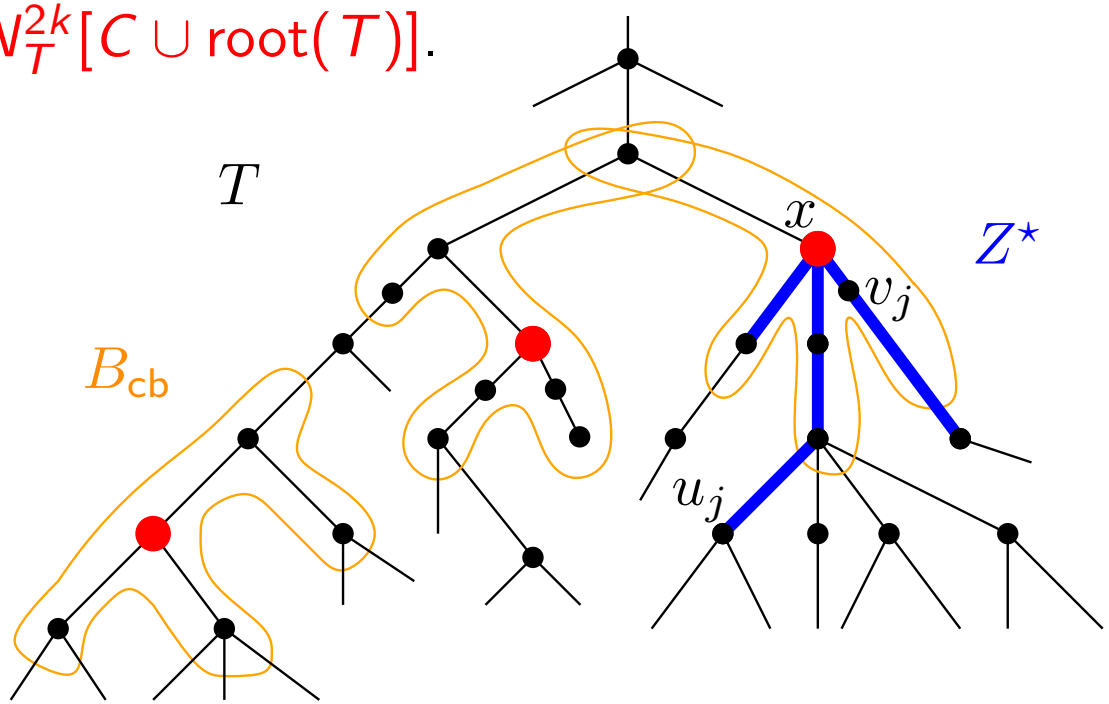

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If $\Delta(T)$ is bounded (in particular, if $\Delta(G)$ is bounded), we are done!

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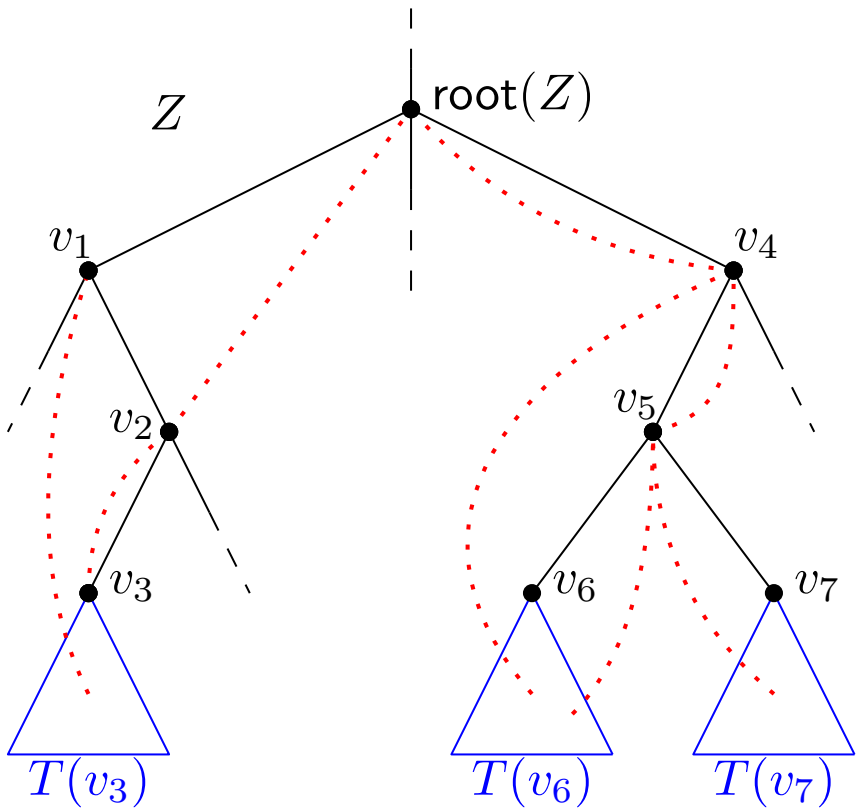

$\mathsf{trace}(T, Z, v_1) = (1)$
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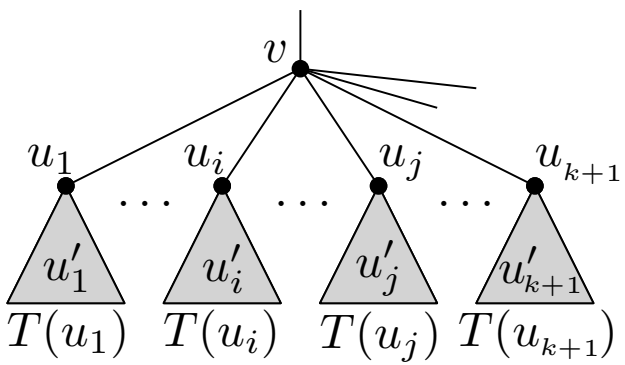

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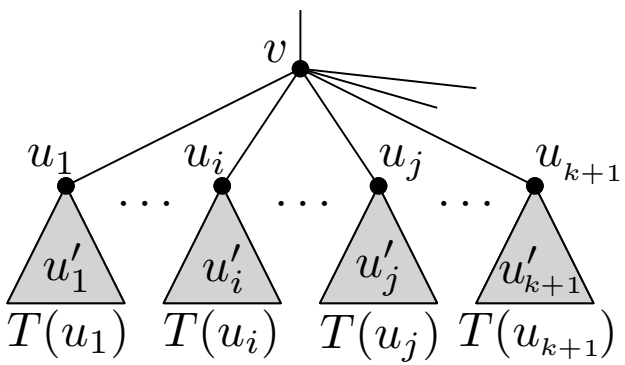

Crucial: enough to keep track of "at least $k+1$", not the actual number.

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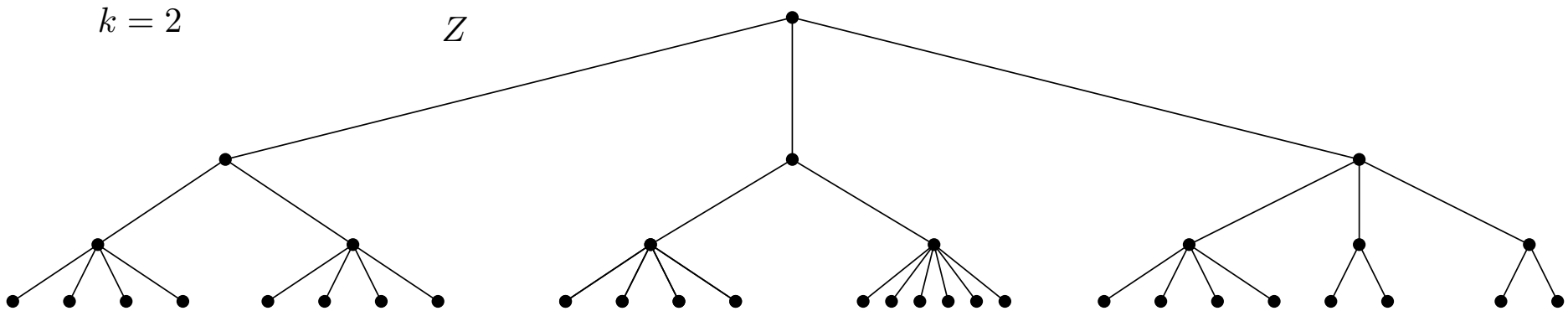

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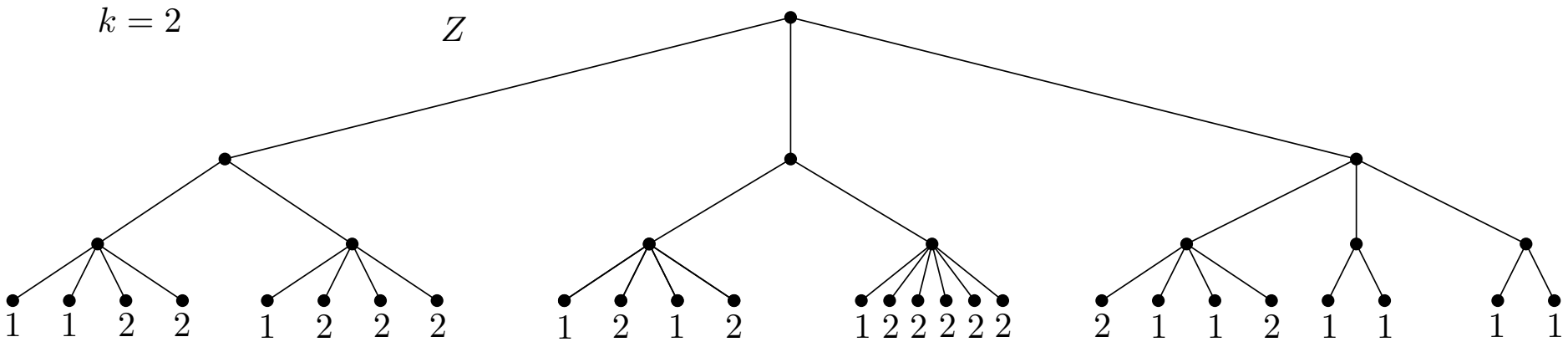

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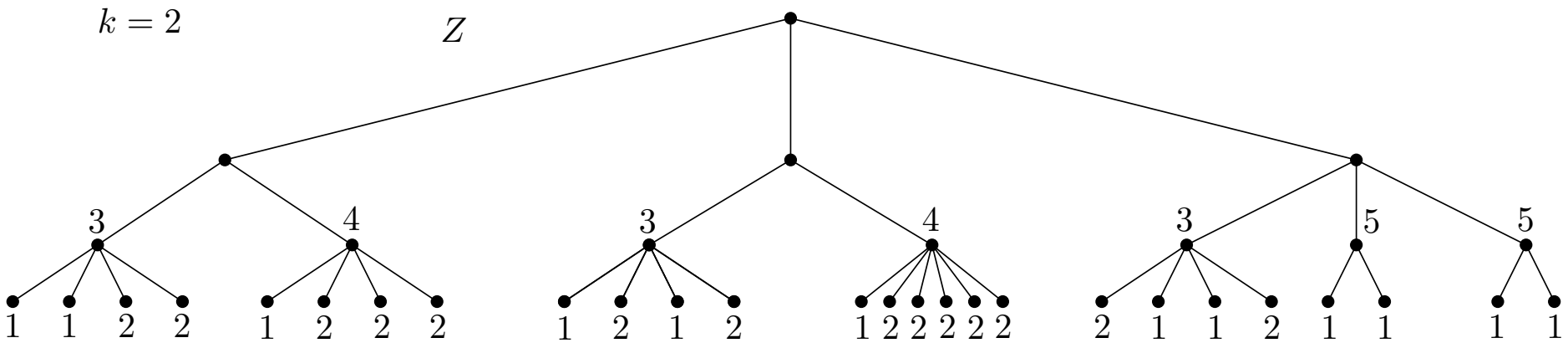

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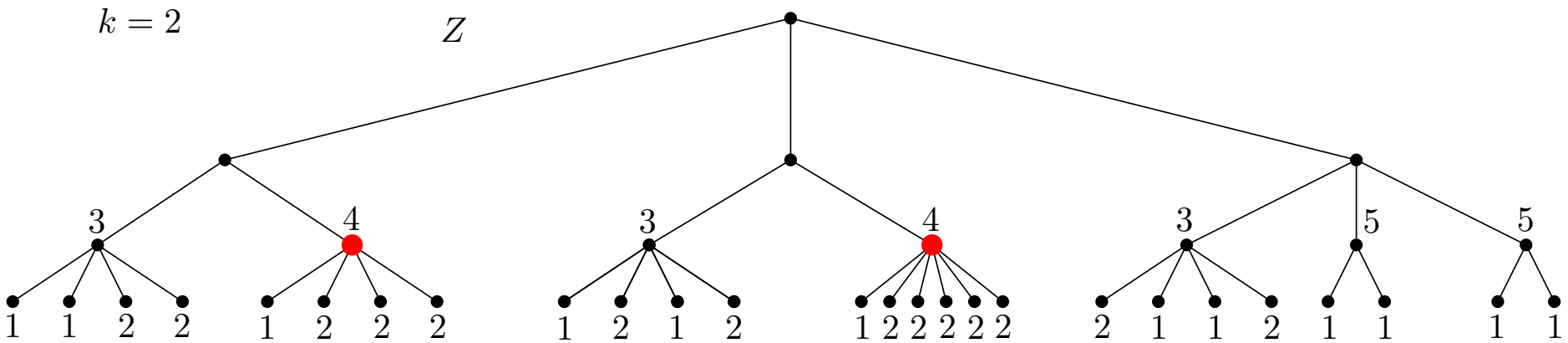

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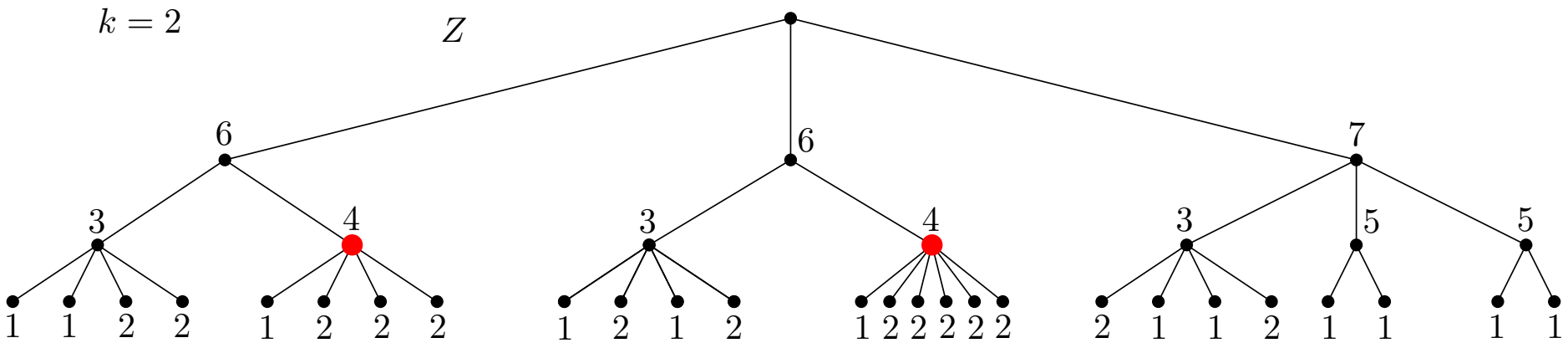

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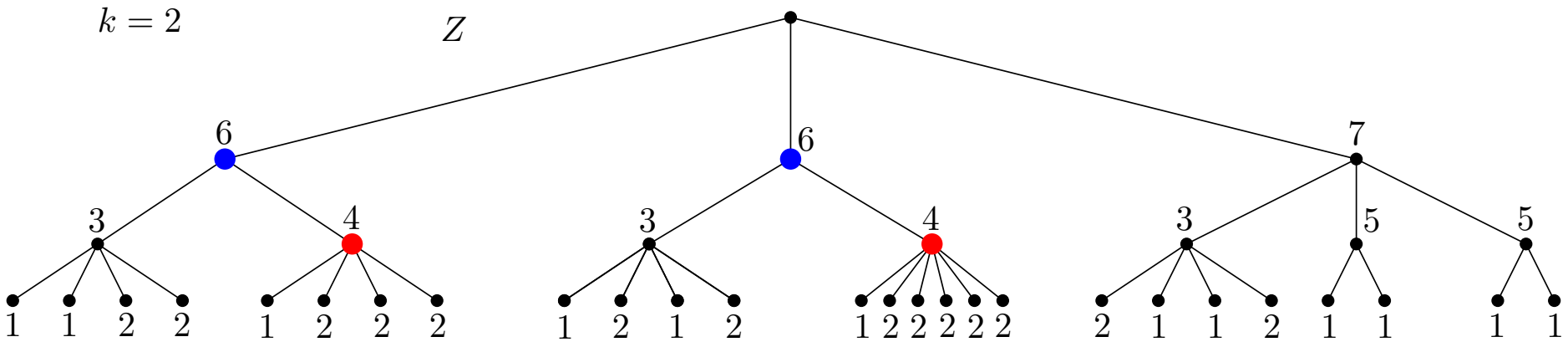

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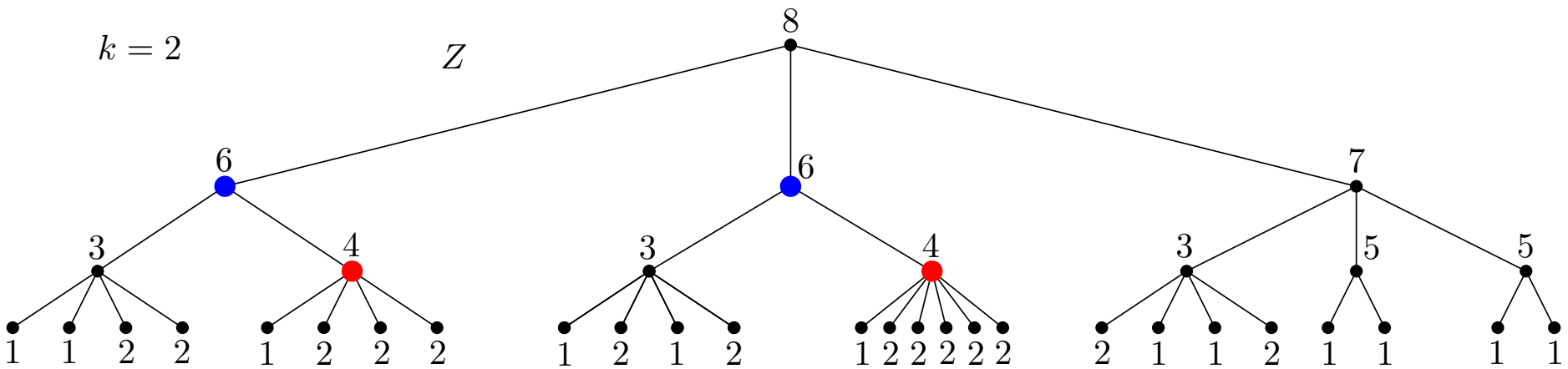

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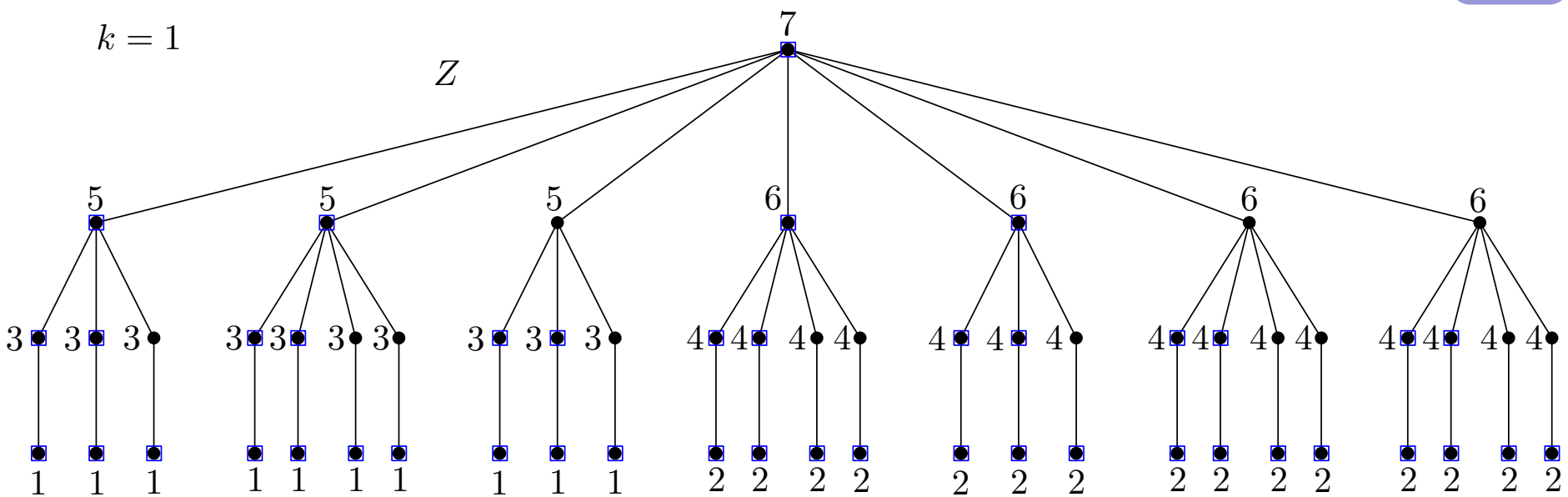

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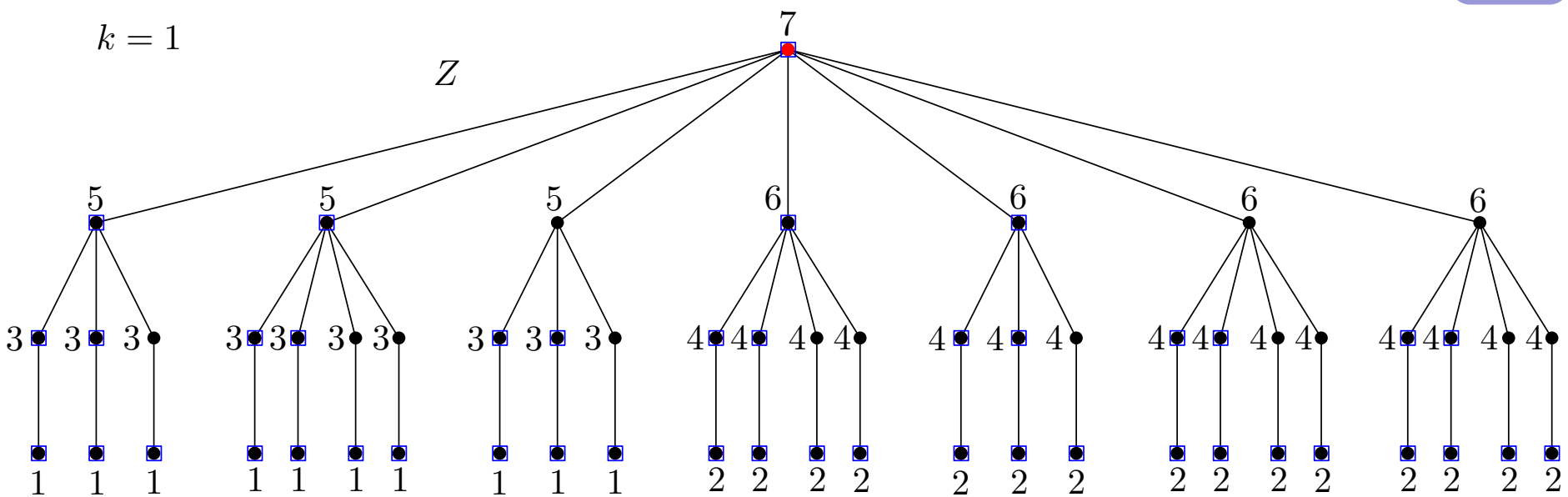

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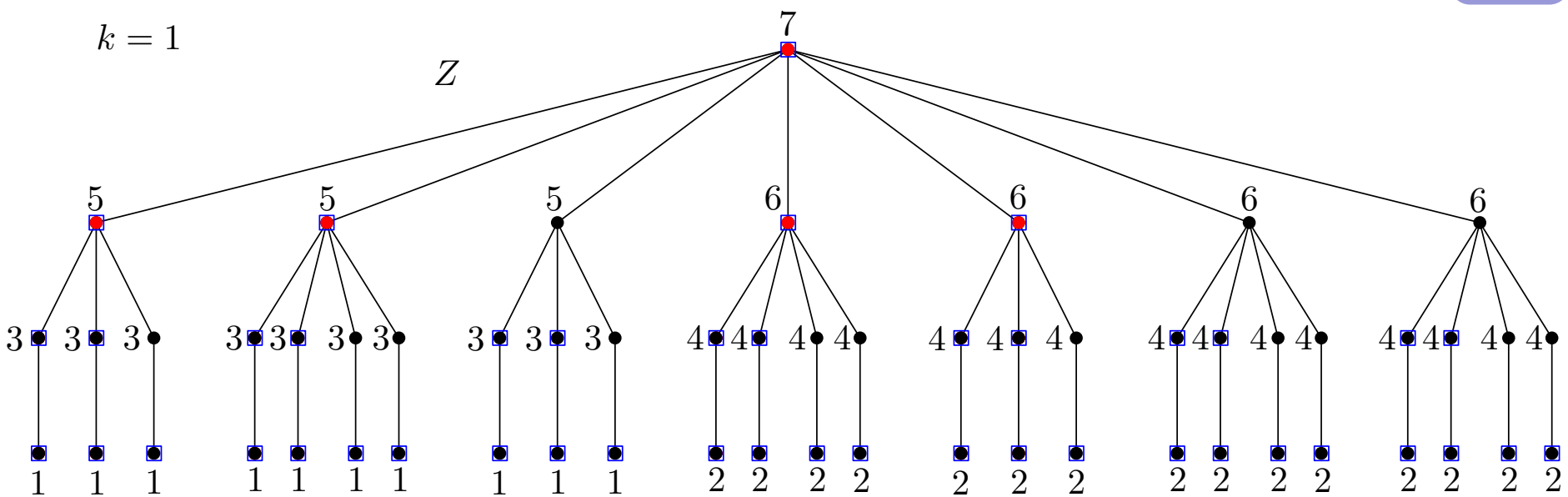

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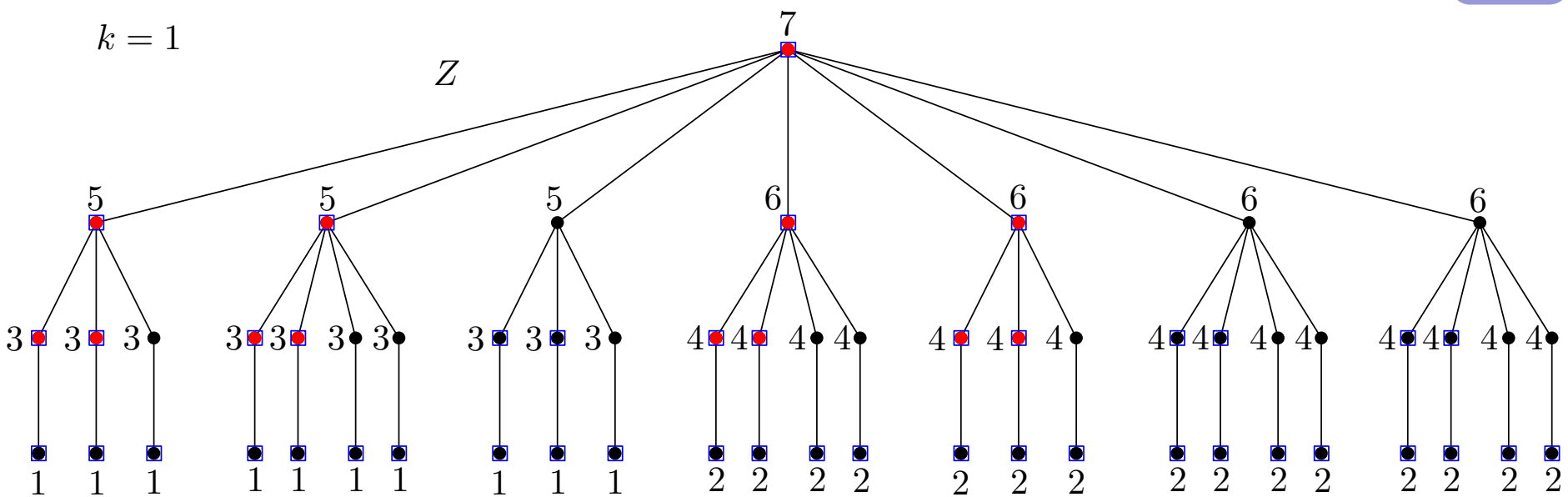

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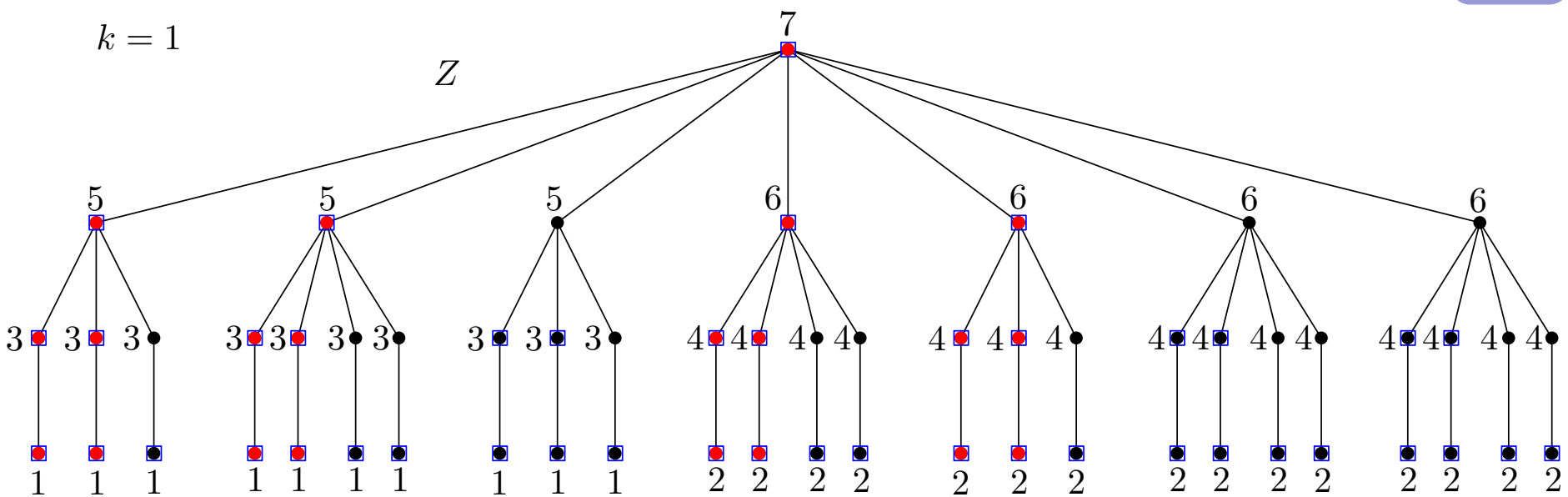

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The set $M \subseteq V(T)$ of marked vertices has size bounded by a function $h(k)$, with the same asymptotic growth as the function $g(k)$ given by the number of types. Moreover, M can be computed in time $h(k) \cdot |V(G)|$.

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- We distinguish two cases...

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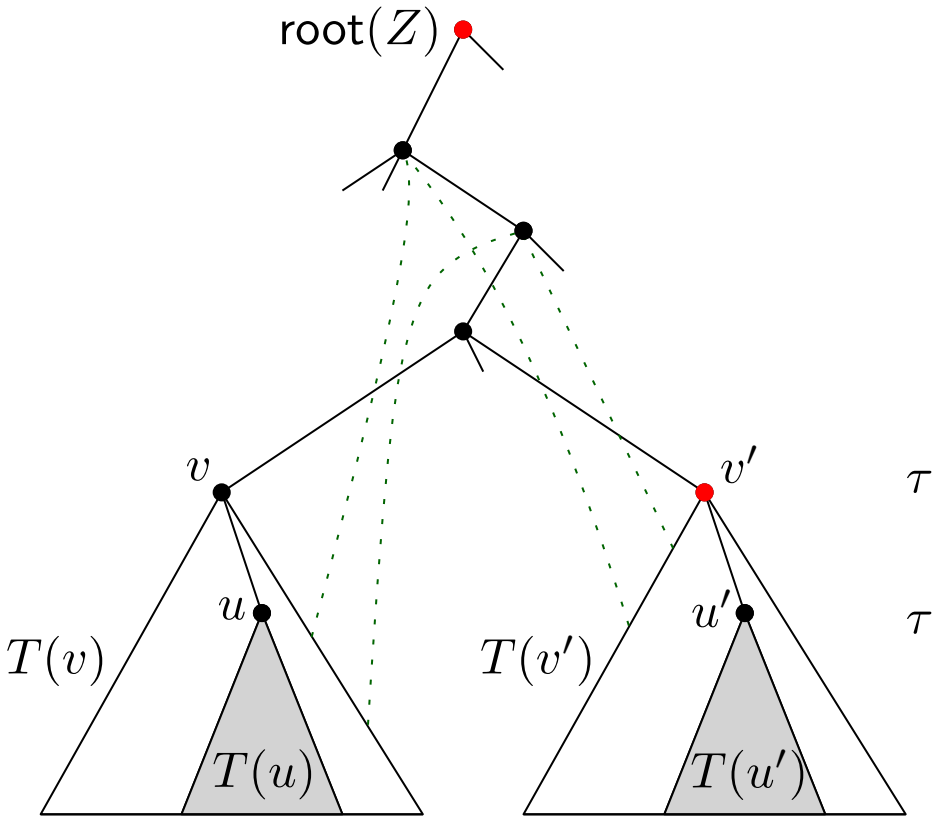

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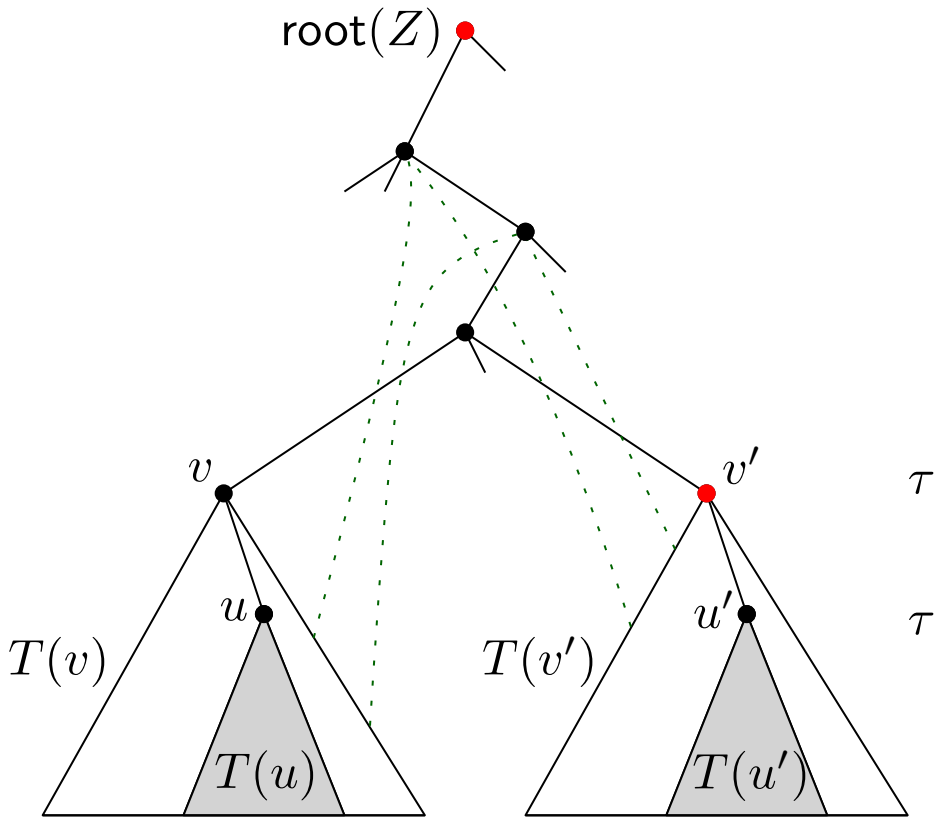

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We define σ' from σ by just replacing v with v' in all the rotations of σ involving v.

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Lemma

There exists a unique vertex $v^\star \in \text{ancestors}(Z, v)$ such that

- *all vertices in $T(v^\star)$ are non-marked,*
- *$v^\star$ has a marked T-sibling v' such that*
  - *$\tau(T, Z, v^\star) = \tau(T, Z, v')$, and*
  - *no vertex in $T(v')$ is used by σ, and*
- *$v^\star$ is the vertex closest to v satisfying the above properties.*

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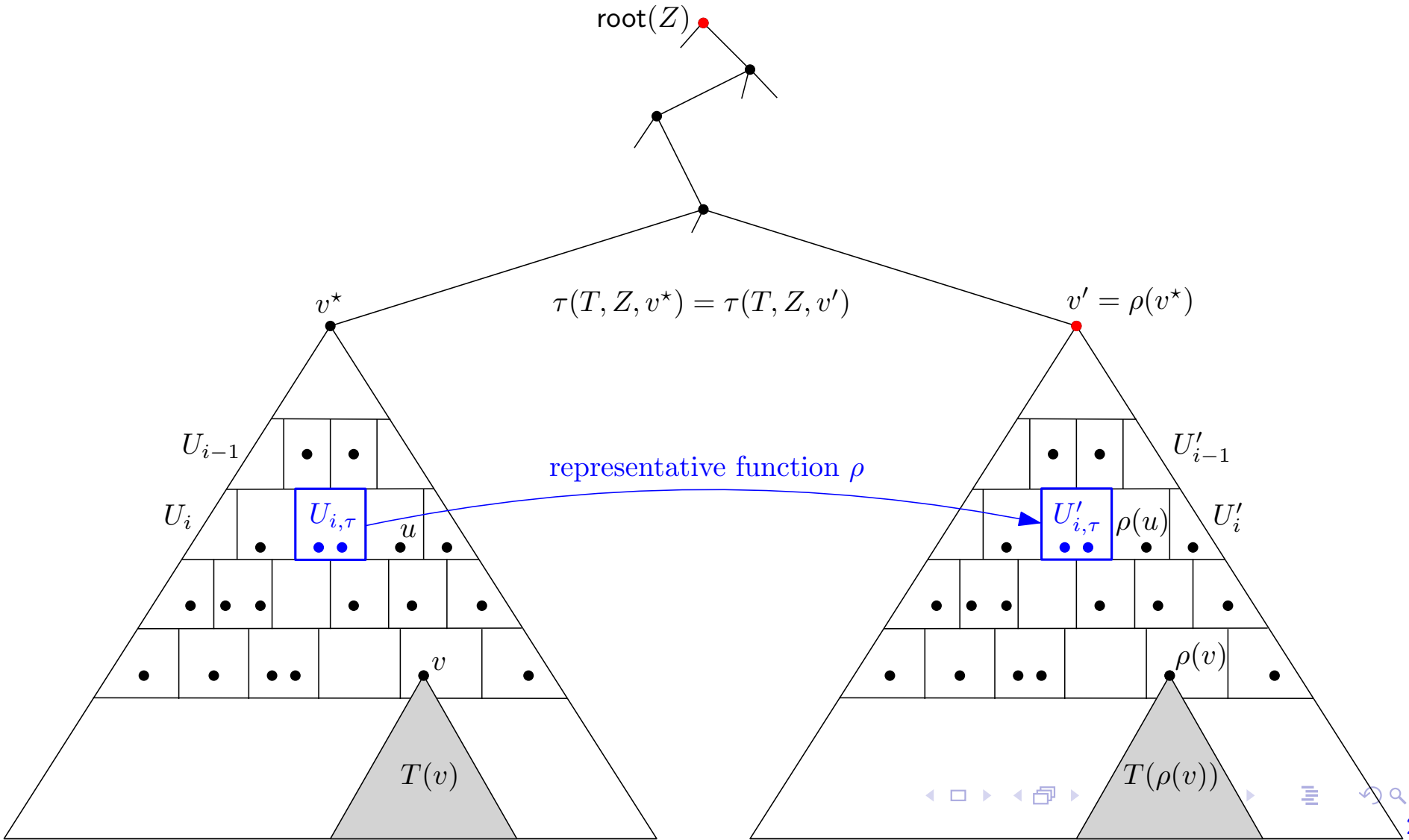

New results: can we go beyond graph associahedra?

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Rotation Distance problem: distances on graph associahedra.

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ROTATION DISTANCE problem: distances on graph associahedra.

Natural generalization: distances on hypergraphic polytopes.

New results: can we go beyond graph associahedra?

Fix a hypergraph H. We define the hypergraphic polytope of H as:

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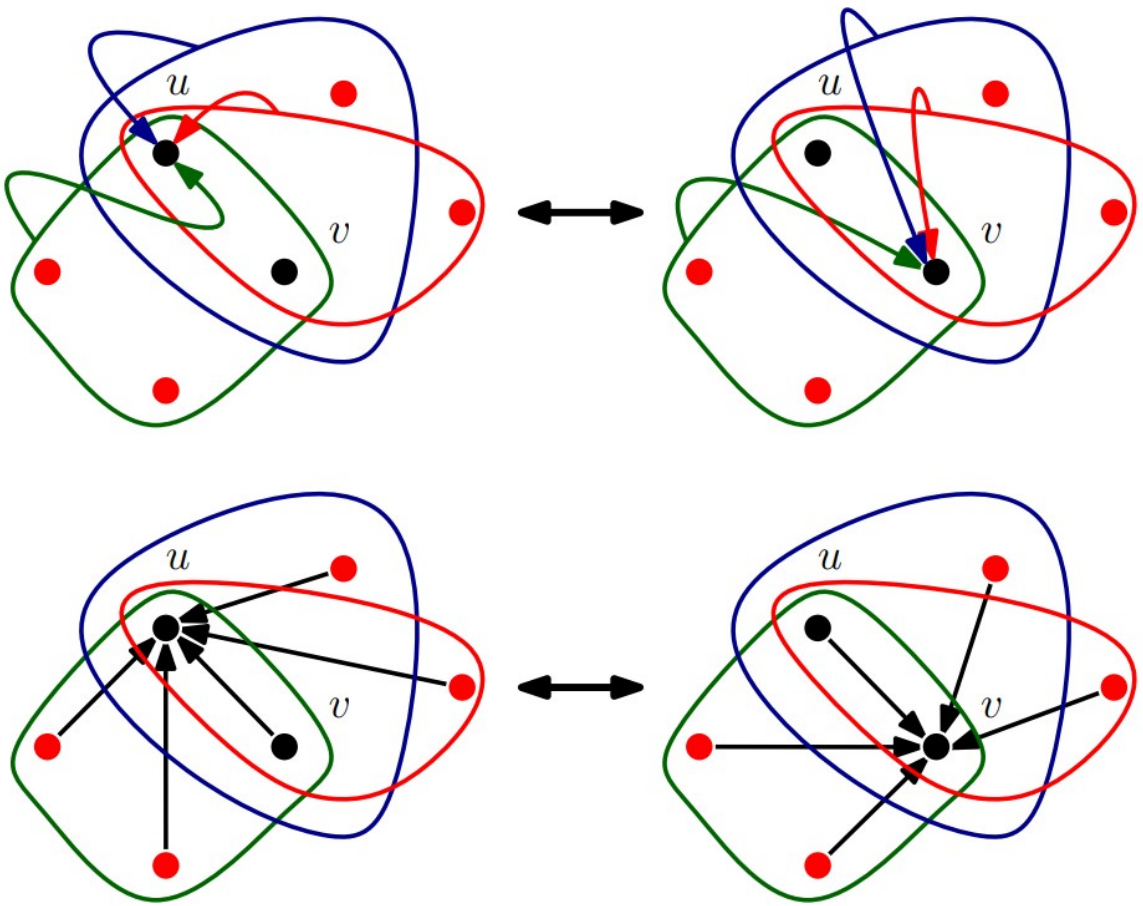
- **Vertices**: all acyclic orientations of H.
- **Edges**: if the two corresponding rotations are related by a flip.



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[Cardinal, Steiner. 2023]

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Computing distances on hypergraphic polytopes is $W[2]$-hard parameterized by the distance.

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We present a parameterized reduction from k-Dominating Set.

Conclusions and further research

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The Rotation Distance *problem can be solved in time* $f(k) \cdot |V(G)|$,

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Rotation Distance	paths	general graphs
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FPT	✓	✓ [this talk]
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Combinatorial Shortest Path on Polymatroids:

- NP-hard. [Ito, Kakimura, Kamiyama, Kobayashi, Maezawa, Nozaki, Okamoto. 2023]
- Is it also FPT?

Gràcies!