On the Erdős-Pósa property for minors of graphs

Ignasi Sau  
CNRS, LIRMM, Montpellier

Joint work with:

Dimitris Chatzidimitriou  
National and Kapodistrian University of Athens, Greece

Samuel Fiorini  
Gwenaël Joret  
Université Libre de Bruxelles, Belgium

Jean-Florent Raymond  
Dimitrios M. Thilikos  
CNRS, LIRMM, Montpellier
Outline of the talk

1. Motivation
2. Vertex version for minors
3. Edge version for minors
4. Vertex version for topological minors
Next section is...

1 Motivation

2 Vertex version for minors

3 Edge version for minors

4 Vertex version for topological minors
König’s min-max theorem in bipartite graphs:

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\min \# \text{ vertices covering all edges} \geq \max \# \text{ of disjoint edges}
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Packing and covering

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König’s min-max theorem in bipartite graphs:

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\begin{align*}
\min \ # \ \text{vertices covering all } H & \in \mathcal{H} \geq \max \ # \ \text{of disjoint } H & \in \mathcal{H} \\
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\[ \min \# \text{ vertices covering all } H \in \mathcal{H} \leq f(\max \# \text{ of disjoint } H \in \mathcal{H}) \]
König’s min-max theorem in bipartite graphs:

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\text{Min Vertex Cover} = \text{Max Matching}
\]

If there exists such \( f \) for all \( G \), then \( \mathcal{H} \) satisfies the \textbf{Erdős-Pósa property}.

\[
\min \# \text{ vertices covering all } H \in \mathcal{H} \leq f(\max \# \text{ of disjoint } H \in \mathcal{H})?
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**Minors and models in graphs**

A **model** in a graph $G$ is a collection $\{S_u : u \in V(H)\}$ such that the $S_u$'s are vertex-disjoint connected subgraphs of $G$, and there is an edge between $S_u$ and $S_v$ in $G$ for every edge $uv \in E(H)$.

$H$ is a **minor** of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.
Minors and models in graphs

$H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.

$H$-model in $G$: collection $\{S_u : u \in V(H)\}$ s.t.
- the $S_u$’s are vertex-disjoint connected subgraphs of $G$, and
- there is an edge between $S_u$ and $S_v$ in $G$ for every edge $uv \in E(H)$.

The $S_u$’s are called vertex images.
Motivation

Vertex version for minors

Edge version for minors

Vertex version for topological minors
Packing and covering $H$-models

Let $H$ be a fixed graph. For a graph $G$, we define:

\[
\text{pack}_H(G) := \text{packing number} = \text{max. number of vertex-disjoint } H\text{-models in } G
\]

\[
\text{cover}_H(G) := \text{covering number} = \text{min. number of vertices hitting all } H\text{-models in } G.
\]

Clearly, $\text{cover}_H(G) \geq \text{pack}_H(G) \forall G$.
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Clearly, $\text{cover}_H(G) \geq \text{pack}_H(G)$ $\forall G$.

For which $H$ $\text{cover}_H(G) \leq f(\text{pack}_H(G))$ $\forall G$, for some function $f$?
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For which $H$ $\text{cover}_H(G) \leq f(\text{pack}_H(G)) \ \forall G$, for some function $f$?

This is called the (vertex) Erdős-Pósa property for $H$-minors.
There exists a complete characterization:

\[ \text{cover}_H(G) \leq f(\text{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is planar} \]

[Robertson, Seymour ’86]
The property does NOT hold if $H$ is not planar

$H = K_5 \times$ 

Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $G$:
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Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $G$:

$
\begin{array}{cccccc}
\text{a} & 1 & 2 & 3 & 4 & 5 \\
\text{b} & & & & & \\
\text{c} & & & & & \\
\text{d} & & & & & \\
\text{e} & & & & & \\
\text{f} & & & & & \\
\end{array}
$

$\text{pack}_H(G) = 1$ \quad \text{but} \quad \text{cover}_H(G) = \Theta(\sqrt{n})$
The property does NOT hold if $H$ is not planar

\[ H = K_5 \quad \checkmark \]

\[ H \text{ not planar} \quad \checkmark \]

Therefore, the result of Robertson and Seymour is best possible.
Erdős-Pósa property of $H$-minors

Complete characterization:

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[Robertson, Seymour '86]

Is it the end of the story? NO!

Known upper bounds for $\text{cover}_H(G) \leq f(\text{pack}_H(G))$ were huge:

\[
f(\text{pack}_H(G)) = O(2^{|H|^2})
\]

This is because Robertson and Seymour's proof uses the excluded grid theorem from Graph Minors.

Natural question: which is the best possible function $f_H(\text{pack}_H(G))$?
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- **Natural question**: which is the \textbf{best possible} function $f_H(\text{pack}_H)$?
Let’s see that if $H$ has a cycle, then $f_H(k) = \Omega(k \log k)$:
Lower bound when $H$ has a cycle

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- Let $G$ be an $n$-vertex (cubic) graph with $\text{tw}(G) = \Omega(n)$ and $\text{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)
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- Any $H$-minor-free graph $F$ satisfies $\text{tw}(F) \leq d$ for some constant $d$, as $H$ is planar. [Robertson, Seymour ’86]

- This implies that (easy to check) $\exists$ constant $b > 0$ such that $f_H(k) > b \cdot k \log k$ (i.e., $f_H(k) = \Omega(k \log k)$).
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- On the other hand, every subgraph $F$ of $G$ containing an $H$-model has a cycle, so $|V(F)| = O(\log n)$, and therefore $\text{pack}_H(G) = O(n/\log n)$. 

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- This implies that (easy to check) $\exists$ constant $b > 0$ such that $f_H(k) > b \cdot k \log k$ (i.e., $f_H(k) = \Omega(k \log k)$).
There exists a function $f_H(k) \iff H$ is planar. The known upper bound was huge: $f_H(k) = O(2^{k^2})$. If $H$ has a cycle, we have a lower bound: $f_H(k) = \Omega(k \log k)$. \cite{Robertson, Seymour '86}

Recent breakthrough: For all graphs $H$, $f_H(k) = O(k \text{polylog } k)$. \cite{Chekuri, Chuzhoy '13}

Question: For $H$ with a cycle, when the optimal $f_H(k) = O(k \log k)$ can be attained?
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$f_H(k) = O(k \log k)$. (optimal)

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$f_H(k) = O(k)$ when $H = \text{forest}$ (optimal). [Fiorini, Joret, Wood '12]
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Brief state of the art of Erdős-Pósa property for minors

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Pumpkins

Can be seen as a natural generalization of a cycle. The $c$-pumpkin is sometimes denoted as $\theta$ in the literature. (N.B: "graph" = multigraph)
$c$-pumpkin:

- Can be seen as a natural generalization of a cycle.
- The $c$-pumpkin is sometimes denoted as $\theta_c$ in the literature.

(N.B: “graph” = multigraph)
Graphs with no $c$-pumpkin minor

- $c=1$: empty graphs
- $c=2$: forests
- $c=3$: no two cycles share an edge

...
Graphs with no $c$-pumpkin minor

- $c = 1$: empty graphs

![Diagram of empty graphs]
Graphs with no $c$-pumpkin minor

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Graphs with no $c$-pumpkin minor

- $c = 1$: empty graphs
- $c = 2$: forests
- $c = 3$: no two cycles share an edge
- etc.
c-pumpkin cover:
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor

\[ c = 3 \]
Covering pumpkins

c-pumpkin cover:
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor

$\theta_C(G)$: min. size of a c-pumpkin cover

★ For $c = 1$: Minimum Vertex Cover
★ For $c = 2$: Minimum Feedback Vertex Set
c-pumpkin packing:
collection of vertex-disjoint subgraphs of $G$, each containing a c-pumpkin minor

$c = 2$

Maximum Matching

Maximum Cycle Packing
**Packing pumpkins**

**c-pumpkin packing:**

collection of vertex-disjoint subgraphs of $G$, each containing a c-pumpkin minor

\[ \text{pack}_{\theta_c}(G): \text{max. cardinality of a c-pumpkin packing} \]

★ For \( c = 1 \): **Maximum Matching**

★ For \( c = 2 \): **Maximum Cycle Packing**
Before the upper bound of $f_H(k) = O(k \text{ polylog} k)$ appeared:

**Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh ’12)**

> For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin cover of size at most $f_{\theta_c}(k) = O(k^2)$.

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Results on Erdős-Pósa property for pumpkins

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- We solve it optimally:

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Their proof uses tree decompositions and brambles.

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For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin cover of size at most $f_{\theta_c}(k) = O(k \log k)$.

Our proof follows and generalizes Erdős-Pósa’s proof for the case $c = 2$. 
Ingredients of the proof for $c$-pumpkins

1. Find relevant reduction rules that preserve the covering and packing numbers of a graph.
   
   For $c = 2$ remove degree-1 vertices and dissolve degree-2 vertices.
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   For $c = 2$, remove degree-1 vertices and dissolve degree-2 vertices.

2. Prove that every $n$-vertex reduced graph contains a $c$-pumpkin model of size $O(\log n)$.
   
   For $c = 2$, if $\delta(G) \geq 3$, then $\text{girth}(G) < 2\log n$. 
Ingredients of the proof for $c$-pumpkins

1. Find relevant reduction rules that preserve the covering and packing numbers of a graph.

For $c = 2$ remove degree-1 vertices and dissolve degree-2 vertices.

2. Prove that every $n$-vertex reduced graph contains a $c$-pumpkin model of size $O(\log n)$.

For $c = 2$ If $\delta(G) \geq 3$, then $\text{girth}(G) < 2 \log n$.

3. Define an appropriate subgraph $H$ of the graph $G$ such that if $|V(H)| \geq d \cdot k \log k$ for some constant $d$ (depending only on $c$), then $H$ contains $k$ vertex-disjoint $c$-pumpkin-models.

For $c = 2$ $H$ = maximal subgraph of $G$ s.t. every vertex has degree 2 or 3.
Ingredients of the proof for $c$-pumpkins (2)

4. Piece everything together:
   - Given $G$, 

\[ G \]
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   - Given $G$, we consider the subgraph $H$ defined in step 3:
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   We can prove that there exists a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, intersecting every $c$-pumpkin-model in $G$. 

   This follows from steps 2+3 applied to the graph $H$. 

   As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless $H$ contains $k$ disjoint $c$-pumpkin-models.
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  ![Diagram](image)

  

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What about if we restrict the class of graphs?

\[ \text{cover}_H(G) \leq f_H(\text{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is planar} \]  

[Robertson, Seymour '86]

For general \( G \), if \( H \) may contain a cycle:

\[ f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k \text{ polylog}k) \]
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★ If \( H \) is planar and \( G \) belongs to a minor-closed graph class, then

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f_H(k) = O(k) \quad \text{(optimal).} \quad \tag{Fomin, Saurabh, Thilikos '10}
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**Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)**

*There exists a function \( g : \mathbb{N} \to \mathbb{N} \) such that for every two positive integers \( c, q \), in every graph \( G \) excluding \( K_q \) as a minor it holds that*

\[ f_{\theta_c}(k) \leq g(c) \cdot k \cdot \log q. \]
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[Fomin, Saurabh, Thilikos '10]

Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)

There exists a function \( g : \mathbb{N} \rightarrow \mathbb{N} \) such that for every two positive integers \( c, q \), in every graph \( G \) excluding \( K_q \) as a minor it holds that

\[ f_{\theta_c}(k) \leq g(c) \cdot k \cdot \log q. \]

▶ For \( q \) fixed, this yields the linear bound for the case of \( H = \theta_c \).
What about if we restrict the class of graphs?

\[
\text{cover}_H(G) \leq f_H(\text{pack}_H(G)) \quad \forall G \iff H \text{ is planar}
\]

[Robertson, Seymour '86]

For general \( G \), if \( H \) may contain a cycle:

\[
f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k \text{ polylog} k)
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**Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)**

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f_{\theta_c}(k) \leq g(c) \cdot k \cdot \log q.
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▶ For \( q \) fixed, this yields the linear bound for the case of \( H = \theta_c \).

▶ For \( q = k \cdot (c + 1) \), this yields the bound of

[Fiorini, Joret, S. '13]
Main open problem

\[
\text{cover}_H(G) \leq f_H(\text{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is planar}
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Main open problem

\begin{align*}
\text{cover}_H(G) & \leq f_H(\text{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is planar} \\
\text{For general } G, \text{ if } H \text{ may contain a cycle:} \\
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\end{align*}

Conjecture

\begin{itemize}
\item For all non-acyclic planar \( H \), we have \( f_H(k) = O(k \log k) \). \quad (\text{optimal})
\end{itemize}
1 Motivation

2 Vertex version for minors

3 Edge version for minors

4 Vertex version for topological minors
Packing and covering $H$-models – vertex version

Let $H$ be a fixed graph. For a graph $G$, we define:

$\text{pack}_H(G) := \text{packing number}$
$= \text{max. number of vertex-disjoint } H\text{-models in } G$

$\text{cover}_H(G) := \text{covering number}$
$= \text{min. number of vertices hitting all } H\text{-models in } G$.

Clearly, $\text{cover}_H(G) \geq \text{pack}_H(G) \forall G$.

For which $H$ $\text{cover}_H(G) \leq f(\text{pack}_H(G)) \forall G$, for some function $f$?

This is called the (vertex) Erdős-Pósa property for $H$-minors.
Packing and covering $H$-models – edge version

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This is called the (edge) Erdős-Pósa property for $H$-minors.
What is known for the edge version?

For the vertex version:

\[ \exists f_{H}(k) \Leftrightarrow H \text{ is planar} \]
What is known for the edge version?

For the **vertex** version:

\[
\text{there exists } f_H(k) \iff H \text{ is planar}
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For the **edge** version:

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\text{there exists } f_H(k) \Rightarrow H \text{ is planar}
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Particular cases of the planar graph $H$

$H = \text{cycle}$: Erdős and Pósa’s original proof can be adapted to the edge version:

$$f_{\theta_2}^e(k) = O(k \log k)$$

[Graph Theory, Chapter 7. Diestel '05]
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Again, we focus on $c$-pumpkins:

**Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)**

There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that for every two positive integers $c, q$, in every graph $G$ excluding $K_q$ as a minor it holds that

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**Theorem (Chatzidimitriou, Raymond, S., Thilikos ‘14)**

$\text{pack}_{\theta_c}$, $\text{pack}^e_{\theta_c}$, $\text{cover}_{\theta_c}$, and $\text{cover}^e_{\theta_c}$ have a (deterministic and poly-time) $f(c) \cdot \log(\text{OPT})$-approximation algorithm.

Improves a $O(\log n)$-approx. for the vertex version.  

[Joret, Paul, S., Saurabh, Thomassé '11]
1 Motivation

2 Vertex version for minors

3 Edge version for minors

4 Vertex version for topological minors
Minors and topological minors

- \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from a subgraph of \( G \) by contracting edges.

\[ \text{G} \quad \rightarrow \quad \text{H} \]

Fixed \( H \):
\( \text{H}-\text{minor-free graphs} \subseteq \text{H}-\text{topological-minor-free graphs} \).
Minors and topological minors

- $H$ is a **minor** of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.

- $H$ is a **topological minor** of $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges with at least one endpoint of degree $\leq 2$. 

Therefore:

$H$ minor of $G$ $\Rightarrow$ $H$ topological minor of $G$. 

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Topological models in graphs

$H$ is a topological minor of $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges with at least one endpoint of degree $\leq 2$.

**$H$-topological model** in $G$: collection $\{v_u : u \in V(H)\} \subseteq V(G)$ s.t.

- $\forall uw \in E(H)$, there exists in $G$ a path between $v_u$ and $v_w$, and
- all these paths are pairwise vertex-disjoint.

A $K_5$-topological model = a subdivision of $K_5$. 
Topological models in graphs

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A $K_5$-topological model = a subdivision of $K_5$. 

![Diagram of a K5-topological model](image-url)
What is known for topological minors?

Erdős-Pósa property for the \textbf{vertex} version for \textbf{minors}:

\[
\text{there exists } f_H(k) \iff H \text{ is planar}
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Erdős-Pósa property for the \textbf{vertex} version for \textbf{topological minors}?
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Erdős-Pósa property for the vertex version for topological minors?

\[ \text{there exists } f_H(k) \Rightarrow H \text{ is planar} \]  

[Robertson, Seymour '84]

Is planarity sufficient?

No! It does not hold even if \( H = \text{tree}. \)  

[Thomassen '88]

That is, there are trees \( T \), such the collection of subdivisions of \( T \) does not satisfy the Erdős-Pósa property (even restricted to planar graphs).
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That is, there are trees \( T \), such the collection of subdivisions of \( T \) does not satisfy the Erdős-Pósa property (even restricted to planar graphs).
Planarity is not sufficient for topological minors

Let $T_1$, $T_2$, $T_3$ be non-isomorphic trees whose vertices have degree 4 or 1, and let $z_i$ be a vertex of degree 4 in $T_i$. 
Let $T$ be defined as above. We claim that the collection of subdivisions of $T$ does not satisfy the Erdős-Pósa property (even in planar graphs).
Planarity is not sufficient for topological minors

We start with the wall of size $k$, we subdivide the above edges, and we add attached copies of the trees $T_1, T_2, T_3$ defined before.
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This (planar) graph contains only one vertex-disjoint subdivision of $T$ (i.e., the packing number is one), but the covering number is arbitrarily large.
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Gràcies!