On the Erdős-Pósa property for minors of graphs

Ignasi Sau
CNRS, LIRMM, Montpellier

Joint work with:

Dimitris Chatzidimitriou
National and Kapodistrian University of Athens, Greece

Samuel Fiorini
Gwenaël Joret
Université Libre de Bruxèlles, Belgium

Jean-Florent Raymond
Dimitrios M. Thilikos
CNRS, LIRMM, Montpellier

Outline of the talk

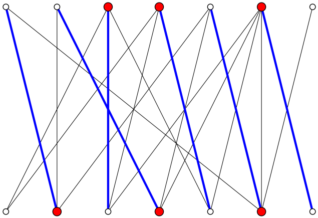
1. Motivation
2. Vertex version for minors
3. Edge version for minors
4. Vertex version for topological minors

Next section is...

Packing and covering

König's min-max theorem in bipartite graphs:

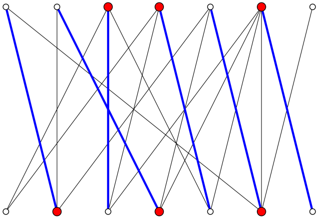
MIN VERTEX COVER = MAX MATCHING



Packing and covering

König's min-max theorem in bipartite graphs:

$$\textsc{Min Vertex Cover} = \textsc{Max Matching}$$

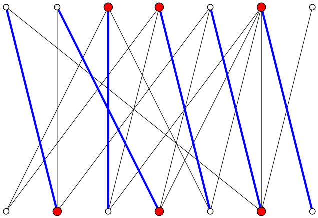


min # vertices covering all edges $\geqslant$ max # of disjoint edges

Packing and covering

König's min-max theorem in bipartite graphs:

$$\text{Min Vertex Cover} = \text{Max Matching}$$



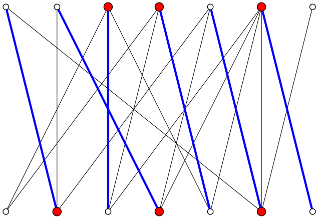
$$\text{min \# vertices covering all edges} \geqslant \text{max \# of disjoint edges}$$

$$\text{min \# vertices covering all edges} \leqslant \text{max \# of disjoint edges}$$

Packing and covering

König's min-max theorem in bipartite graphs:

$$\textsc{Min Vertex Cover} = \textsc{Max Matching}$$

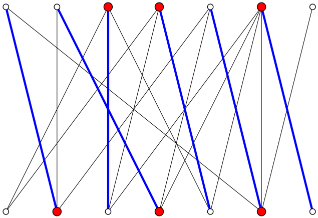


min # vertices covering all $\boxed{\text{edges}}$ $\geqslant$ max # of disjoint $\boxed{\text{edges}}$

Packing and covering

König's min-max theorem in bipartite graphs:

$$\text{Min Vertex Cover} = \text{Max Matching}$$

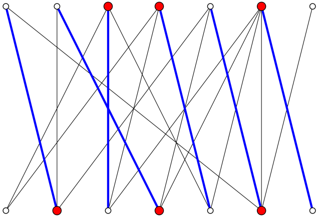


min # vertices covering all $\boxed{H \in \mathcal{H}}$ $\geqslant$ max # of disjoint $\boxed{H \in \mathcal{H}}$

Packing and covering

König's min-max theorem in bipartite graphs:

$$\textsc{Min Vertex Cover} = \textsc{Max Matching}$$



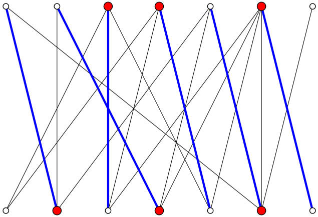
min # vertices covering all $\boxed{H \in \mathcal{H}}$ $\geqslant$ max # of disjoint $\boxed{H \in \mathcal{H}}$

min # vertices covering all $\boxed{H \in \mathcal{H}}$ $\leqslant$ max # of disjoint $\boxed{H \in \mathcal{H}}$?

Packing and covering

König's min-max theorem in bipartite graphs:

$$\textcolor{red}{\textsc{Min Vertex Cover}} = \textcolor{blue}{\textsc{Max Matching}}$$



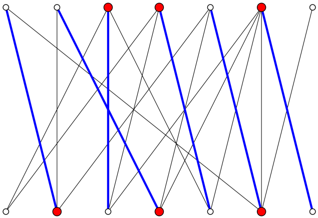
min # vertices covering all $\boxed{H \in \mathcal{H}}$ $\geqslant$ max # of disjoint $\boxed{H \in \mathcal{H}}$

min # vertices covering all $\boxed{H \in \mathcal{H}}$ $\leqslant$ f(max # of disjoint $\boxed{H \in \mathcal{H}}$) ?

Packing and covering

König's min-max theorem in bipartite graphs:

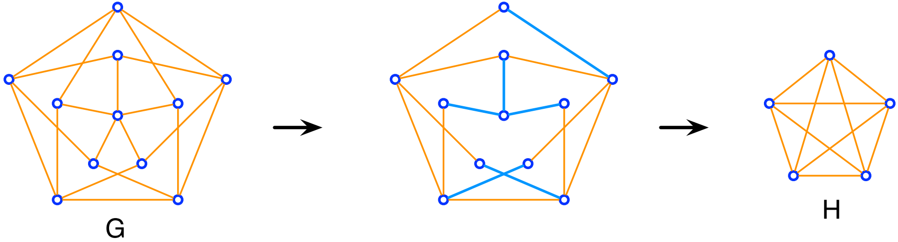
$$\textcolor{red}{\textsc{Min Vertex Cover}} = \textcolor{blue}{\textsc{Max Matching}}$$



If there exists such f for all G, then $\mathcal{H}$ satisfies the **Erdős-Pósa property**.

min # vertices covering all $\boxed{H \in \mathcal{H}}$ $\leqslant$ f(max # of disjoint $\boxed{H \in \mathcal{H}}$) ?

Minors and models in graphs



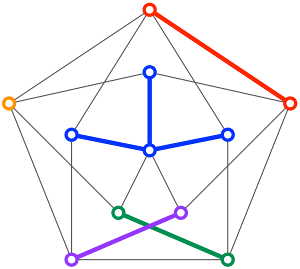
H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

Minors and models in graphs

H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

H-model in G: collection $\{S_u : u \in V(H)\}$ s.t.

- the S_u's are vertex-disjoint connected subgraphs of G, and
- there is an edge between S_u and S_v in G for every edge $uv \in E(H)$.



A K_5-model

The S_u's are called vertex images.

Next section is...

Packing and covering H-models

Let H be a **fixed** graph. For a graph G, we define:

$\mathtt{pack}_H(G) :=$ packing number
$\quad = $ max. number of **vertex**-disjoint H-models in G

$\mathtt{cover}_H(G) :=$ covering number
$\quad = $ min. number of **vertices** hitting all H-models in G.

Clearly, $\mathtt{cover}_H(G) \geqslant \mathtt{pack}_H(G) \quad \forall G.$

Packing and covering H-models

Let H be a **fixed** graph. For a graph G, we define:

$\mathtt{pack}_H(G) :=$ packing number
$\quad = $ max. number of **vertex**-disjoint H-models in G

$\mathtt{cover}_H(G) :=$ covering number
$\quad = $ min. number of **vertices** hitting all H-models in G.

Clearly, $\quad \mathtt{cover}_H(G) \geqslant \mathtt{pack}_H(G) \quad \forall G.$

For which H $\quad \boxed{\mathtt{cover}_H(G) \leqslant f(\mathtt{pack}_H(G))} \ \forall G,$ for some function f ?

Packing and covering H-models

Let H be a **fixed** graph. For a graph G, we define:

$\texttt{pack}_H(G) :=$ packing number
$\quad = \text{max. number of } \textbf{vertex}\text{-disjoint } H\text{-models in } G$

$\texttt{cover}_H(G) :=$ covering number
$\quad = \text{min. number of } \textbf{vertices} \text{ hitting all } H\text{-models in } G.$

Clearly, $\quad \texttt{cover}_H(G) \geqslant \texttt{pack}_H(G) \quad \forall G.$

For which H $\quad \boxed{\texttt{cover}_H(G) \leqslant f(\texttt{pack}_H(G))} \ \forall G, \quad$ for some function f ?

This is called the **(vertex) Erdős-Pósa property for H-minors**.

Erdős-Pósa property of H-minors

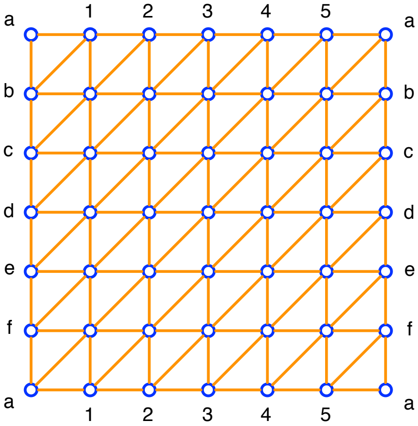
There exists a complete characterization:

$$\boxed{\texttt{cover}_H(G) \leqslant f(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is } \textbf{planar}}$$ [Robertson, Seymour '86]

The property does NOT hold if H is not planar

$\boxed{H = K_5}$ ✗

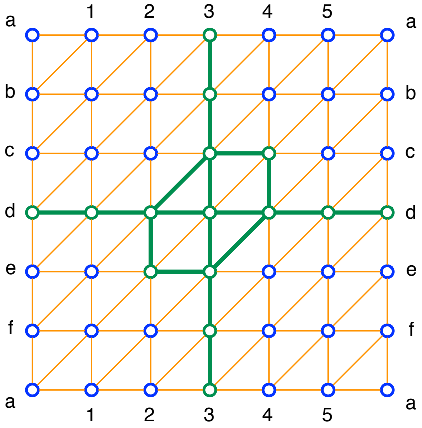
Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid G:



The property does NOT hold if H is not planar

$\boxed{H = K_5}$ ✗

Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid G:

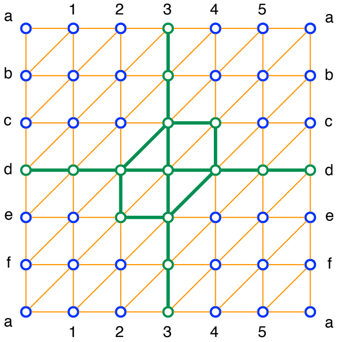


$\texttt{pack}_H(G) = 1$ but $\texttt{cover}_H(G) = \Theta(\sqrt{n})$

The property does NOT hold if H is not planar

$H = K_5$ ✗

H not planar ✗



Therefore, the result of Robertson and Seymour is best possible.

Complete characterization:

$$\boxed{\mathtt{cover}_H(G) \leqslant f(\mathtt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is } \textbf{planar}}$$

[Robertson, Seymour '86]

Erdős-Pósa property of H-minors

Complete characterization:

$$\boxed{\texttt{cover}_H(G) \leqslant f(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is } \textbf{planar}}$$ [Robertson, Seymour '86]

Is it the end of the story?

Erdős-Pósa property of H-minors

Complete characterization:

$$\boxed{\texttt{cover}_H(G) \leqslant f(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is } \textbf{planar}}$$

[Robertson, Seymour '86]

Is it the end of the story? NO!

- Known upper bounds $\texttt{cover}_H \leqslant f(\texttt{pack}_H)$ were **huge**:

$$f(\texttt{pack}_H) = O(2^{\texttt{pack}_H{}^2})$$

This is because Robertson and Seymour's proof uses the excluded grid theorem from Graph Minors.

Erdős-Pósa property of H-minors

Complete characterization:

$$\boxed{\texttt{cover}_H(G) \leqslant f(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is } \textbf{planar}}$$ [Robertson, Seymour '86]

Is it the end of the story? NO!

- Known upper bounds $\texttt{cover}_H \leqslant f(\texttt{pack}_H)$ were **huge**:

$$f(\texttt{pack}_H) = O(2^{\texttt{pack}_H{}^2})$$

  This is because Robertson and Seymour's proof uses the excluded grid theorem from Graph Minors.

- Natural question: which is the best possible function $f_H(\texttt{pack}_H)$?

Lower bound when H has a cycle

Let's see that if H has a cycle, then $f_H(k) = \Omega(k \log k)$:

Lower bound when H has a cycle

Let's see that if H has a cycle, then $f_H(k) = \Omega(k \log k)$:

- Let G be an n-vertex (cubic) graph with $\mathrm{tw}(G) = \Omega(n)$ and $\mathrm{girth}(G) = \Omega(\log n)$.

(such graphs are well-known to exist)

Lower bound when H has a cycle

Let's see that if H has a cycle, then $f_H(k) = \Omega(k \log k)$:

- Let G be an n-vertex (cubic) graph with $\mathrm{tw}(G) = \Omega(n)$ and $\mathrm{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)
- Any H-minor-free graph F satisfies $\mathrm{tw}(F) \leqslant d$ for some constant d, as H is planar. [Robertson, Seymour '86]

Lower bound when H has a cycle

Let's see that if H has a cycle, then $f_H(k) = \Omega(k \log k)$:

- Let G be an n-vertex (cubic) graph with $\mathrm{tw}(G) = \Omega(n)$ and $\mathrm{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)
- Any H-minor-free graph F satisfies $\mathrm{tw}(F) \leqslant d$ for some constant d, as H is planar. [Robertson, Seymour '86]
- Thus $\mathrm{tw}(G - X) \leqslant d$ for any H-hitting set X, and therefore $\mathtt{cover}_H(G) = \Omega(n)$.

Lower bound when H has a cycle

Let's see that if H has a cycle, then $f_H(k) = \Omega(k \log k)$:

- Let G be an n-vertex (cubic) graph with $\mathrm{tw}(G) = \Omega(n)$ and $\mathrm{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)
- Any H-minor-free graph F satisfies $\mathrm{tw}(F) \leqslant d$ for some constant d, as H is planar. [Robertson, Seymour '86]
- Thus $\mathrm{tw}(G - X) \leqslant d$ for any H-hitting set X, and therefore $\mathtt{cover}_H(G) = \Omega(n)$.
- On the other hand, every subgraph F of G containing an H-model has a cycle, so $|V(F)| = O(\log n)$, and therefore $\mathtt{pack}_H(G) = O(n/\log n)$.

Lower bound when H has a cycle

Let's see that if H has a cycle, then $f_H(k) = \Omega(k \log k)$:

- Let G be an n-vertex (cubic) graph with $\mathrm{tw}(G) = \Omega(n)$ and $\mathrm{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)
- Any H-minor-free graph F satisfies $\mathrm{tw}(F) \leqslant d$ for some constant d, as H is planar. [Robertson, Seymour '86]
- Thus $\mathrm{tw}(G - X) \leqslant d$ for any H-hitting set X, and therefore $\mathtt{cover}_H(G) = \Omega(n)$.
- On the other hand, every subgraph F of G containing an H-model has a cycle, so $|V(F)| = O(\log n)$, and therefore $\mathtt{pack}_H(G) = O(n/\log n)$.
- This implies that (easy to check) $\exists$ constant $b > 0$ such that $f_H(k) > b \cdot k \log k$ (i.e., $f_H(k) = \Omega(k \log k)$).

Brief state of the art of Erdős-Pósa property for minors

- There exists a function $f_H(k) \Leftrightarrow H$ is planar [Robertson, Seymour '86]

  The known upper bound was huge: $f_H(k) = O(2^{k^2})$.
  If H has a cycle, we have a lower bound: $f_H(k) = \Omega(k \log k)$.

Brief state of the art of Erdős-Pósa property for minors

- There exists a function $f_H(k) \Leftrightarrow H$ is planar [Robertson, Seymour '86]

  The known upper bound was huge: $f_H(k) = O(2^{k^2})$.
  If H has a cycle, we have a lower bound: $f_H(k) = \Omega(k \log k)$.

- Erdős and Pósa original result for $H =$ cycle:
  $f_H(k) = O(k \log k)$. (optimal) [Erdős, Pósa '65]

Brief state of the art of Erdős-Pósa property for minors

- There exists a function $f_H(k)$ $\Leftrightarrow$ H is planar [Robertson, Seymour '86]

  The known upper bound was huge: $f_H(k) = O(2^{k^2})$.
  If H has a cycle, we have a lower bound: $f_H(k) = \Omega(k \log k)$.

- Erdős and Pósa original result for H = cycle:
  $f_H(k) = O(k \log k)$. (optimal) [Erdős, Pósa '65]

- $f_H(k) = O(k)$ when H = forest (optimal). [Fiorini, Joret, Wood '12]

Brief state of the art of Erdős-Pósa property for minors

- There exists a function $f_H(k)$ $\Leftrightarrow$ H is planar [Robertson, Seymour '86]

  The known upper bound was huge: $f_H(k) = O(2^{k^2})$.
  If H has a cycle, we have a lower bound: $f_H(k) = \Omega(k \log k)$.

- Erdős and Pósa original result for H = cycle:
  $f_H(k) = O(k \log k)$. (optimal) [Erdős, Pósa '65]

- $f_H(k) = O(k)$ when H = forest (optimal). [Fiorini, Joret, Wood '12]

★ Recent breakthrough:
For all graphs H, $\boxed{f_H(k) = O(k\,\text{polylog}\,k)}$. [Chekuri, Chuzhoy '13]

Brief state of the art of Erdős-Pósa property for minors

- There exists a function $f_H(k) \Leftrightarrow H$ is planar [Robertson, Seymour '86]

  The known upper bound was huge: $f_H(k) = O(2^{k^2})$.
  If H has a cycle, we have a lower bound: $\boxed{f_H(k) = \Omega(k \log k)}$.

- Erdős and Pósa original result for $H =$ cycle:
  $f_H(k) = O(k \log k)$. (optimal) [Erdős, Pósa '65]

- $f_H(k) = O(k)$ when $H =$ forest (optimal). [Fiorini, Joret, Wood '12]

★ Recent breakthrough:
For all graphs H, $\boxed{f_H(k) = O(k \operatorname{polylog} k)}$. [Chekuri, Chuzhoy '13]

Brief state of the art of Erdős-Pósa property for minors

- There exists a function $f_H(k)$ $\Leftrightarrow$ H is planar [Robertson, Seymour '86]

  The known upper bound was huge: $f_H(k) = O(2^{k^2})$.
  If H has a cycle, we have a lower bound: $\boxed{f_H(k) = \Omega(k \log k)}$.

- Erdős and Pósa original result for H = cycle:
  $f_H(k) = O(k \log k)$. (optimal) [Erdős, Pósa '65]

- $f_H(k) = O(k)$ when H = forest (optimal). [Fiorini, Joret, Wood '12]

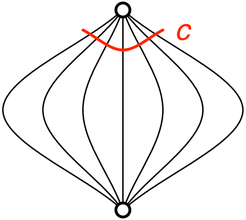
★ Recent breakthrough:
For all graphs H, $\boxed{f_H(k) = O(k \operatorname{polylog} k)}$. [Chekuri, Chuzhoy '13]

Question For H with a cycle, when the optimal $f_H(k) = O(k \log k)$ can be attained?

Pumpkins 

Pumpkins 

c-pumpkin:



⋆ Can be seen as a natural generalization of a cycle.

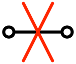
⋆ The c-pumpkin is sometimes denoted as θ_c in the literature.

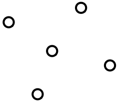
(N.B: "graph" $=$ multigraph)

Graphs with no c-pumpkin minor

Graphs with no c-pumpkin minor

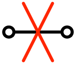
- $c = 1$: empty graphs

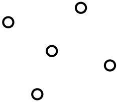




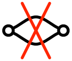
Graphs with no c-pumpkin minor

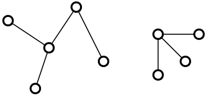
- $c = 1$: empty graphs





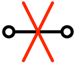
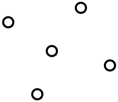
- $c = 2$: forests



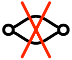
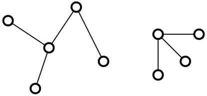


Graphs with no c-pumpkin minor

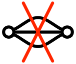
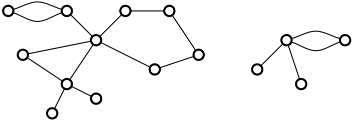
- $c = 1$: empty graphs

- $c = 2$: forests

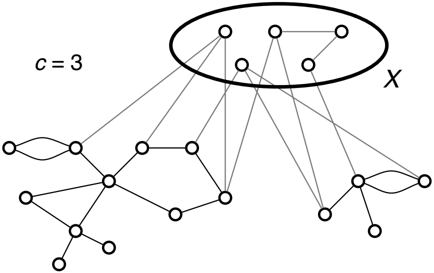
- $c = 3$: no two cycles share an edge

- etc.

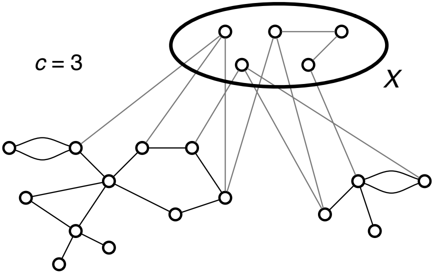
Covering pumpkins

c-pumpkin cover:
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor



Covering pumpkins

c-pumpkin cover:
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor

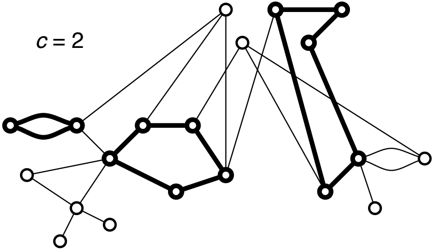


$\texttt{cover}_{\theta_c}(G)$: min. size of a c-pumpkin cover

- ⋆ For $c = 1$: Minimum Vertex Cover
- ⋆ For $c = 2$: Minimum Feedback Vertex Set

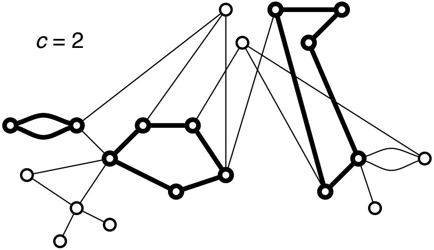
Packing pumpkins

c-pumpkin packing:
collection of vertex-disjoint subgraphs of G, each containing a c-pumpkin minor



Packing pumpkins

c-pumpkin packing:
collection of vertex-disjoint subgraphs of G, each containing a c-pumpkin minor



$\texttt{pack}_{\theta_c}(G)$: max. cardinality of a c-pumpkin packing

- ⋆ For $c = 1$: Maximum Matching
- ⋆ For $c = 2$: Maximum Cycle Packing

Results on Erdős-Pósa property for pumpkins

- Before the upper bound of $f_H(k) = O(k\,\mathrm{polylog}\,k)$ appeared:

Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh '12)

For any fixed integer $c \geqslant 1$ and given an integer $k \geqslant 1$, every graph G either contains k vertex-disjoint c-pumpkins-models, or has a c-pumpkin cover of size at most $f_{\theta_c}(k) = O(k^2)$.

Results on Erdős-Pósa property for pumpkins

- Before the upper bound of $f_H(k) = O(k \operatorname{polylog} k)$ appeared:

Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh '12)

For any fixed integer $c \geqslant 1$ and given an integer $k \geqslant 1$, every graph G either contains k vertex-disjoint c-pumpkins-models, or has a c-pumpkin cover of size at most $f_{\theta_c}(k) = O(k^2)$.

- We solve it optimally:

Theorem (Fiorini, Joret, S. '13)

For any fixed integer $c \geqslant 1$ and given an integer $k \geqslant 1$, every graph G either contains k vertex-disjoint c-pumpkins-models, or has a c-pumpkin cover of size at most $f_{\theta_c}(k) = O(k \log k)$.

Results on Erdős-Pósa property for pumpkins

- Before the upper bound of $f_H(k) = O(k \operatorname{polylog} k)$ appeared:

Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh '12)

For any fixed integer $c \geqslant 1$ and given an integer $k \geqslant 1$, every graph G either contains k vertex-disjoint c-pumpkins-models, or has a c-pumpkin cover of size at most $f_{\theta_c}(k) = O(k^2)$.

⋆ Their proof uses tree decompositions and brambles.

- We solve it optimally:

Theorem (Fiorini, Joret, S. '13)

For any fixed integer $c \geqslant 1$ and given an integer $k \geqslant 1$, every graph G either contains k vertex-disjoint c-pumpkins-models, or has a c-pumpkin cover of size at most $f_{\theta_c}(k) = O(k \log k)$.

⋆ Our proof follows and generalizes Erdős-Pósa's proof for the case $c = 2$.

Ingredients of the proof for c-pumpkins

1. Find relevant reduction rules that preserve the covering and packing numbers of a graph.

   $\boxed{\text{For } c = 2}$ remove degree-1 vertices and dissolve degree-2 vertices.

Ingredients of the proof for c-pumpkins

1. Find relevant reduction rules that preserve the covering and packing numbers of a graph.

   $\boxed{\text{For } c = 2}$ remove degree-1 vertices and dissolve degree-2 vertices.

2. Prove that every n-vertex reduced graph contains a c-pumpkin model of size $O(\log n)$.

   $\boxed{\text{For } c = 2}$ If $\delta(G) \geqslant 3$, then $\mathrm{girth}(G) < 2 \log n$.

Ingredients of the proof for c-pumpkins

1. Find relevant reduction rules that preserve the covering and packing numbers of a graph.

   For $c = 2$ remove degree-1 vertices and dissolve degree-2 vertices.

2. Prove that every n-vertex reduced graph contains a c-pumpkin model of size $O(\log n)$.

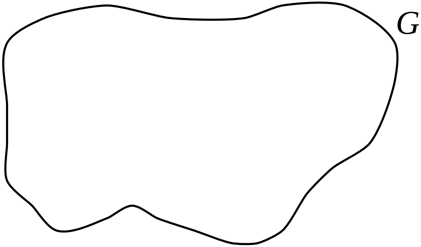
   For $c = 2$ If $\delta(G) \geqslant 3$, then $\mathrm{girth}(G) < 2 \log n$.

3. Define an appropriate subgraph H of the graph G such that if $|V(H)| \geqslant d \cdot k \log k$ for some constant d (depending only on c), then H contains k vertex-disjoint c-pumpkin-models.

   For $c = 2$ $H =$ maximal subgraph of G s.t. every vertex has degree 2 or 3.

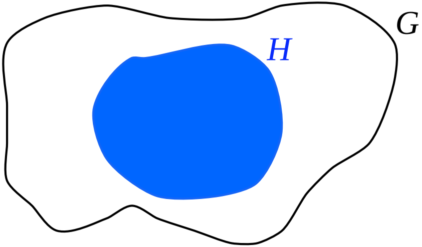
Ingredients of the proof for c-pumpkins (2)

4. Piece everything together:
    - Given G,



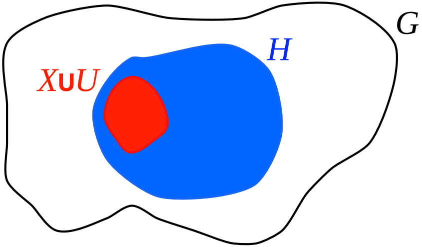
Ingredients of the proof for c-pumpkins (2)

4. Piece everything together:
    - Given G, we consider the subgraph H defined in step 3:



Ingredients of the proof for c-pumpkins (2)

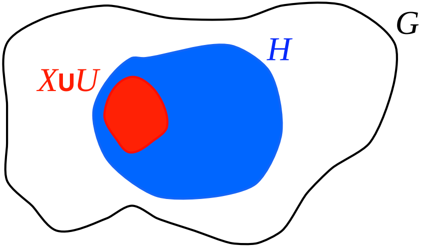
4. Piece everything together:
   - Given G, we consider the subgraph H defined in step 3:



   - We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, intersecting every c-pumpkin-model in G.

Ingredients of the proof for c-pumpkins (2)

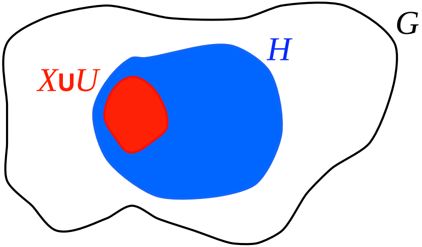
4. Piece everything together:
   - Given G, we consider the subgraph H defined in step 3:



   - We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, intersecting every c-pumpkin-model in G.
   - As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless H contains k disjoint c-pumpkin-models.

Ingredients of the proof for c-pumpkins (2)

4. Piece everything together:
   - Given G, we consider the subgraph H defined in step 3:



   - We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, intersecting every c-pumpkin-model in G.
   - As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless H contains k disjoint c-pumpkin-models.
   - This follows from steps 2+3 applied to the graph H.

What about if we restrict the class of graphs?

$\boxed{\texttt{cover}_H(G) \leqslant f_H(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow H \text{ is } \textbf{planar}}$ [Robertson, Seymour '86]

For general G, if H may contain a cycle:

$$f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k\,\mathrm{polylog}\,k)$$

What about if we restrict the class of graphs?

$\boxed{\texttt{cover}_H(G) \leqslant f_H(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow H \text{ is } \textbf{planar}}$ [Robertson, Seymour '86]

For general G, if H may contain a cycle:

$$f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k\, \text{polylog} k)$$

★ If H is planar and G belongs to a minor-closed graph class, then

$$f_H(k) = O(k) \qquad \text{(optimal).}$$ [Fomin, Saurabh, Thilikos '10]

What about if we restrict the class of graphs?

$\boxed{\mathtt{cover}_H(G) \leqslant f_H(\mathtt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow H \text{ is } \textbf{planar}}$ [Robertson, Seymour '86]

For general G, if H may contain a cycle:

$$f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k\, \mathrm{polylog} k)$$

★ If H is planar and G belongs to a minor-closed graph class, then

$$f_H(k) = O(k) \qquad \text{(optimal)}.$$

[Fomin, Saurabh, Thilikos '10]

Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)

There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that for every two positive integers c, q, in every graph G excluding K_q as a minor it holds that

$$f_{\theta_c}(k) \leqslant g(c) \cdot k \cdot \log q.$$

What about if we restrict the class of graphs?

$$\boxed{\texttt{cover}_H(G) \leqslant f_H(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow H \text{ is } \textbf{planar}}$$

[Robertson, Seymour '86]

For general G, if H may contain a cycle:

$$f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k \operatorname{polylog} k)$$

★ If H is planar and G belongs to a minor-closed graph class, then

$$f_H(k) = O(k) \qquad \text{(optimal)}.$$

[Fomin, Saurabh, Thilikos '10]

Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)

There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that for every two positive integers c, q, in every graph G excluding K_q as a minor it holds that

$$f_{\theta_c}(k) \leqslant g(c) \cdot k \cdot \log q.$$

- For q fixed, this yields the linear bound for the case of $H = \theta_c$.

What about if we restrict the class of graphs?

$\boxed{\mathtt{cover}_H(G) \leqslant f_H(\mathtt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow H \text{ is } \textbf{planar}}$ [Robertson, Seymour '86]

For general G, if H may contain a cycle:

$$f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k\,\mathrm{polylog}\,k)$$

★ If H is planar and G belongs to a minor-closed graph class, then

$$f_H(k) = O(k) \qquad \text{(optimal)}.$$ [Fomin, Saurabh, Thilikos '10]

Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)

There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that for every two positive integers c, q, in every graph G excluding K_q as a minor it holds that

$$f_{\theta_c}(k) \leqslant g(c) \cdot k \cdot \log q.$$

- For q fixed, this yields the linear bound for the case of $H = \theta_c$.
- For $q = k \cdot (c+1)$, this yields the bound of [Fiorini, Joret, S. '13]

Main open problem

$$\boxed{\texttt{cover}_H(G) \leqslant f_H(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow H \text{ is } \textbf{planar}}$$ [Robertson, Seymour '86]

For general G, if H may contain a cycle:

$$f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k \operatorname{polylog} k)$$

Main open problem

$$\boxed{\texttt{cover}_H(G) \leqslant f_H(\texttt{pack}_H(G)) \quad \forall G \quad \Leftrightarrow H \text{ is } \textbf{planar}}$$

[Robertson, Seymour '86]

For general G, if H may contain a cycle:

$$f_H(k) = \Omega(k \log k) \quad \text{and} \quad f_H(k) = O(k \,\text{polylog}\, k)$$

Conjecture

For all non-acyclic planar H, we have $f_H(k) = O(k \log k)$. *(optimal)*

Next section is...

Packing and covering H-models – vertex version

Let H be a **fixed** graph. For a graph G, we define:

$\mathtt{pack}_H(G)$:= packing number
$\quad = $ max. number of **vertex**-disjoint H-models in G

$\mathtt{cover}_H(G)$:= covering number
$\quad = $ min. number of **vertices** hitting all H-models in G.

Clearly, $\mathtt{cover}_H(G) \geqslant \mathtt{pack}_H(G) \forall G$.

For which H $\boxed{\mathtt{cover}_H(G) \leqslant f(\mathtt{pack}_H(G))}$ $\forall G$, for some function f ?

This is called the **(vertex) Erdős-Pósa property for H-minors**.

Packing and covering H-models – edge version

Let H be a **fixed** graph. For a graph G, we define:

$\texttt{pack}_H(G) :=$ packing number
$= \text{max. number of } \textbf{edge}\text{-disjoint } H\text{-models in } G$

$\texttt{cover}_H(G) :=$ covering number
$= \text{min. number of } \textbf{edges} \text{ hitting all } H\text{-models in } G.$

Clearly, $\texttt{cover}_H(G) \geqslant \texttt{pack}_H(G) \quad \forall G.$

For which H $\boxed{\texttt{cover}_H(G) \leqslant f(\texttt{pack}_H(G))}$ $\forall G$, for some function f ?

This is called the **(edge) Erdős-Pósa property for H-minors**.

What is known for the edge version?

For the **vertex** version:

$$\boxed{\text{there exists } f_H(k) \Leftrightarrow H \text{ is planar}}$$

What is known for the edge version?

For the **vertex** version:

$$\boxed{\text{there exists } f_H(k) \Leftrightarrow H \text{ is planar}}$$

For the **edge** version:

$$\boxed{\text{there exists } f_H(k) \Rightarrow H \text{ is planar}}$$

What is known for the edge version?

For the **vertex** version:

$$\boxed{\text{there exists } f_H(k) \Leftrightarrow H \text{ is planar}}$$

For the **edge** version:

$$\boxed{\text{there exists } f_H(k) \Rightarrow H \text{ is planar}}$$

$$\boxed{\text{there exists } f_H(k) \Leftarrow H \text{ is planar}} \text{ ???}$$

Particular cases of the planar graph H

H = cycle: Erdős and Pósa's original proof can be adapted to the edge version:

$$f^{\mathbf{e}}_{\theta_2}(k) = O(k \log k)$$

[Graph Theory, Chapter 7. Diestel '05]

Particular cases of the planar graph H

H = cycle: Erdős and Pósa's original proof can be adapted to the edge version:

$$f^{\mathbf{e}}_{\theta_2}(k) = O(k \log k)$$

[Graph Theory, Chapter 7. Diestel '05]

Again, we focus on c-pumpkins:

Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)

There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that for every two positive integers c, q, in every graph G excluding K_q as a minor it holds that

$$f^{\mathbf{e}}_{\theta_c}(k) \leqslant g(c) \cdot k \cdot \log q.$$

Particular cases of the planar graph H

$H =$ cycle: Erdős and Pósa's original proof can be adapted to the edge version:

$$f^{\mathbf{e}}_{\theta_2}(k) = O(k \log k)$$ [Graph Theory, Chapter 7. Diestel '05]

Again, we focus on c-pumpkins:

Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)

There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that for every two positive integers c, q, in every graph G excluding K_q as a minor it holds that

$$f^{\mathbf{e}}_{\theta_c}(k) \leqslant g(c) \cdot k \cdot \log q.$$

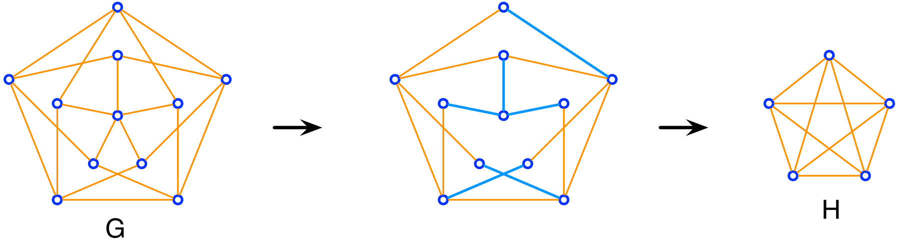
Theorem (Chatzidimitriou, Raymond, S., Thilikos '14)

$\mathbf{pack}_{\theta_c}$, $\mathbf{pack}^{\mathbf{e}}_{\theta_c}$, $\mathbf{cover}_{\theta_c}$, *and* $\mathbf{cover}^{\mathbf{e}}_{\theta_c}$ *have a (deterministic and poly-time)* $f(c) \cdot \log(\mathrm{OPT})$*-approximation algorithm.*

Improves a $O(\log n)$-approx. for the vertex version. [Joret, Paul, S., Saurabh, Thomassé '11]

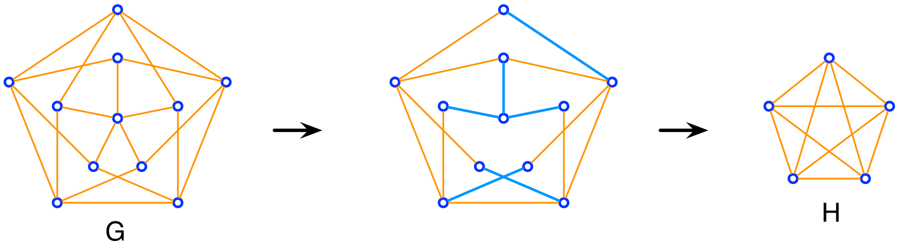
Next section is...

Minors and topological minors



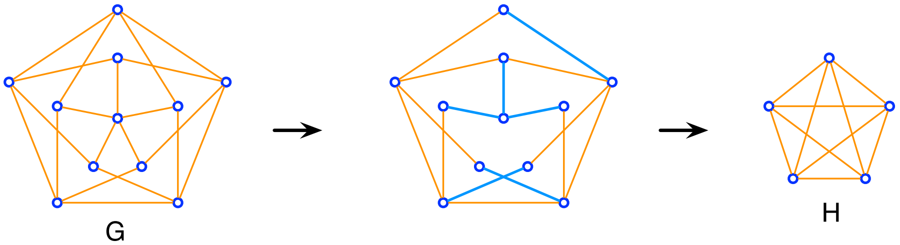
- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

Minors and topological minors

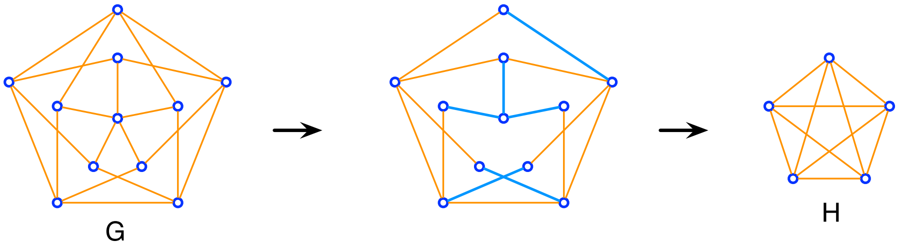


- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.
- H is a topological minor of G if H can be obtained from a subgraph of G by contracting edges with at least one endpoint of deg $\leqslant 2$.

Minors and topological minors



- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.
- H is a topological minor of G if H can be obtained from a subgraph of G by contracting edges with at least one endpoint of deg $\leqslant 2$.
- Therefore: $\boxed{H \text{ minor of } G \Rightarrow H \text{ topological minor of } G}$.



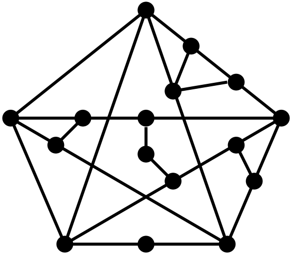
- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.
- H is a topological minor of G if H can be obtained from a subgraph of G by contracting edges with at least one endpoint of deg $\leqslant 2$.
- Therefore: $\boxed{H \text{ minor of } G \not\Leftarrow H \text{ topological minor of } G}$.
- Fixed H: $\boxed{H\text{-minor-free graphs} \subseteq H\text{-topological-minor-free graphs}}$.

Topological models in graphs

H is a topological minor of G if H can be obtained from a subgraph of G by contracting edges with at least one endpoint of deg $\leqslant 2$.

H-topological model in G: collection $\{v_u : u \in V(H)\} \subseteq V(G)$ s.t.

- $\forall\ uw \in E(H)$, there exists in G a path between v_u and v_w, and
- all these paths are pairwise vertex-disjoint.



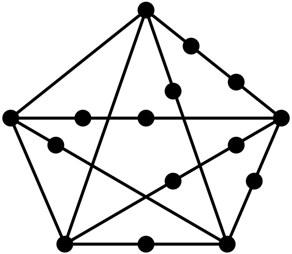
A K_5-topological model = a subdivision of K_5

Topological models in graphs

H is a topological minor of G if H can be obtained from a subgraph of G by contracting edges with at least one endpoint of deg $\leqslant 2$.

H-topological model in G: collection $\{v_u : u \in V(H)\} \subseteq V(G)$ s.t.

- $\forall\ uw \in E(H)$, there exists in G a path between v_u and v_w, and
- all these paths are pairwise vertex-disjoint.



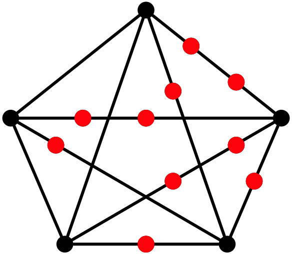
A K_5-topological model = a subdivision of K_5

Topological models in graphs

H is a topological minor of G if H can be obtained from a subgraph of G by contracting edges with at least one endpoint of deg $\leqslant 2$.

H-topological model in G: collection $\{v_u : u \in V(H)\} \subseteq V(G)$ s.t.

- $\forall\ uw \in E(H)$, there exists in G a path between v_u and v_w, and
- all these paths are pairwise vertex-disjoint.



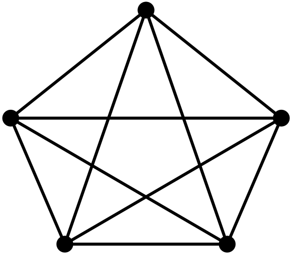
A K_5-topological model = a subdivision of K_5

Topological models in graphs

H is a topological minor of G if H can be obtained from a subgraph of G by contracting edges with at least one endpoint of deg $\leqslant 2$.

H-topological model in G: collection $\{v_u : u \in V(H)\} \subseteq V(G)$ s.t.

- $\forall\ uw \in E(H)$, there exists in G a path between v_u and v_w, and
- all these paths are pairwise vertex-disjoint.



A K_5-topological model = a subdivision of K_5

What is known for topological minors?

Erdős-Pósa property for the **vertex** version for **minors**:

$$\boxed{\text{there exists } f_H(k) \Leftrightarrow H \text{ is planar}}$$

Erdős-Pósa property for the **vertex** version for **topological minors**?

What is known for topological minors?

Erdős-Pósa property for the **vertex** version for **minors**:

$$\boxed{\text{there exists } f_H(k) \Leftrightarrow H \text{ is planar}}$$

Erdős-Pósa property for the **vertex** version for **topological minors**?

$$\boxed{\text{there exists } f_H(k) \Rightarrow H \text{ is planar}}$$

[Robertson, Seymour '84]

Is planarity sufficient?

What is known for topological minors?

Erdős-Pósa property for the **vertex** version for **minors**:

$$\boxed{\text{there exists } f_H(k) \Leftrightarrow H \text{ is planar}}$$

Erdős-Pósa property for the **vertex** version for **topological minors**?

$$\boxed{\text{there exists } f_H(k) \Rightarrow H \text{ is planar}}$$ [Robertson, Seymour '84]

Is planarity sufficient?

No! It does not hold even if $H =$ tree. [Thomassen '88]

What is known for topological minors?

Erdős-Pósa property for the **vertex** version for **minors**:

$$\boxed{\text{there exists } f_H(k) \Leftrightarrow H \text{ is planar}}$$

Erdős-Pósa property for the **vertex** version for **topological minors**?

$$\boxed{\text{there exists } f_H(k) \Rightarrow H \text{ is planar}}$$

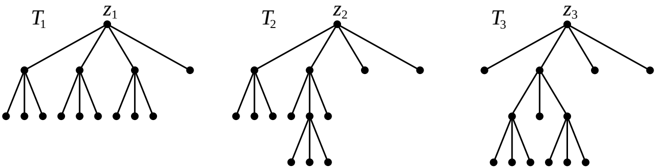
[Robertson, Seymour '84]

Is planarity sufficient?

No! It does not hold even if $H =$ tree. [Thomassen '88]

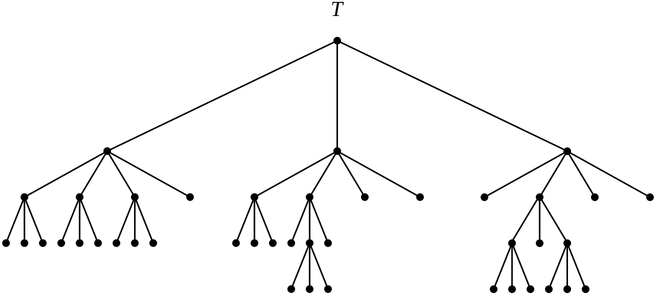
That is, there are trees T, such the collection of subdivisions of T does not satisfy the Erdős-Pósa property (even restricted to planar graphs).

Planarity is not sufficient for topological minors



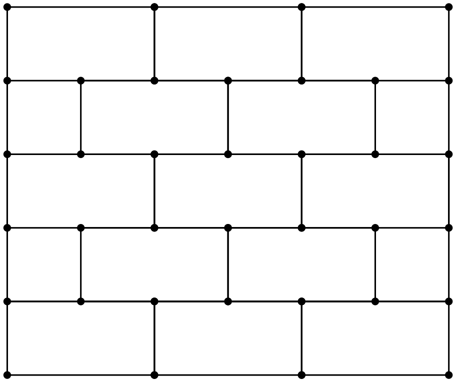
Let T_1, T_2, T_3 be non-isomorphic trees whose vertices have degree 4 or 1, and let z_i be a vertex of degree 4 in T_i.

Planarity is not sufficient for topological minors



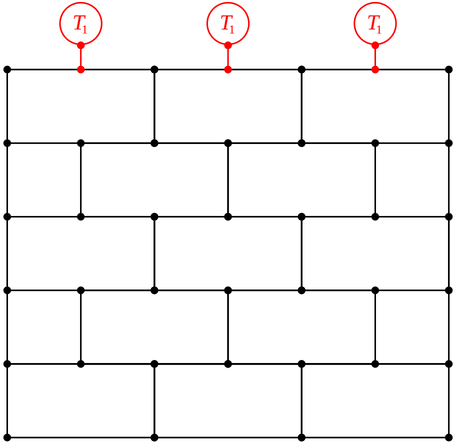
Let T be defined as above. We claim that the collection of subdivisions of T does not satisfy the Erdős-Pósa property (even in planar graphs).

Planarity is not sufficient for topological minors



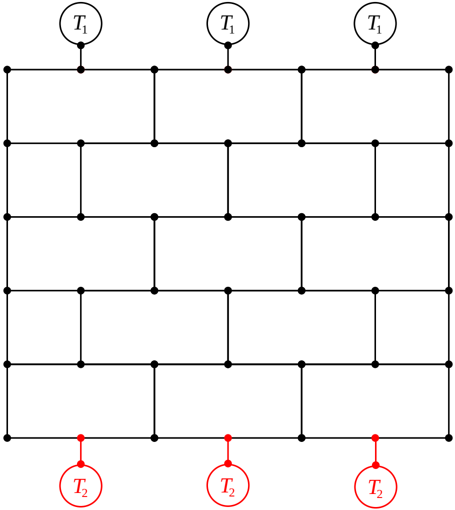
We start with the wall of size k, we subdivide the above edges, and we add attached copies of the trees T_1, T_2, T_3 defined before.

Planarity is not sufficient for topological minors



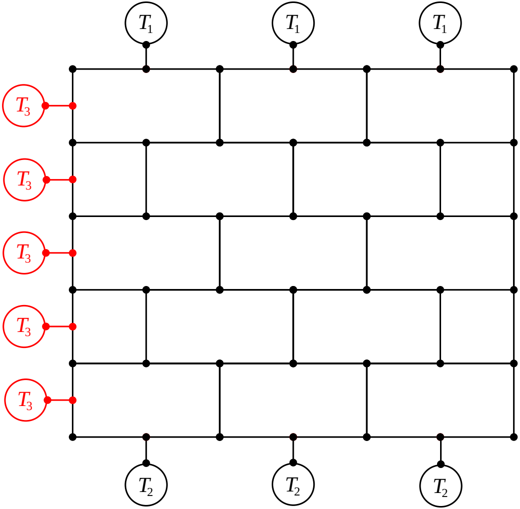
We start with the wall of size k, we subdivide the above edges, and we add attached copies of the trees T_1, T_2, T_3 defined before.

Planarity is not sufficient for topological minors



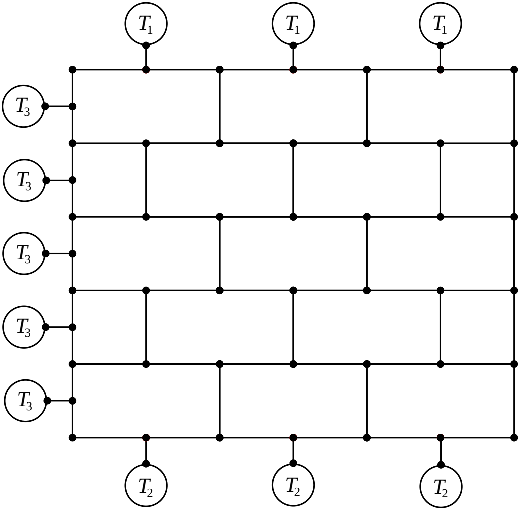
We start with the wall of size k, we subdivide the above edges, and we add attached copies of the trees T_1, T_2, T_3 defined before.

Planarity is not sufficient for topological minors



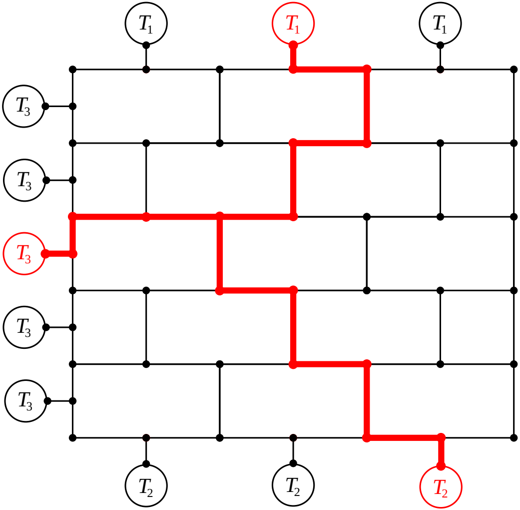
We start with the wall of size k, we subdivide the above edges, and we add attached copies of the trees T_1, T_2, T_3 defined before.

Planarity is not sufficient for topological minors



This (planar) graph contains only one vertex-disjoint subdivision of T (i.e., the packing number is one), but the covering number is arbitrarily large.

Planarity is not sufficient for topological minors



This (planar) graph contains only one vertex-disjoint subdivision of T (i.e., the packing number is one), but the covering number is arbitrarily large.

Gràcies!