Dynamic programming in sparse graphs

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Outline

Next section is...

1. Motivation
2. Graphs on surfaces
   - Preliminaries
   - Main ideas of our approach
3. Extension to H-minor-free graphs
4. Some recent results

Some words on parameterized complexity

- Idea: given an NP-hard problem, fix one parameter of the input to see if the problem gets more "tractable".

  Example: the size of a VERTEX COVER.

- Given a (NP-hard) problem with input of size n and a parameter k, a fixed-parameter tractable (FPT) algorithm runs in

$$f(k) \cdot n^{\mathcal{O}(1)}, \quad \text{for some function } f.$$

  Examples: k-VERTEX COVER, k-LONGEST PATH.

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FPT and single-exponential algorithms

- Courcelle's theorem (1988):

  Graph problems expressible in Monadic Second Order Logic can be solved in time $f(k) \cdot n^{\mathcal{O}(1)}$ in graphs with **tw** $\leq k$.

- **Problem**: $f(k)$ can be huge!!! (for instance, $f(k) = 2^{3^{4^{5^{6^k}}}}$)

- A single-exponential parameterized algorithm is a FPT algo s.t.

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> Objective:
> build a framework to obtain **single-exponential algorithms** for a class of NP-hard problems in **sparse graphs**.

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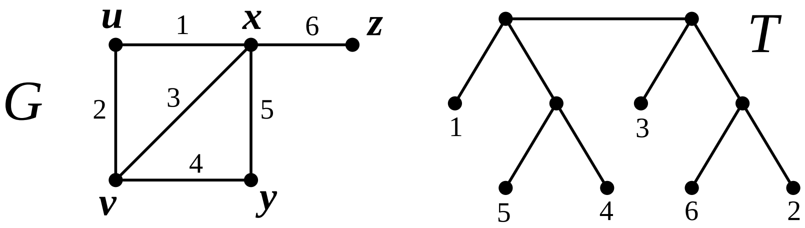
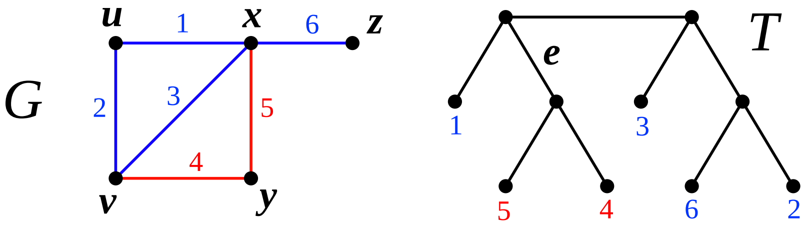
Branch decompositions and branchwidth



A branch decomposition of a graph G is a pair (T, μ):

- T is a tree where all internal vertices have degree 3.
- μ is a bijection between the leaves of T and $E(G)$.

Branch decompositions and branchwidth



Each edge $e \in T$ partitions $E(G)$ into two sets A_e and B_e.

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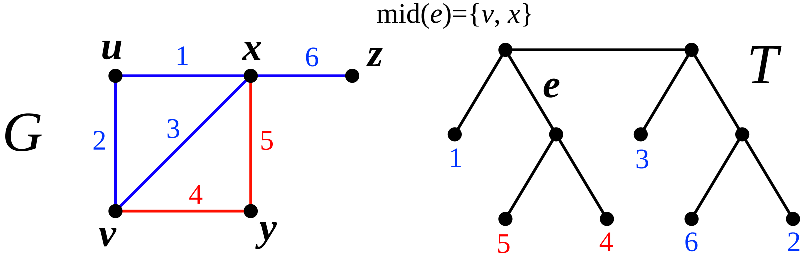

For each $e \in E(T)$, we define **mid**$(e) = V(A_e) \cap V(B_e)$.

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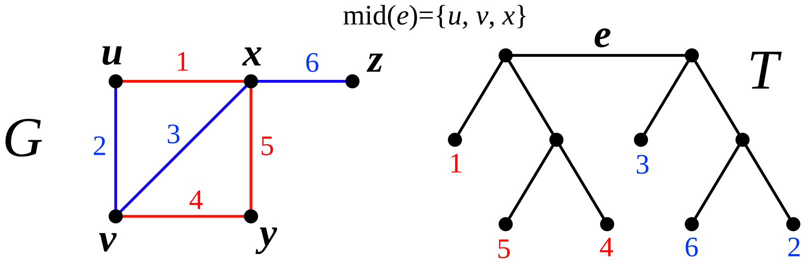
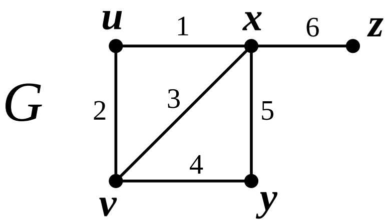
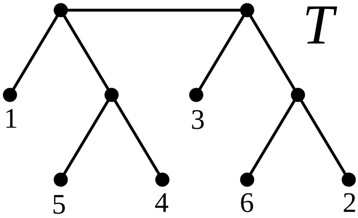


The width of (T, μ) is $\max_{e \in E(T)} |\mathbf{mid}(e)|$.

Branch decompositions and branchwidth





The branchwidth of a graph G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G:

$$\mathbf{bw}(G) = \min_{(T,\mu)} \max_{e \in E(T)} |\mathbf{mid}(e)|.$$

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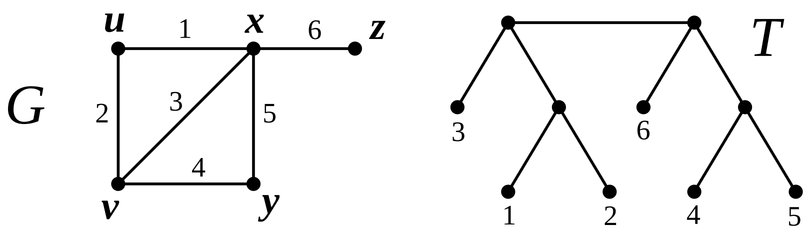

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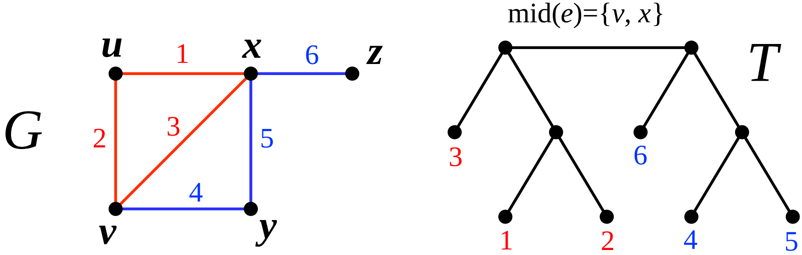

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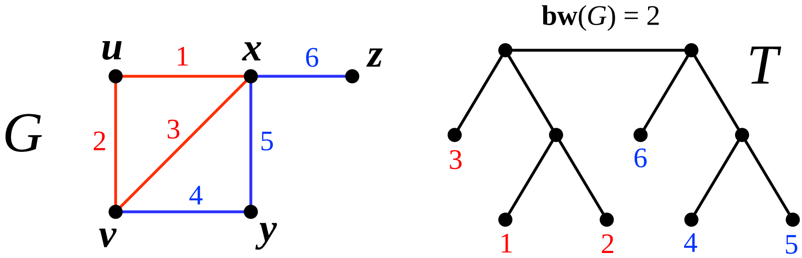

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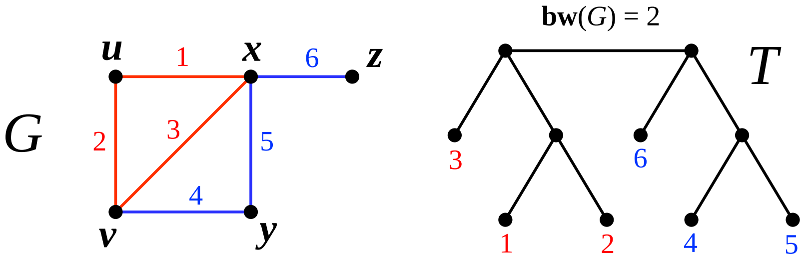

We have the following relationship for graphs G such that $|E(G)| \geq 3$:

$$\mathbf{bw}(G) \;\leq\; \mathbf{tw}(G) + 1 \;\leq\; \frac{3}{2}\mathbf{bw}(G).$$

[Robertson i Seymour. *JCTSB'91*]

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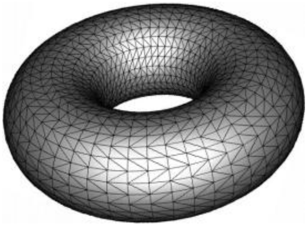
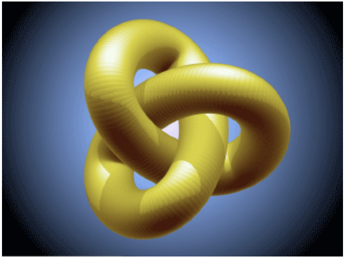
Surfaces

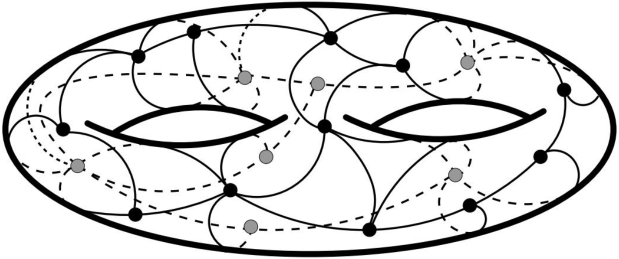
- **SURFACE** = TOPOLOGICAL SPACE, LOCALLY “FLAT”





Graphs on surfaces

Embedded graph: graph drawn on a surface, with no edge-crossings.



- The Euler genus of a graph G, denoted by $\mathbf{eg}(G)$, is the least Euler genus of the surfaces in which G can be embedded.

Dynamic programming (DP)

- Applied in a bottom-up fashion on a rooted branch decomposition of the input graph G.
- For each graph problem, DP requires the suitable definition of tables encoding how potential (global) solutions are restricted to a middle set $\mathbf{mid}(e)$.
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A classification of graph optimization problems

How can we certificate a solution in a middle set **mid**(e)?

1. A subset of vertices of **mid**(e) (not restricted by some global condition).
   Examples: VERTEX COVER, DOMINATING SET.
   The size of the tables is bounded by $2^{\mathcal{O}(k)}$.

2. A *connected pairing* of vertices of **mid**(e).
   Examples: LONGEST PATH, CYCLE PACKING, HAMILTONIAN CYCLE.
   The # of pairings in a set of k elements is $k^{\Theta(k)} = 2^{\Theta(k \log k)}$...

   OK for planar graphs [Dorn, Penninkx, Bodlaender, Fomin. *ESA'05*];
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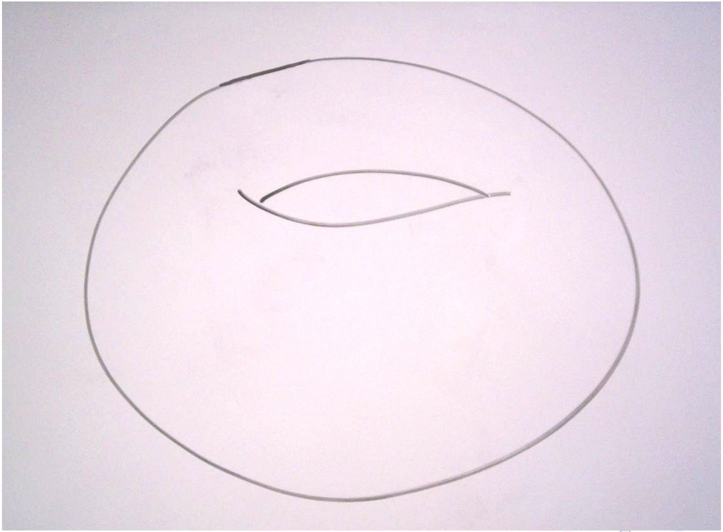

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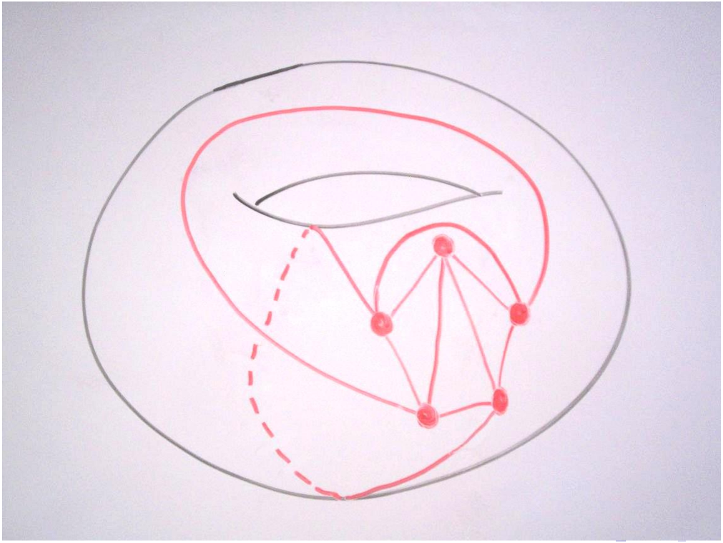

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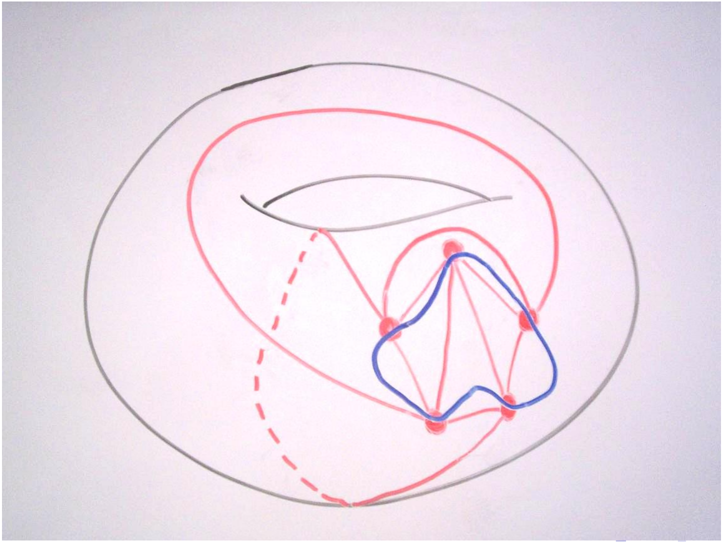

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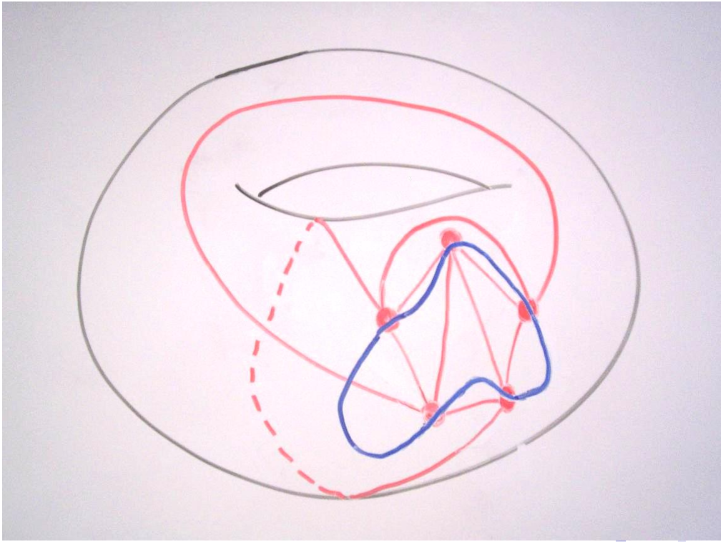

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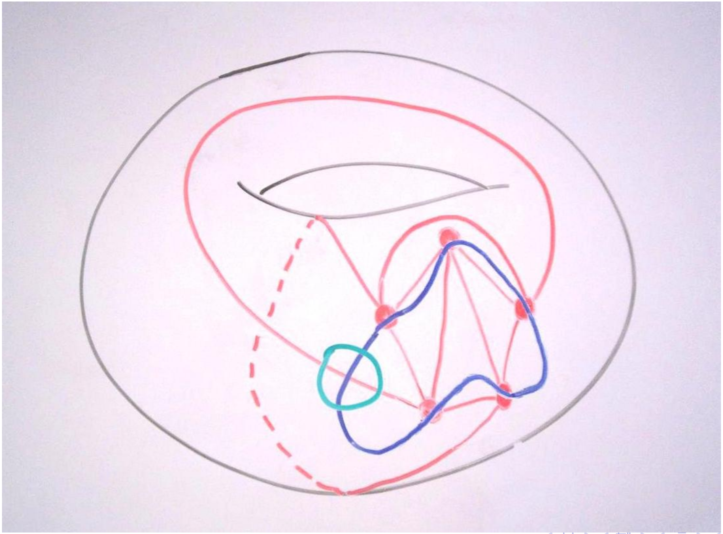

Sphere cut decompositions

Key idea for planar graphs [Dorn *et al.* *ESA'05*]:

- Sphere cut decomposition: Branch decomposition where the vertices in each **mid**(e) are situated around a noose. [Seymour and Thomas. *Combinatorica'94*]
- Recall that the size of the tables of a DP algorithm depends on how many ways a partial solution can intersect **mid**(e).
- In how many ways can we draw polygons inside a circle such that they touch the circle only on its k vertices and they do not intersect?
- Exactly the number of *non-crossing partitions* over k elements, which is given by the k-th Catalan number:

$$\text{CN}(k) = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}} \approx 4^k.$$

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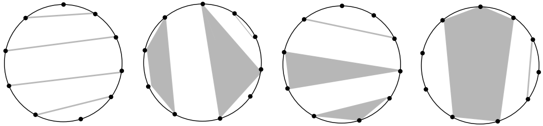

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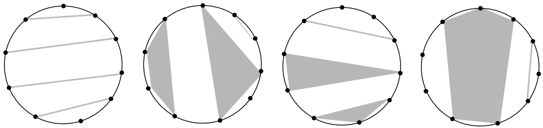

- Exactly the number of *non-crossing partitions* over k elements, which is given by the k-th Catalan number:

$$\mathrm{CN}(k) = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}} \approx 4^k.$$

"Old" idea for graphs on surfaces

Key idea for graphs on surfaces [Dorn *et al.* *SWAT'06*]:

- Perform a planarization of the input graph by splitting the potential solutions into a number of pieces depending on the surface.
- Then, apply the sphere cut decomposition technique to a more complicated version of the problem where the number of pairings is still bounded by some Catalan number.
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Sphere cut ⇝ surface cut

Our approach is based on a new type of branch decomposition, called surface cut decomposition.

- Surface cut decompositions for graphs on surfaces **generalize** sphere cut decompositions for planar graphs.
  [Seymour and Thomas. *Combinatorica'94*]
- That is, we exploit directly the combinatorial structure of the potential solutions in the surface (**without planarization**).
- Using surface cut decompositions, we provide in a **unified** way single-exponential algorithms for **connected packing-encodable** problems, and with **better genus** dependence.

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Surface cut decompositions (simplified version)

Let G be a graph embedded in a surface Σ, with $\mathbf{eg}(\Sigma) = \mathbf{g}$.

A surface cut decomposition of G is a branch decomposition (T, μ) of G and a subset $A \subseteq V(G)$, with $|A| = \mathcal{O}(\mathbf{g})$, s.t. for all $e \in E(T)$

- either $|\mathbf{mid}(e) \setminus A| \leq 2$,
- or
  - ⋆ the vertices in $\mathbf{mid}(e) \setminus A$ are contained in a set $\mathcal{N}$ of $\mathcal{O}(\mathbf{g})$ nooses;
  - ⋆ these nooses intersect in $\mathcal{O}(\mathbf{g})$ vertices;
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Main results

1. Surface cut decompositions can be **efficiently computed**:

Theorem (Rué, Thilikos, and S.)

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  1. Let **P** be a **connected packing-encodable** problem on a surface-embedded graph G.
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Next section is...

Structure of minor-free graphs

- Idea: use the structure of H-minor-free graphs.

- Some (simplified) preliminaries:
  - h-clique-sum of two graphs G_1 and G_2:
    choose cliques $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ with $|V(K_1)| = |V(K_2)| = h$,
    identify them, and possibly remove some edges of that clique.
  - Apex in an embedded graph:
    add a vertex with any neighbors in the embedded graph.
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- Structure Theorem [Robertson and Seymour (1983-2012)]:
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Extension to H-minor-free graphs

- Strategy: use an extension of surface cut decomposition in each almost-embeddable graph, and then merge them.
- The clique-sums and the apices are “easy” to deal with, but the vortices are more complicated...
- We can capture their combinatorial behavior with h-triangulations: partition in the disk in which no subset of $h+1$ blocks pairwise intersect. *(A non-crossing partition is a 1-triangulation.)*
- It is known that the # of h-triangulations on k elements satisfies

$$T_h(k) \leq_{k \to \infty} \frac{h!}{\pi^{h/2}} \cdot k^{-3h/2} \cdot 4^{hk}$$

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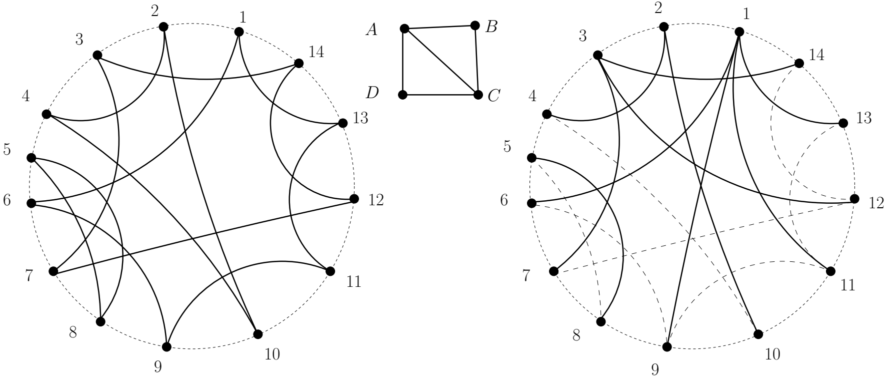
Example of a 3-triangulation

A 3-triangulation of the disc $\mathbb{D}_{14}$ with four blocks $A = \{1, 6, 9, 11, 13\}$, $B = \{2, 4, 10\}$, $C = \{3, 7, 12, 14\}$, and $D = \{5, 8\}$.



A partition is an h-triangulation iff its incidence graph has clique size $\leq h$.

H-minor-free cut decompositions

- In order to define *H*-minor-free cut decompositions, we first need a suitable version of the Robertson & Seymour Structure Theorem, in which every *h*-almost-embeddable piece is embedded in a polyhedral way: it is 3-vertex-connected, and the shortest non-contractible noose has length ≥ 3.
- Then, *H*-minor-free cut decompositions are defined in the "natural" way (quite technical)...
- We just give some intuition about how to deal with the vortices.
- Connected packing: collection of vertex-disjoint connected subgraphs of the input graph. We are interested in their intersection with the middle sets.

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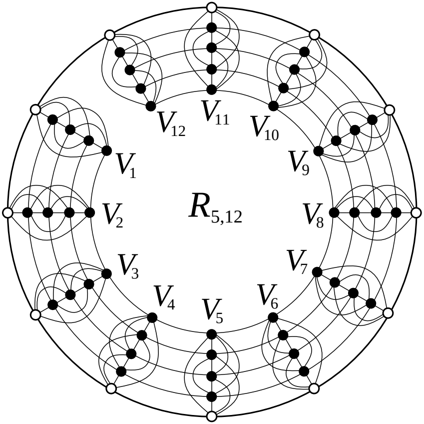
Vortex patterns

Vortex of depth h in an embedded graph:
paste a graph of pathwidth at most h in a face of the embedding.

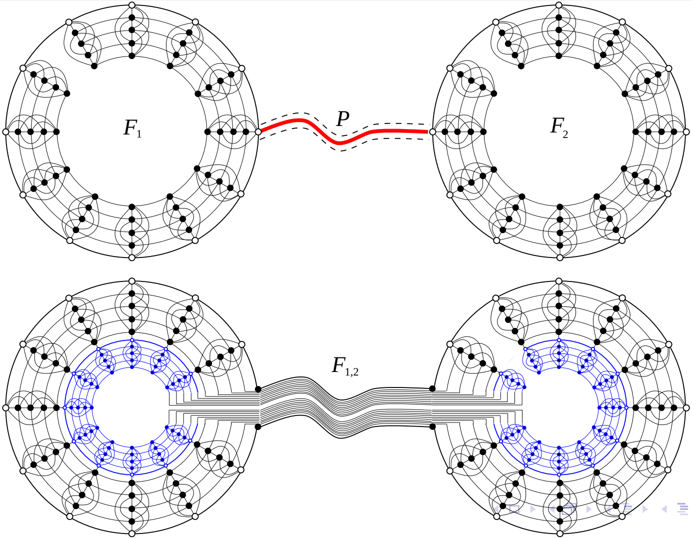


It can be easily seen that each vortex is a minor of a vortex pattern (preserving the vertices in the face of the embedding).

Merging vortices

Lemma

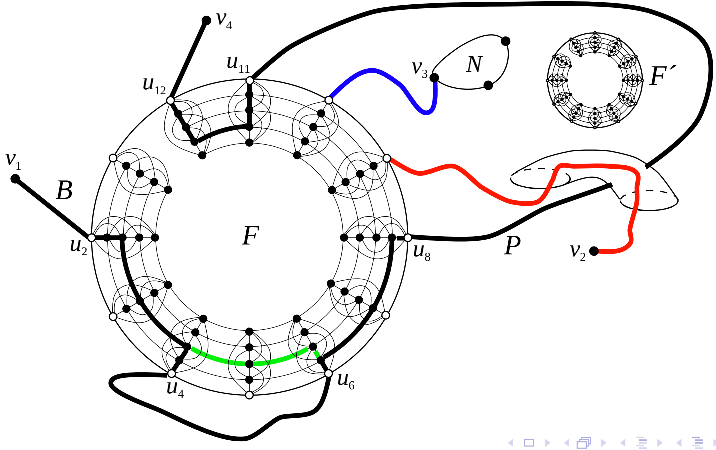
We may assume that each connected subgraph meets at most one vortex.



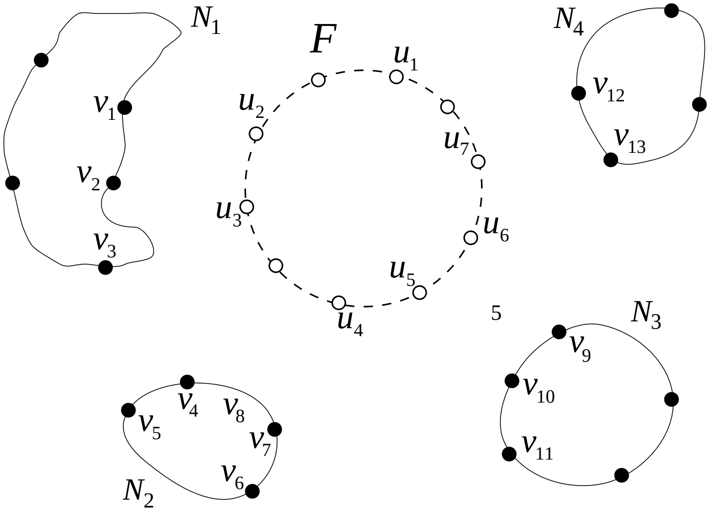
How connected subgraphs can cross a vortex

Lemma

We may assume that the total number of times that the subgraphs in a connected packing meet each vortex is $O_h(k)$.

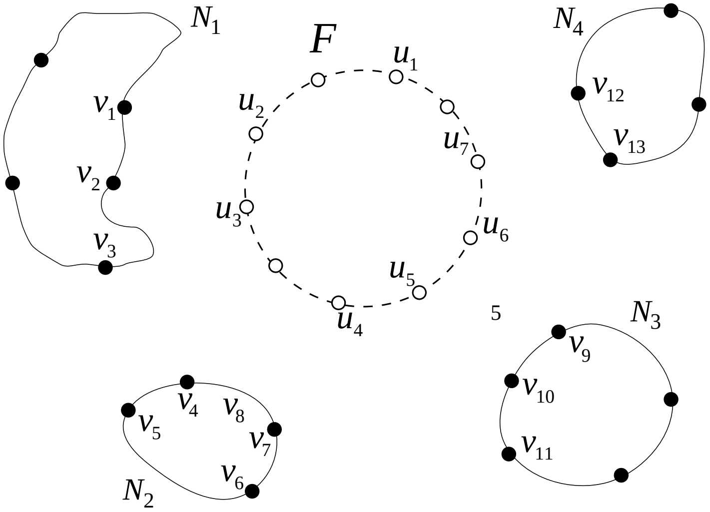


Simulating the behavior of a vortex



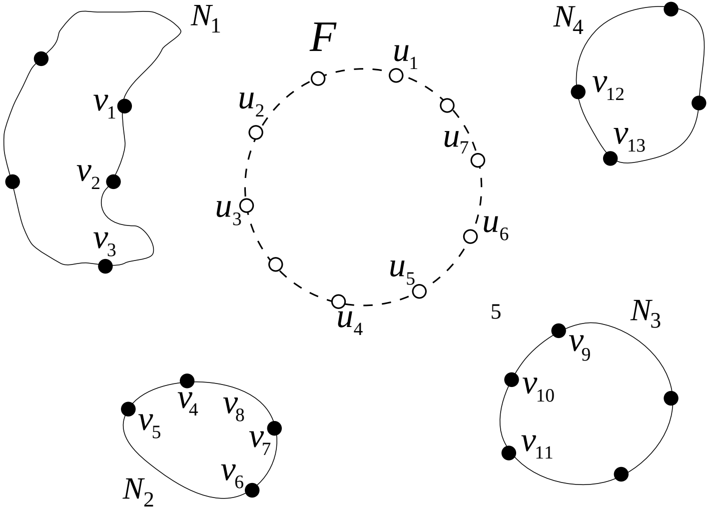
Example (in the plane) of our approach to simulate the behavior of the vortices.

Simulating the behavior of a vortex



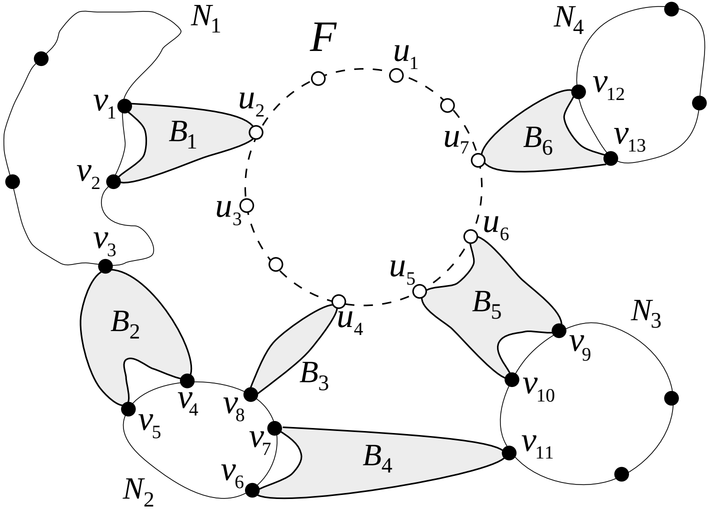
There are four nooses N_1, N_2, N_3, N_4 (drawn with full lines), and one vortex F of depth 2 (drawn with a dashed circle).

Simulating the behavior of a vortex



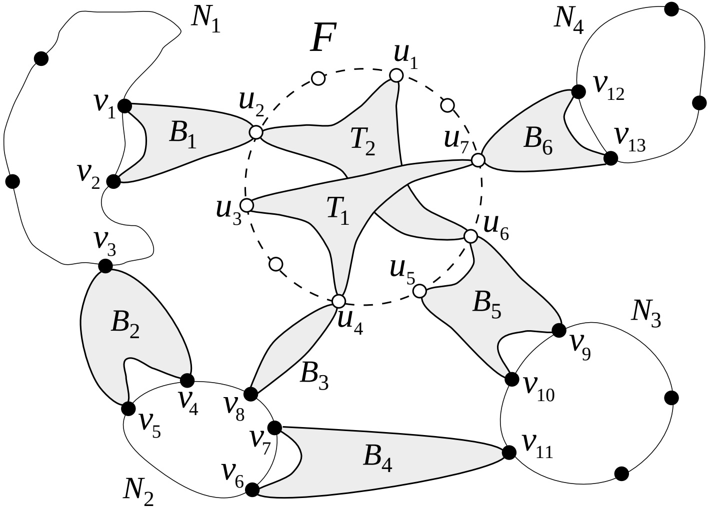
Black vertices correspond to vertices in the separator S (thus, in the nooses), while white vertices belong to the base set of the vortex.

Simulating the behavior of a vortex



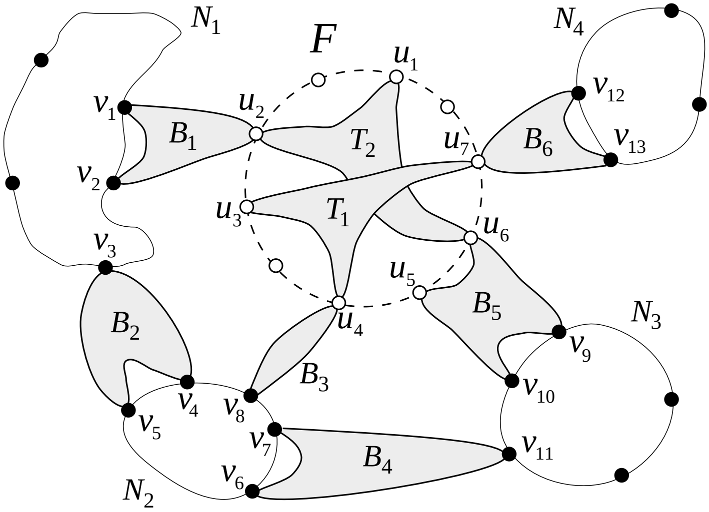
The non-crossing packing in Σ has six connected subgraphs B_1, B_2, B_3, B_4, B_5, and B_6.

Simulating the behavior of a vortex



The 2-triangulation of the vortex F has two connected subgraphs T_1 and T_2.

Simulating the behavior of a vortex



With the two subgraphs T_1 and T_2 corresponding to a 2-triangulation of the vortex, subgraphs B_1 and B_5 (resp. B_6 and B_6) get merged.

... the final result

Theorem

Every connected packing-encodable problem whose input is an n-vertex graph G that excludes an h-vertex graph H as a minor, and has branchwidth at most k, can be solved by a DP algorithm on an H-minor-free cut decomposition of G with tables of size $2^{O_h(k)} \cdot n^{O(1)}$.

We prove that, given an H-minor-free graph G, an H-minor-free cut decomposition of G of width $O_h(\mathbf{bw}(G))$ can be constructed in $O_h(n^3)$ time. Therefore, we conclude the following result.

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Next section is...

1. Motivation
2. Graphs on surfaces
   - Preliminaries
   - Main ideas of our approach
3. Extension to H-minor-free graphs
4. Some recent results

Some recent results

1. For an FPT problem, is it **always** possible to obtain algorithms with running time $2^{\mathcal{O}(\mathbf{tw})} \cdot n^{\mathcal{O}(1)}$?

   [Lokshtanov, Marx, Saurabh. *SODA'11*]
   If 3SAT cannot be solved in time $2^{o(n)}$, then DISJOINT PATHS cannot be solved in time $2^{o(\mathbf{tw} \log \mathbf{tw})} \cdot n^{\mathcal{O}(1)}$ in general graphs.

   - HAMILTONIAN PATH, FVS, CONNECTED VERTEX COVER, ... Is $2^{\mathcal{O}(\mathbf{tw} \log \mathbf{tw})} \cdot n^{\mathcal{O}(1)}$ best possible?

2. Randomized algorithms for connected packing-encodable problems in general graphs in time $2^{\mathcal{O}(\mathbf{tw})} \cdot n^{\mathcal{O}(1)}$.

   [Cygan, Nederlof, (Pilipczuk)^2, van Rooij, Wojtaszczyk. *FOCS'11*]

   - They introduce a DP technique called Cut&Count. *(It relies on a probabilistic result called the Isolation Lemma.)*
   - Can these algorithms be derandomized?

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Gràcies!

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