Graph modification problems with forbidden minors

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Outline of the talk

1. Introduction
2. Hitting forbidden minors: survey of known results
   - Parameterized by treewidth
   - Parameterized by solution size
3. Some ingredients of the proofs
   - Parameterized by treewidth
   - Irrelevant vertex technique
   - Parameterized by solution size
4. More general modification operations
5. Further research

Next section is...

Graph modification problems

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(vertex deletion, edge deletion/addition/contraction, ...).

$\mathcal{M}$-Modification to $\mathcal{C}$	
Input:	A graph G and an integer k.
Question:	Can we transform G to a graph in $\mathcal{C}$ by applying at most k operations from $\mathcal{M}$?

This meta-problem has a huge expressive power.

Many possible interesting variants

- $\mathcal{M}$ = vertex deletion, $\mathcal{C}$ = forbidden induced subgraphs.

  [S., Souza. 2020: arXiv 2004.08324]

- $\mathcal{M}$ = vertex deletion, $\mathcal{C}$ = generalization of bipartite graphs.

  [Baste, Faria, Klein, S. 2015: arXiv 1504.05515]

- $\mathcal{M}$ = edge contraction, $\mathcal{C}$ = graph transversal parameters.

  [Lima, dos Santos, S., Souza. 2020: arXiv 2005.01460]

  [Lima, dos Santos, S., Souza, Tale. 2022: arXiv 2202.03322]

- . . . and many more!

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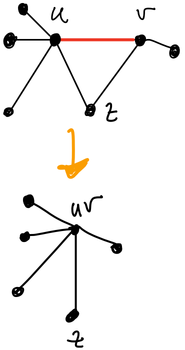
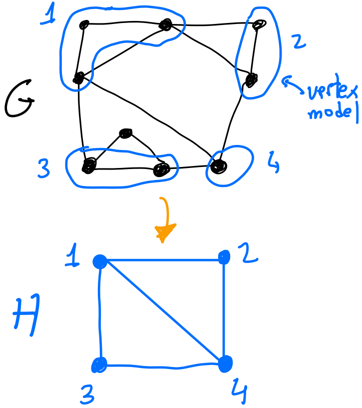
Graph minors

A graph H is a minor of a graph G, denoted by $H \leqslant_m G$, if H can be obtained from a subgraph of G by contracting edges.





Minor-closed graph classes

A graph class $\mathcal{C}$ is minor-closed (or closed under minors) if

$$G \in \mathcal{C} \;\Rightarrow\; H \in \mathcal{C} \text{ for every } H \leqslant_m G.$$

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Every minor-closed graph class $\mathcal{C}$ can be characterized by excluded minors:

> List all the graphs $\mathcal{F}_{\mathcal{C}} := \{G_1, G_2, \ldots\}$ that do not belong to $\mathcal{C}$, and then $\mathcal{C} = \text{exc}(\mathcal{F}_{\mathcal{C}})$.

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Note that, in general, this list $\mathcal{F}_{\mathcal{C}} = \{G_1, G_2, \ldots\}$ may be infinite.

Forbidden minors for some minor-closed graph classes

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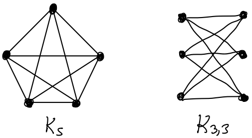

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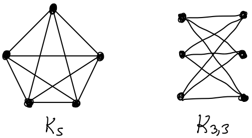

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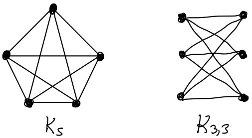

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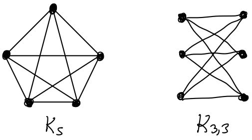

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Wagner's conjecture

Conjecture (Wagner. 1970)

For every minor-closed graph class $\mathcal{C}$, there exists a finite set of graphs $\mathcal{F}_{\mathcal{C}}$ such that $\mathcal{C} = \text{exc}(\mathcal{F}_{\mathcal{C}})$.

Wagner's conjecture... now Robertson-Seymour's theorem

Theorem (Robertson, Seymour. 1983-2004)

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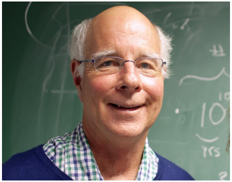
Parameterized complexity in a nutshell

Idea Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established and very active area.

Parameterized problems

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed, finite alphabet.

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- VERTEX k-COLORING: Can the vertices of a graph be colored with $\leq k$ colors, so that any two adjacent vertices get different colors?

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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- k-Clique: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
- Vertex k-Coloring: NP-hard for fixed $k = 3$.

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  The problem is para-NP-hard

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Hitting forbidden minors

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- $\mathcal{F} = \{K_2\}$: Vertex Cover.
- $\mathcal{F} = \{K_3\}$: Feedback Vertex Set.
- $\mathcal{F} = \{K_5, K_{3,3}\}$: Vertex Planarization.
- $\mathcal{F}$ = {diamond}: Cactus Vertex Deletion.

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We consider the following two parameterizations of $\mathcal{F}$-M-Deletion:

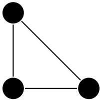
1. Structural parameter: $\text{tw}(G)$.
2. Solution size: k.

Joint work with Julien Baste, Laure Morelle, Giannos Stamoulis, and Dimitrios M. Thilikos.

Next subsection is...

Treewidth via k-trees

Example of a 2-tree:



[Figure by Julien Baste]

For $k \geq 1$, a k-tree is a graph that can be built starting from a $(k+1)$-clique and then iteratively adding a vertex connected to a k-clique.

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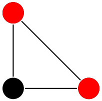

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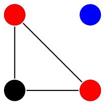

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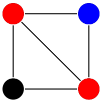

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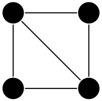

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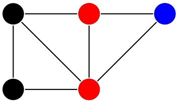

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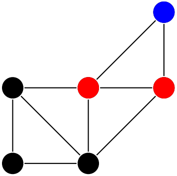

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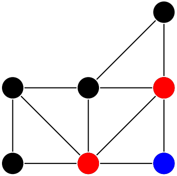

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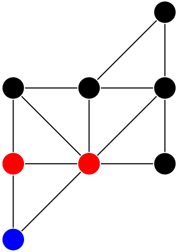

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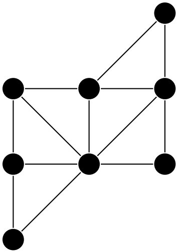

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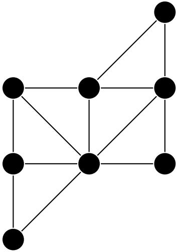

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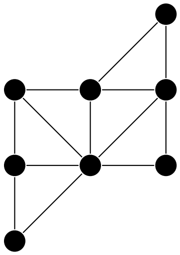

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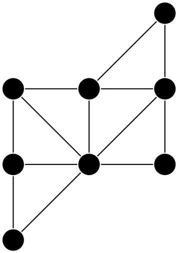

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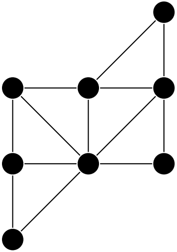

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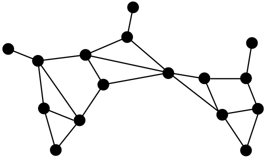
Construction suggests the notion of tree decomposition: small separators.

Treewidth measures the tree-likeness of a graph



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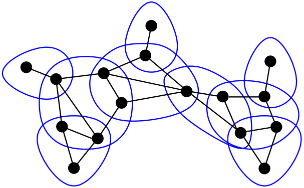

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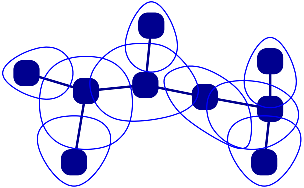

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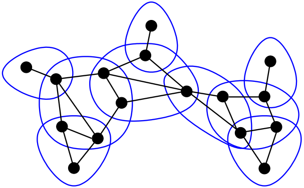

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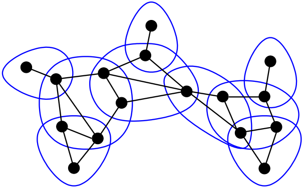

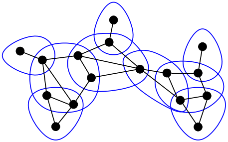

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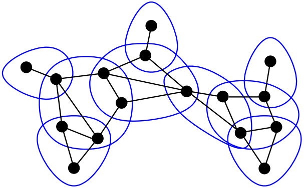

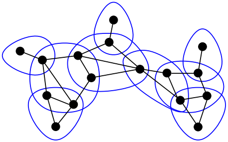

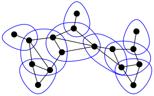

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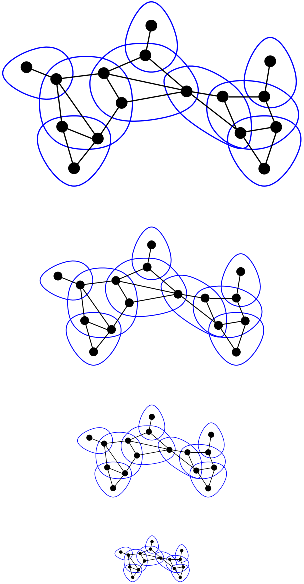

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ETH: The 3-SAT problem on n variables cannot be solved in time $2^{o(n)}$.

[Impagliazzo, Paturi. 1999]

What was known for particular collections $\mathcal{F}$

Let $\mathcal{F}$ be a fixed finite collection of graphs.

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- $\mathcal{F} = \{K_5, K_{3,3}\}$: Vertex Planarization.
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  [Jansen, Lokshtanov, Saurabh. 2014 + Pilipczuk. 2015]

With Julien Baste and Dimitrios M. Thilikos (2016-2020)

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-Deletion on n-vertex graphs can be solved in time

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- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. I. General upper bounds**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. II. Single-exponential algorithms**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. III. Lower bounds**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. IV. An optimal algorithm**. 2021]

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A dichotomy for hitting a connected minor

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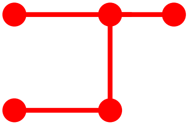

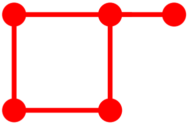

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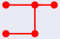
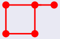
- $2^{\mathcal{O}(\mathsf{tw})} \cdot n^{\mathcal{O}(1)}$,   *if* $H \leqslant_c$  *or* $H \leqslant_c$ .

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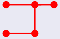
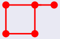
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_{\mathrm{c}}$  *or* $H \leqslant_{\mathrm{c}}$  .
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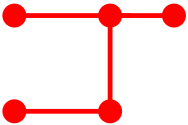

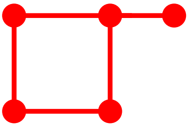

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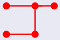
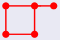
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_{\mathrm{c}}$  *or* $H \leqslant_{\mathrm{c}}$ .
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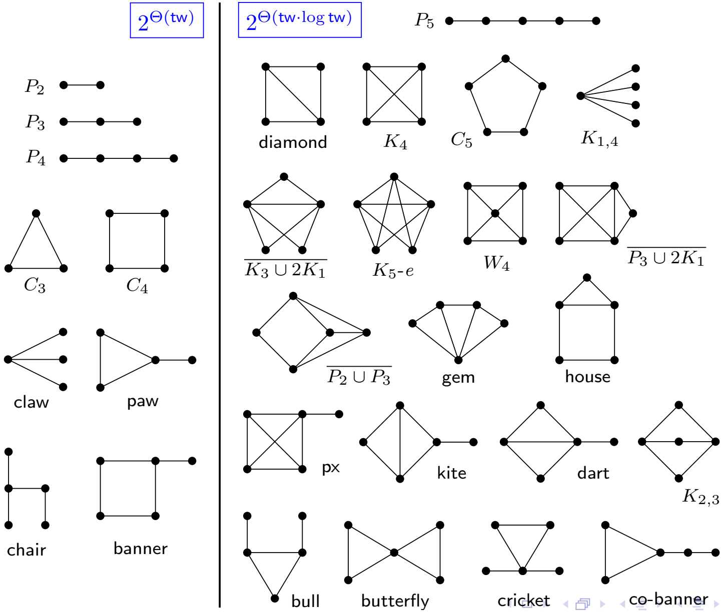
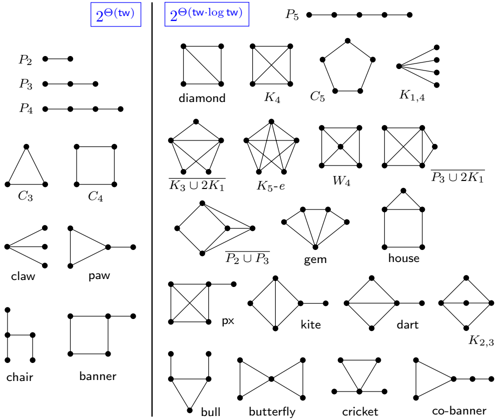
In both cases, the running time is asymptotically optimal under the ETH.

skip

Complexity of hitting a single connected minor H



A compact statement for a single connected graph



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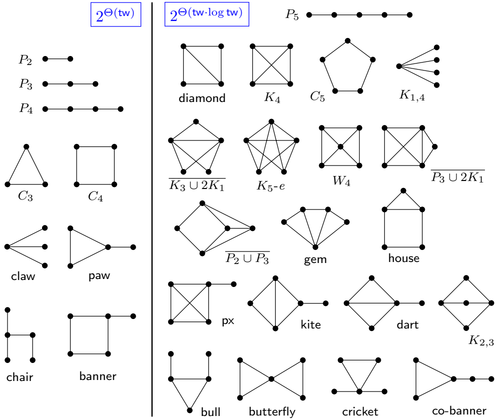

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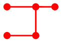
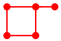
- All graphs on the left are contractions of  or 

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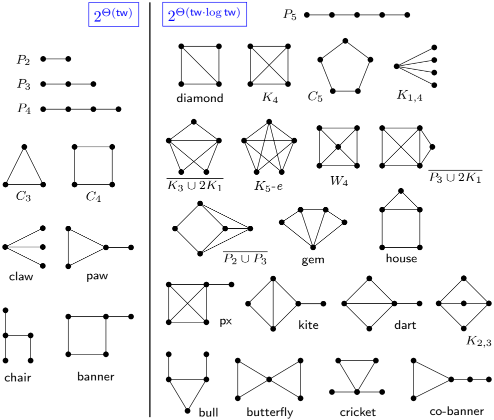

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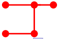
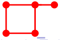
- All graphs on the right are not contractions of  or 

Next subsection is...

We parameterize by the size of the desired solution

$\mathcal{F}$-M-Deletion

Input: A graph G and an integer k.

Parameter: k.

Question: Does G contain a set $S \subseteq V(G)$ with $|S| \leqslant k$ such that $G \setminus S$ does not contain any of the graphs in $\mathcal{F}$ as a minor?

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$$\mathcal{C}_k = \{G \mid (G, k) \text{ is a positive instance of } \mathcal{F}\text{-M-Deletion}\}$$

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But... only existential, non-uniform, $f(\mathcal{C}_k)$ astronomical.

Can we do better?

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- For every $\mathcal{F}$, some enormous explicit function $f_{\mathcal{F}}(k)$ can be derived from an FPT algorithm for hitting topological minors:

$$f_{\mathcal{F}}(k) \cdot n^{\mathcal{O}(1)}.$$

[Fomin, Lokshtanov, Panolan, Saurabh, Zehavi. 2020]

Our results

Theorem (S., Stamoulis, Thilikos. 2020)

For all $\mathcal{F}$, the $\mathcal{F}$-M-DELETION problem can be solved in time $2^{\mathsf{poly}(k)} \cdot n^3$.

Here, $\mathsf{poly}(k)$ is a polynomial whose degree depends on $\mathcal{F}$.

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Theorem (Morelle, S., Stamoulis, Thilikos. 2022)

For all $\mathcal{F}$, the $\mathcal{F}$-M-Deletion problem can be solved in time $2^{\mathsf{poly}(k)} \cdot n^2$.

skip

Next section is...

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Recall the statement of the problem

Let $\mathcal{F}$ be a fixed finite collection of graphs.

$\mathcal{F}$-M-Deletion	
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Parameter:	The treewidth tw of G.
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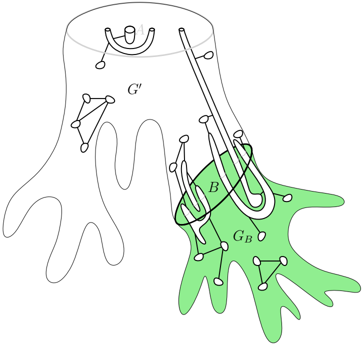

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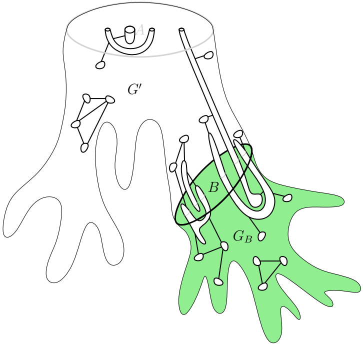

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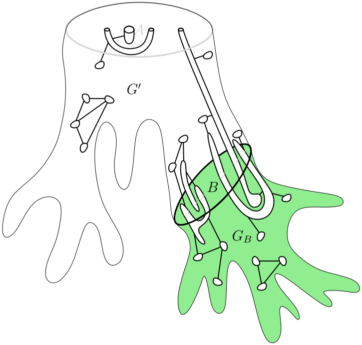

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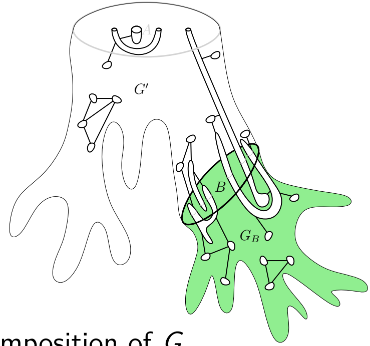

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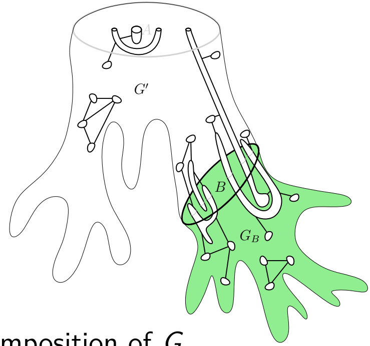

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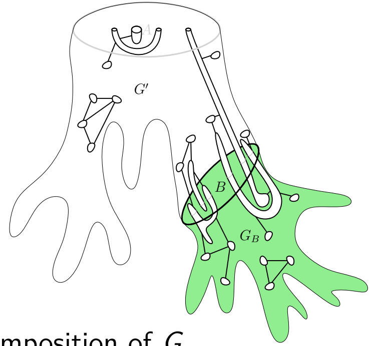

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- Goal: Bound the number of representatives: $|\mathcal{R}^{(\mathcal{F},t)}| = 2^{\mathcal{O}_{\mathcal{F}}(\mathrm{tw} \cdot \log \mathrm{tw})}$.

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Next subsection is...

Basic principle of the irrelevant vertex technique

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Disjoint Paths

Input: a graph G and k pairs of vertices $T = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$.

Question: does G contain k vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i to t_i?

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Strategy:

1. If $\mathrm{tw}(G) > f(k)$, find an irrelevant vertex:

   A vertex $v \in V(G)$ such that (G, T, k) and $(G \setminus v, T, k)$ are equivalent instances.

Basic principle of the irrelevant vertex technique

This technique was invented in [Robertson and Seymour. 1995]

Disjoint Paths

Input: a graph G and k pairs of vertices $T = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$.

Question: does G contain k vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i to t_i?

Strategy:

1. If $\text{tw}(G) > f(k)$, find an irrelevant vertex:

   A vertex $v \in V(G)$ such that (G, T, k) and $(G \setminus v, T, k)$ are equivalent instances.

2. Otherwise, if $\text{tw}(G) \leq f(k)$, solve the problem using dynamic programming (by Courcelle).

How to find an irrelevant vertex when the treewidth is large?

How to find an irrelevant vertex when the treewidth is large?

By using the Grid Exclusion Theorem!

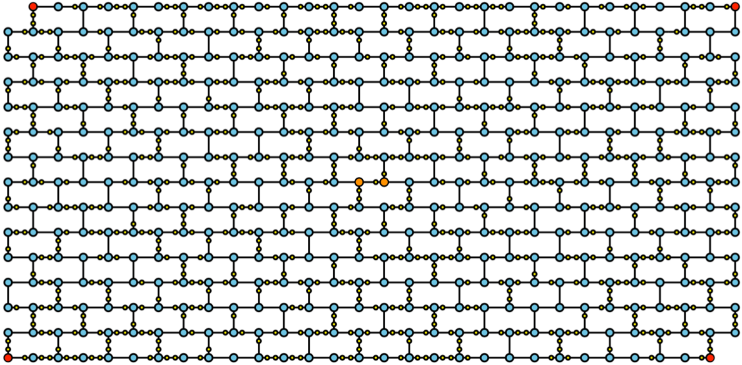
How to find an irrelevant vertex when the treewidth is large?

By using the Wall Exclusion Theorem!

How to find an irrelevant vertex when the treewidth is large?

Theorem (Robertson and Seymour. 1986)

For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains an ℓ-wall as a minor.



[Figure by Dimitrios M. Thilikos]

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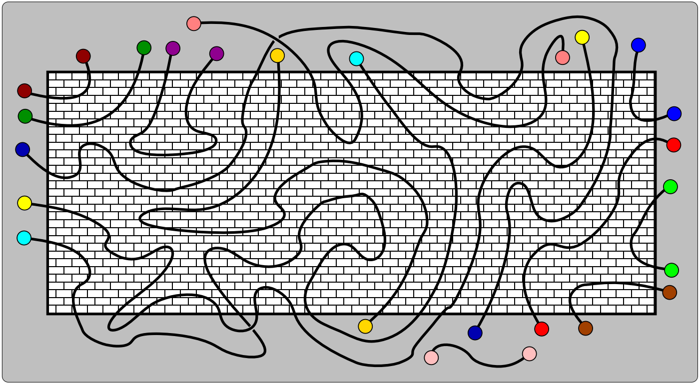
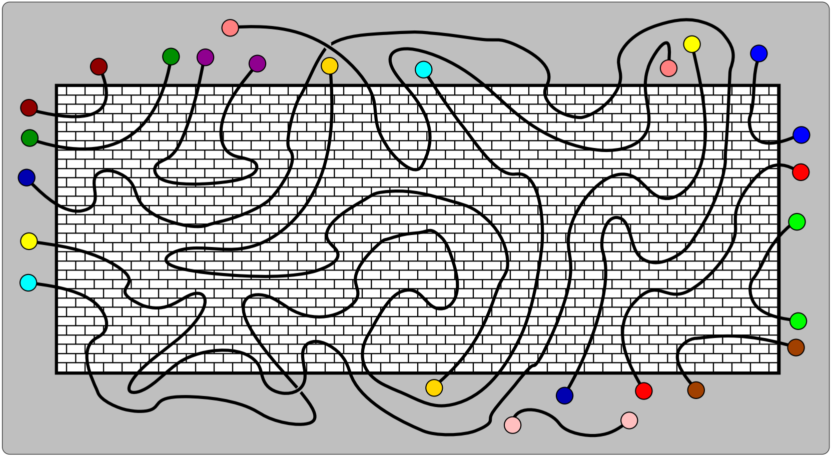


[Figure by Dimitrios M. Thilikos]

Goal: declare one of the central vertices of the wall irrelevant.



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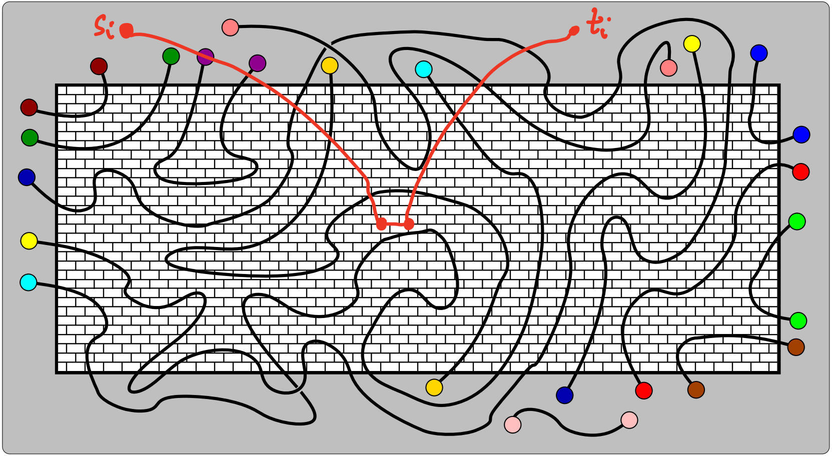
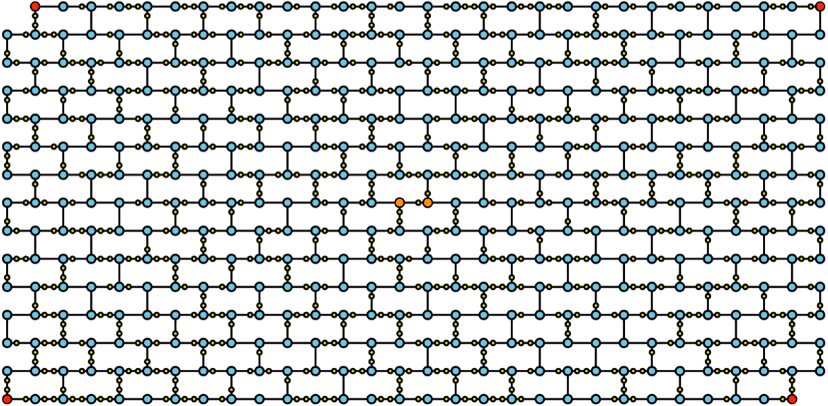


This is only possible if the wall is insulated from the exterior!

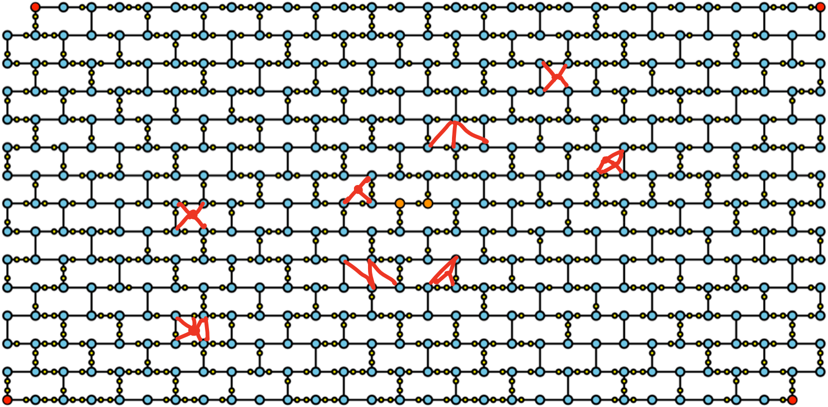
Flat walls

Goal: enrich the notion of wall so that we can insulate it from the exterior.



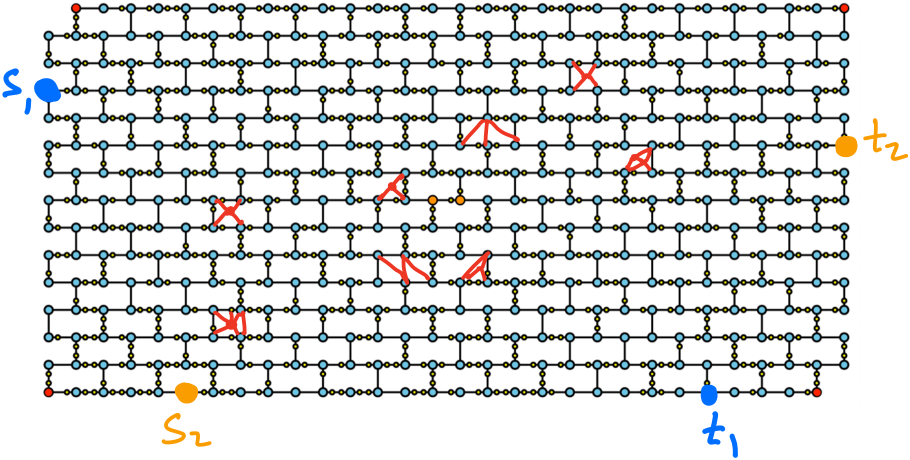
Flat walls

We need to allow some extra edges in the interior of the wall.



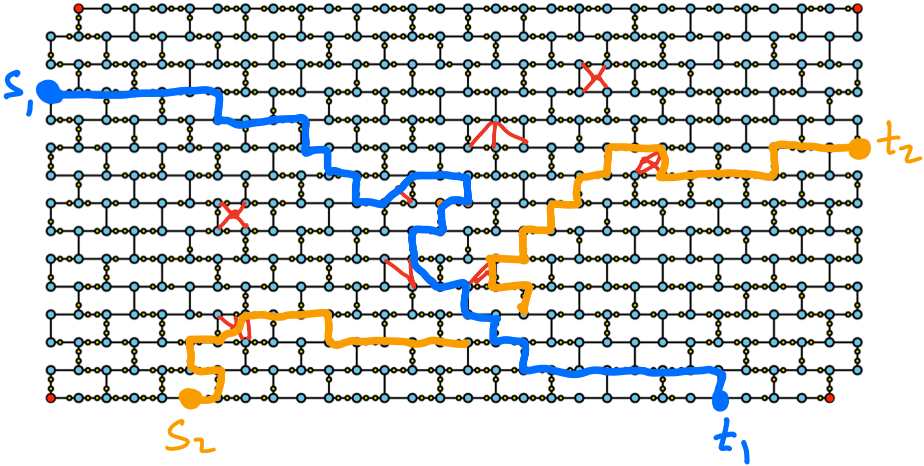
Flat walls

We impose a topological property that defines the "flatness" of the wall.



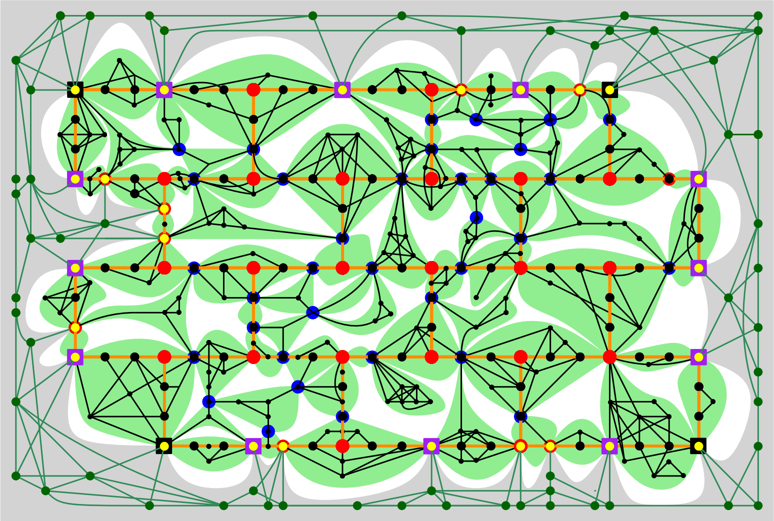
Flat walls

There are no crossing paths $s_1 - t_1$ and $s_2 - t_2$ from/to the perimeter.

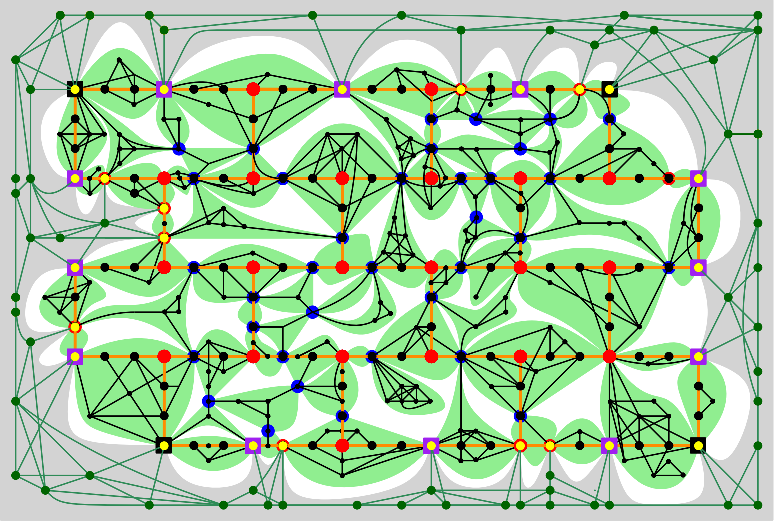


Flat walls

A real flat wall can be quite wild... [Figure by Dimitrios M. Thilikos]



Flat walls: a bit more formal



[Figures by Dimitrios M. Thilikos]

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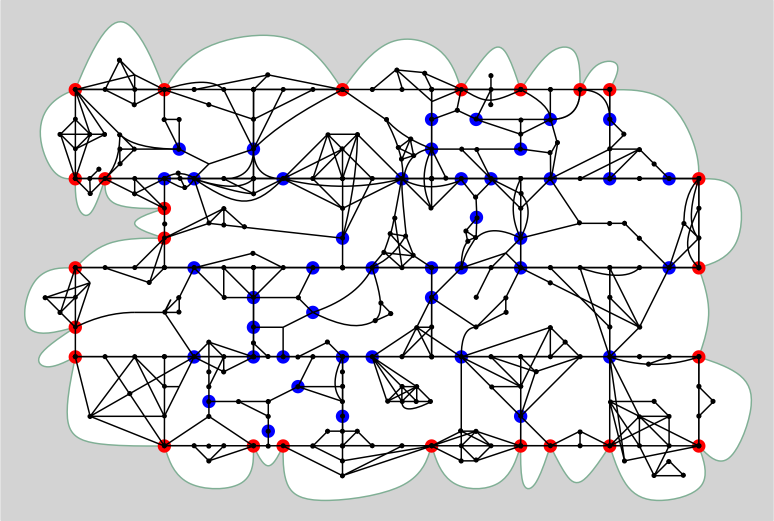

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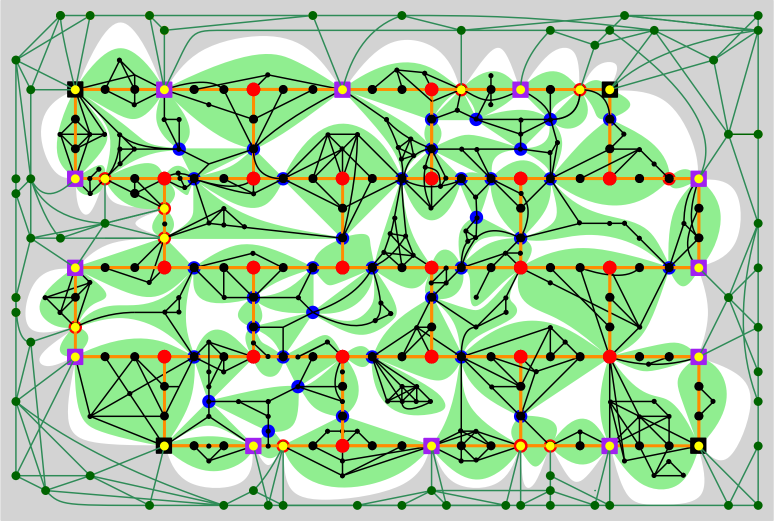

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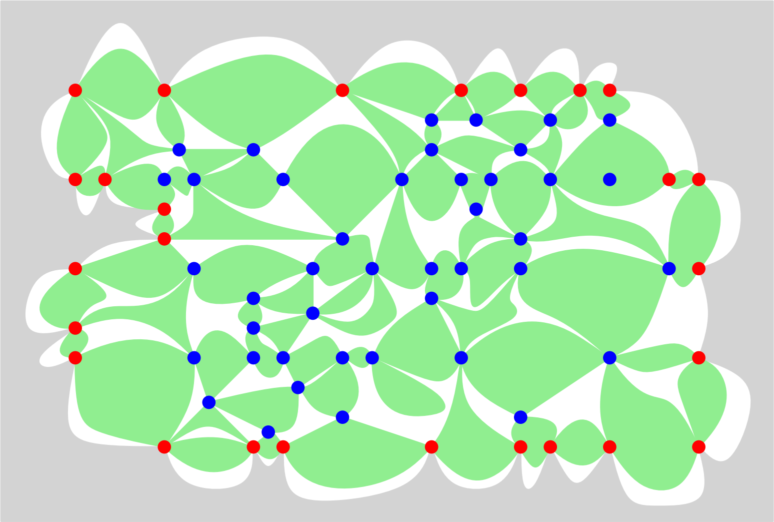

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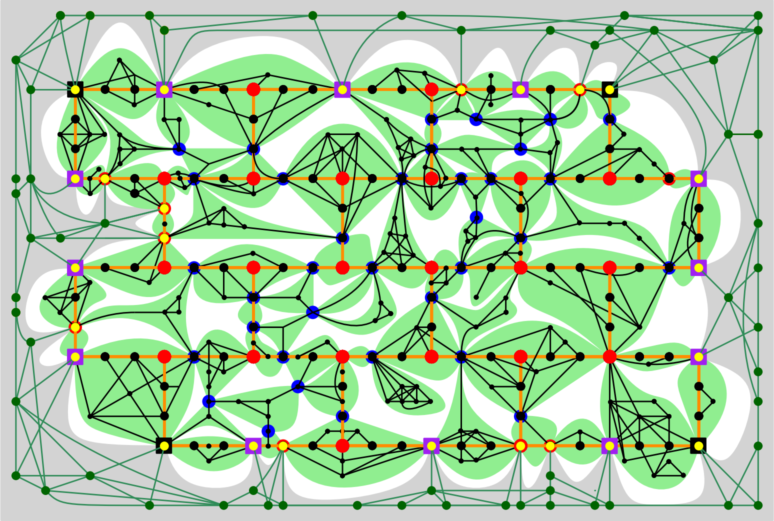

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The Flat Wall Theorem

Theorem (Robertson and Seymour. 1995)

There exist recursive functions $f_1 : \mathbb{N}^2 \rightarrow \mathbb{N}$ *and* $f_2 : \mathbb{N} \rightarrow \mathbb{N}$, *such that for every graph* G *and every* $q, r \in \mathbb{N}$, *one of the following holds:*

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There are several different variants and optimizations of this theorem...

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Important: possible to find one of the outputs in time $f(q, r) \cdot |V(G)|$.

Back to the Disjoint Paths problem

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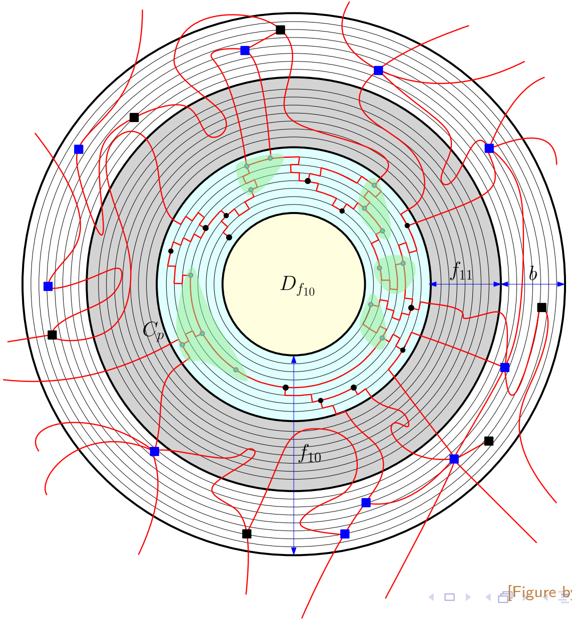
The irrelevant vertex technique has been applied to many problems... usually with a lot of technical pain.

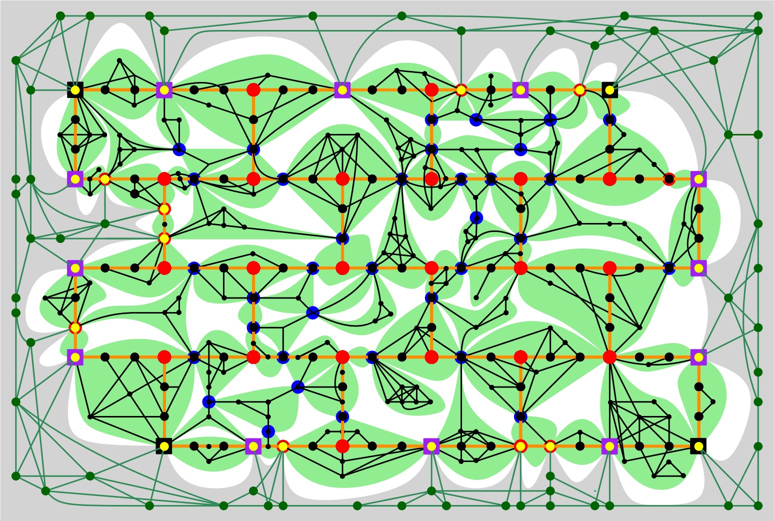
Rerouting inside a big flat wall...



[Figure by Dimitrios M. Thilikos]

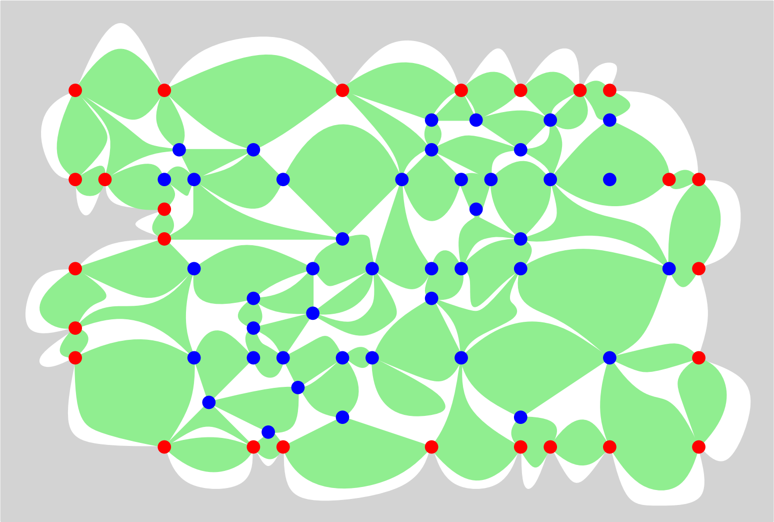
Crucial notion: homogeneity

In order to declare a vertex irrelevant for some problem, usually we need to consider a homogenous flat wall, which we proceed to define.



Crucial notion: homogeneity

We consider a flap-coloring encoding the relevant information of our favorite problem inside each flap (similar to tables of DP).



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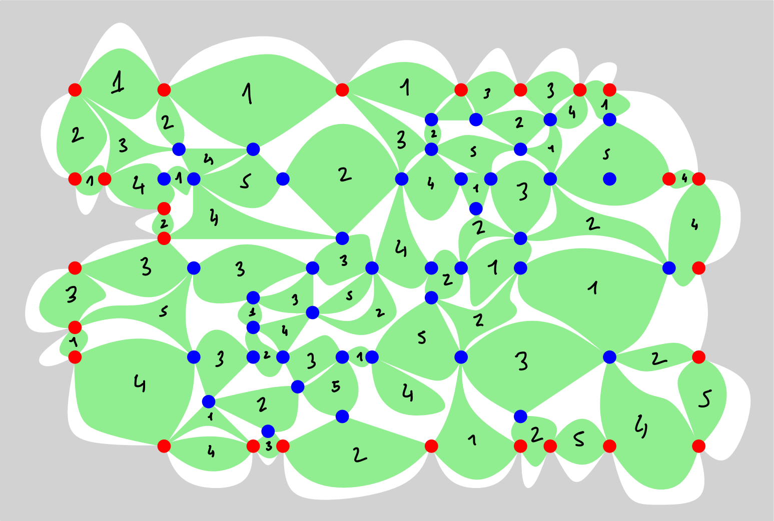
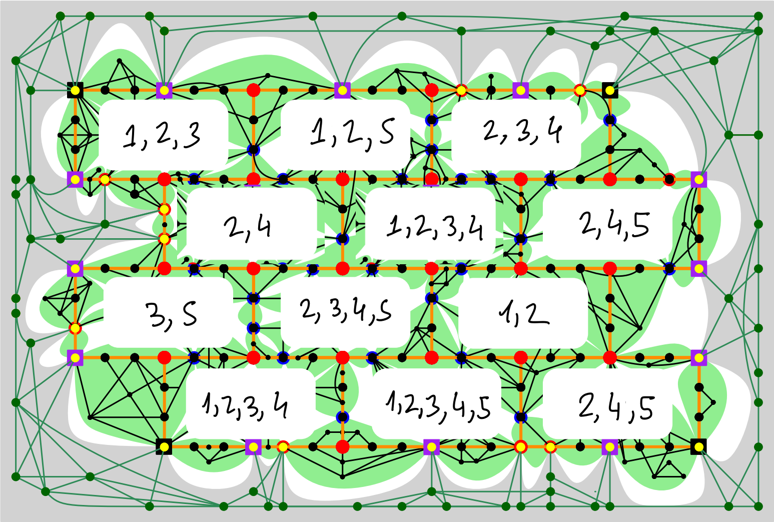


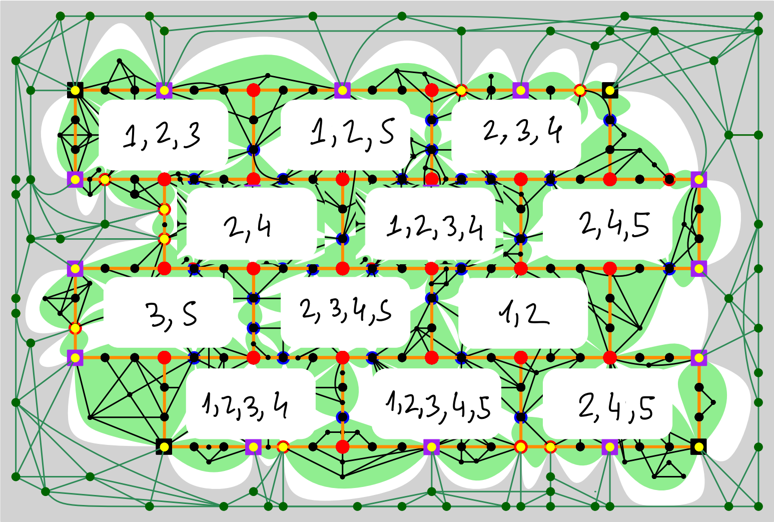
Crucial notion: homogeneity

For every brick of the wall, we define its palette as the colors appearing in the flaps it contains.



Crucial notion: homogeneity

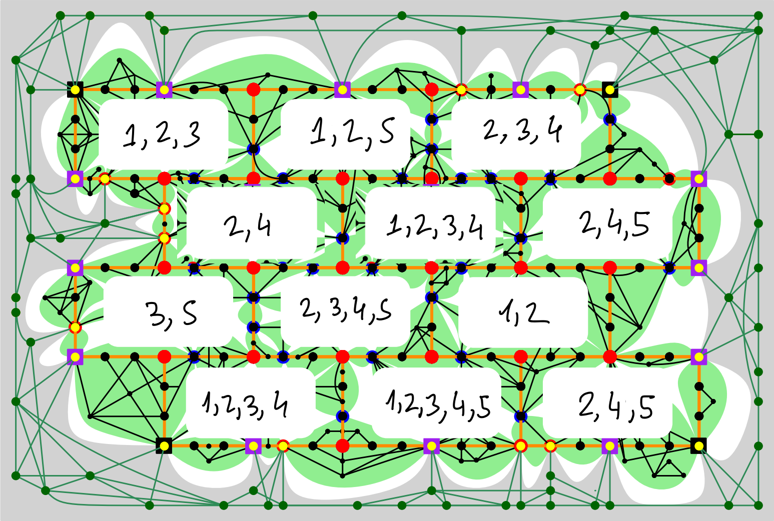
A flat wall is homogenous if every (internal) brick has the same palette.
Fact: every brick of a homogenous flat wall has the same "behavior".



Crucial notion: homogeneity

Price of homogeneity to obtain a homogenous flat r-wall (zooming):

If we have c colors, we need to start with a flat r^c-wall. (why?)



Next subsection is...

Recall the statement of the problem

Let $\mathcal{F}$ be a fixed finite collection of graphs.

$\mathcal{F}$-M-DELETION	
Input:	A graph G and an integer k.
Parameter:	k.
Question:	Does G contain a set $S \subseteq V(G)$ with $\lvert S\rvert \leqslant k$ such that $G \setminus S$ does not contain any of the graphs in $\mathcal{F}$ as a minor?

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Theorem (S., Stamoulis, Thilikos. 2020)

For all $\mathcal{F}$, the $\mathcal{F}$-M-Deletion problem can be solved in time $2^{\mathsf{poly}(k)} \cdot n^3$.

General scheme of the algorithm:

[whole slide shamelessly borrowed from Giannos Stamoulis]

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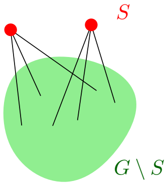

Iterative compression: given solution S of size $k + 1$, search solution of size k.

General scheme of the algorithm:

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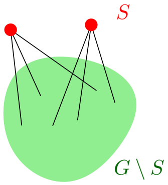

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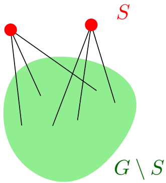

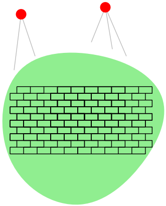

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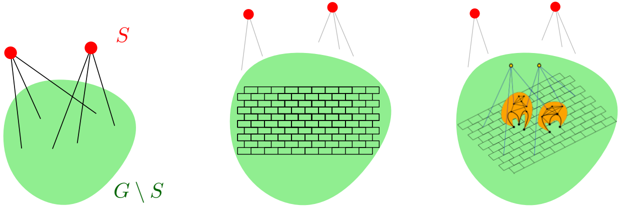

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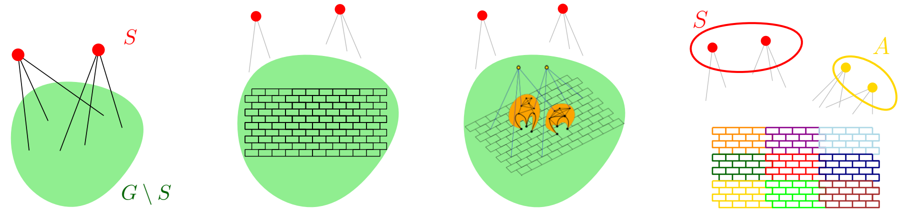

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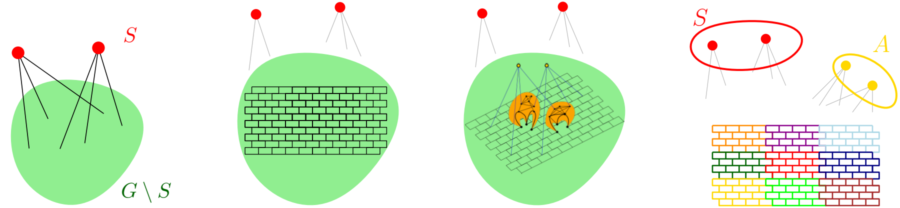

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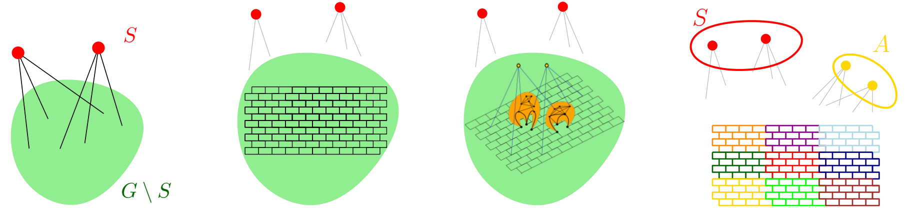

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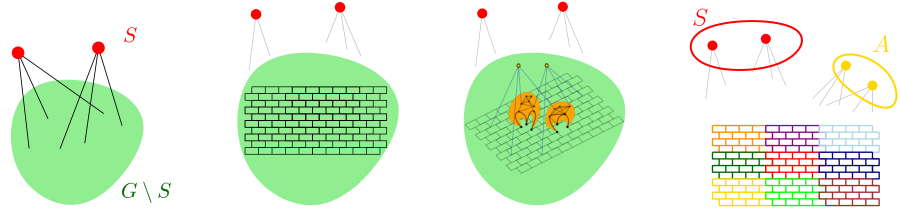

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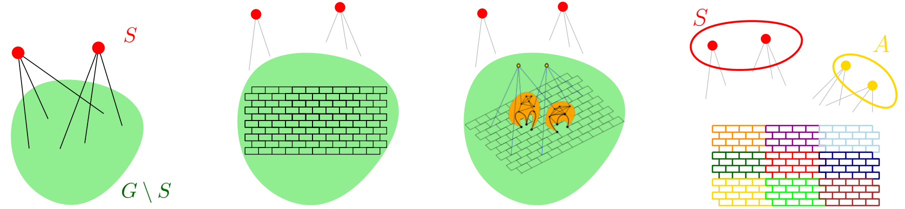

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     Find an irrelevant vertex v inside this flat subwall.
     Update $G = G \setminus v$ and repeat.

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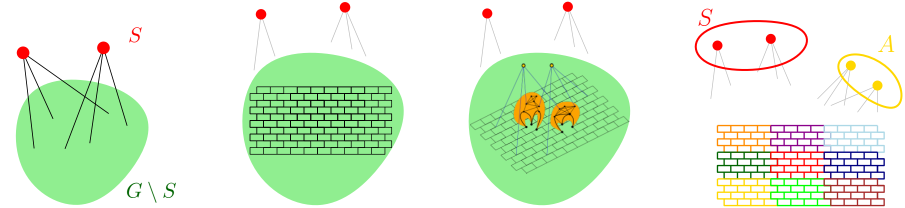

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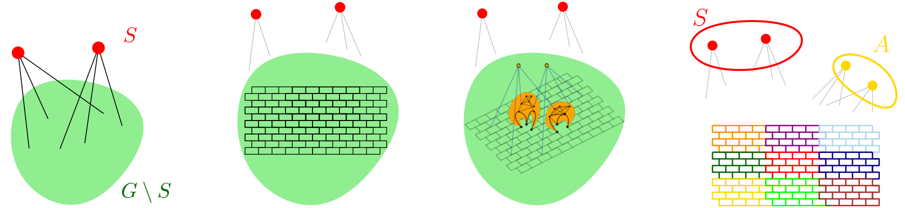
     Update $G = G \setminus v$ and repeat.

Thus, $\mathrm{tw}(G \setminus S) = k^{\mathcal{O}_{\mathcal{F}}(1)}$.

General scheme of the algorithm:

[whole slide shamelessly borrowed from Giannos Stamoulis]



Iterative compression: given solution S of size $k+1$, search solution of size k.

If treewidth of $G \setminus S$ is "large enough" (as a polynomial function of k):

1. Find a "very very large" wall in $G \setminus S$.
2. Find a "very large" flat wall W of $G \setminus S$ with few apices A.
3. Find in W a packing of $\mathcal{O}_{\mathcal{F}}(k^4)$ disjoint "large" subwalls:
   - If every subwall has at least $|A|+1$ neighbors in $S \cup A$:
     Every solution intersects $S \cup A$ $\rightarrow$ we can branch!
   - If one of these subwalls has at most $|A|$ neighbors in $S \cup A$:
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Thus, $\mathrm{tw}(G \setminus S) = k^{\mathcal{O}_{\mathcal{F}}(1)}$: our previous FPT algo gives $2^{k^{\mathcal{O}_{\mathcal{F}}(1)}} \cdot n^2$.

Main idea of our improved algorithm

Theorem (Morelle, S., Stamoulis, Thilikos. 2022)

For all $\mathcal{F}$, the $\mathcal{F}$-M-DELETION problem can be solved in time $2^{\text{poly}(k)} \cdot n^2$.

Improvement from n^3 to n^2: avoiding iterative compression.

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Improvement from n^3 to n^2: avoiding iterative compression.

How to achieve it?

We are able to detect a vertex that must belong to every solution.

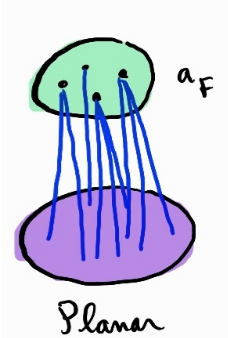
Approach inspired by [Marx, Schlotter. 2012] [S., Stamoulis, Thilikos. 2020]

skip

Finding a vertex belonging to every solution of size k

Let $\mathcal{F}$ be a finite collection of graphs.

The **apex number** $a_{\mathcal{F}}$ is the smallest number of vertices that can be removed from a graph of $\mathcal{F}$ such that the remaining graph is planar.



[Figure by Laure Morelle]

$a_{\mathcal{F}} = 1 \rightarrow$ apex graph

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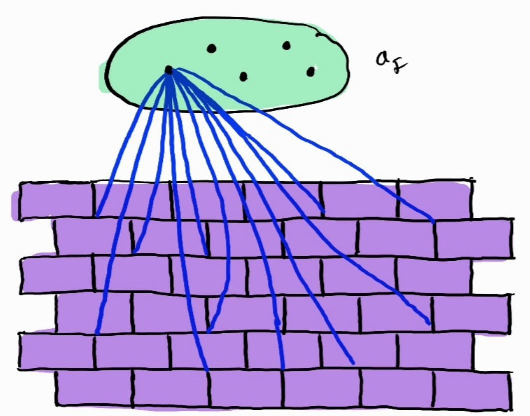

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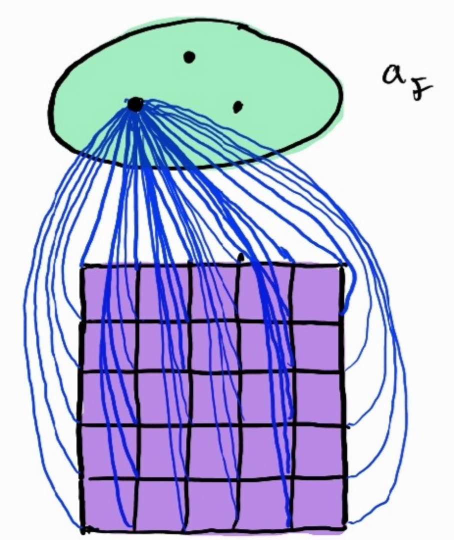

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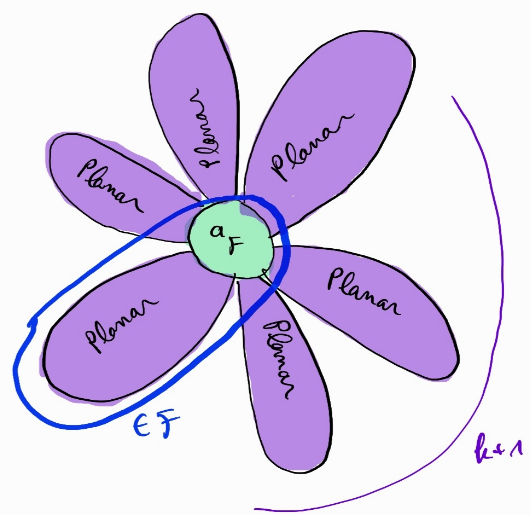

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    (Branching tree is of size $a_{\mathcal{F}}^k$, so we do *not* get an extra factor n).

skip

Next section is...

Motivation: distance from triviality

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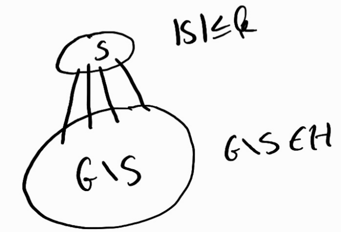

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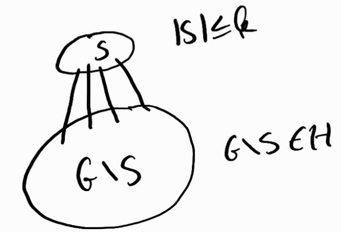

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The **elimination distance** of a graph G to a graph class $\mathcal{H}$ is:

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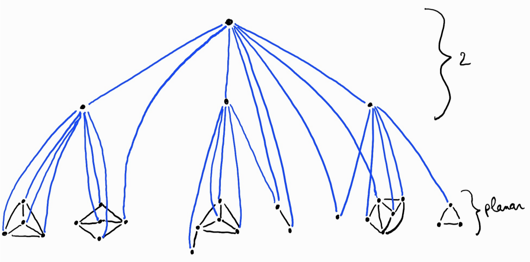

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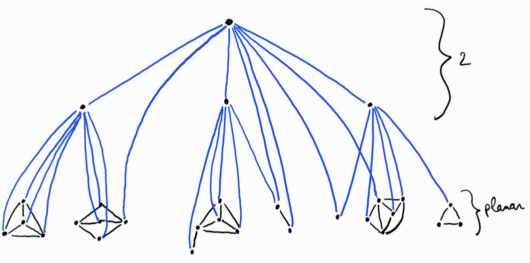

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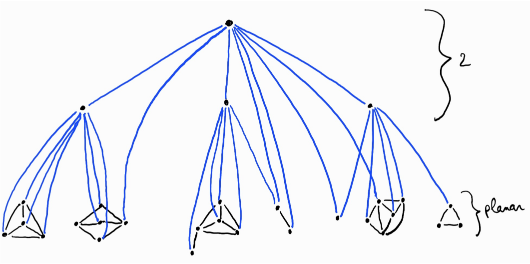

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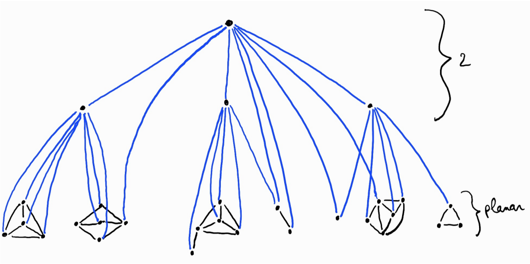

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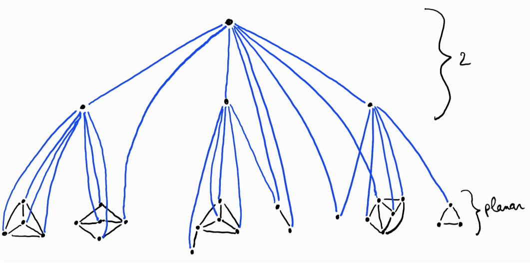

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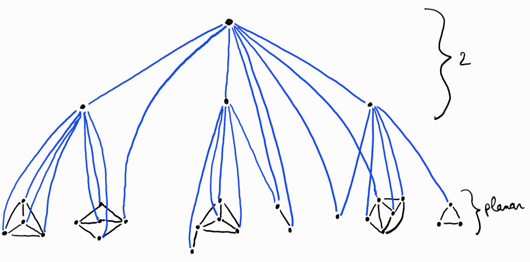

[Figure by Laure Morelle]

Elimination Distance to $\mathcal{H}$

Input: A graph G and a $k \in \mathbb{N}$.

Question: Is $\text{ed}_{\mathcal{H}}(G) \leq k$?

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Can we provide an explicit function $f(k)$?

Taking the treewidth as the parameter

If $\mathcal{H} = \{\emptyset\}$ (treedepth): [Reidl, Rossmanith, Sanchez Villaamil, Sikdar. 2014]

Dynamic programming algorithm parameterized by treewidth in $2^{\mathcal{O}(k \cdot \mathrm{tw})} \cdot n$.

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Theorem (Morelle, S., Stamoulis, Thilikos. 2022)

Given a graph G on n vertices and with treewidth at most tw, and $k \in \mathbb{N}$, there is an algorithm that solves Elimination Distance to $\mathcal{H}$ *for the instance (G, k) in time $2^{\mathcal{O}_{\mathcal{H}}(k \cdot \mathsf{tw} + \mathsf{tw} \log \mathsf{tw})} \cdot n$.*

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$\to$ algorithm in time $n^{\mathcal{O}_{\mathcal{H}}(\mathsf{tw}^2)}$ for Elimination Distance to $\mathcal{H}$.

Our results for Elimination Distance to $\mathcal{H}$

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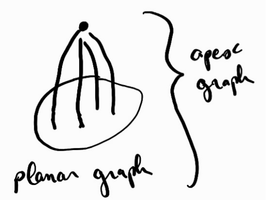

[Figure by Laure Morelle]

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If $\mathrm{Obs}(\mathcal{H})$ contains an apex graph, given a graph G on n vertices and $k \in \mathbb{N}$, there is an algorithm that solves Elimination Distance to $\mathcal{H}$ *for the instance (G, k) in time $2^{\mathrm{poly}_{\mathcal{H}}(k)} \cdot n^3$.*

Our results for ELIMINATION DISTANCE TO $\mathcal{H}$

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Given a graph G on n vertices and $k \in \mathbb{N}$, there is an algorithm that solves ELIMINATION DISTANCE TO $\mathcal{H}$ *for the instance (G, k) in time*

- $2^{2^{2^{\mathrm{poly}_{\mathcal{H}}(k)}}} \cdot n^2$ *for a general minor-closed class $\mathcal{H}$,*
- $2^{2^{\mathrm{poly}_{\mathcal{H}}(k)}} \cdot n^2$ *if $\mathrm{Obs}(\mathcal{H})$ contains an apex graph.*

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Main challenge compared to VERTEX DELETION TO $\mathcal{H}$:

The size of a k-elimination set may be unbounded, so we cannot branch!

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Main challenge compared to Vertex Deletion to $\mathcal{H}$:

The size of a k-elimination set may be unbounded, so we cannot branch!
We always have to find an irrelevant vertex: larger treewidth bounds.

Next section is...

What's next about $\mathcal{F}$-M-Vertex-Deletion?

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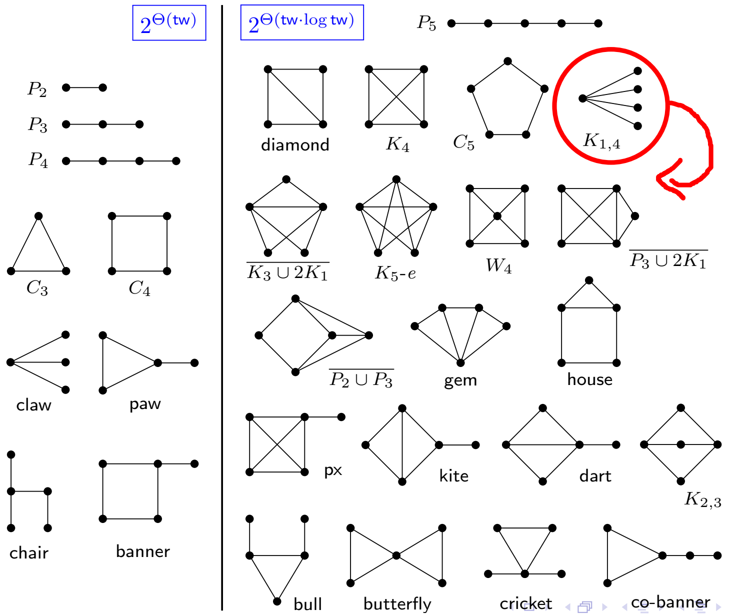
Is the price of homogeneity unavoidable?

skip

For topological minors, there is (at least) one change



Gràcies!