Algorithmes paramétrés efficaces

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Équipe **AlGCo**
Algorithmes, **G**raphes et **Co**mbinatoire

CIEL
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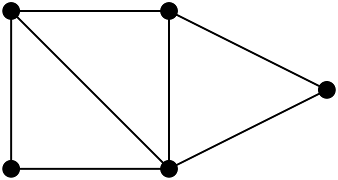
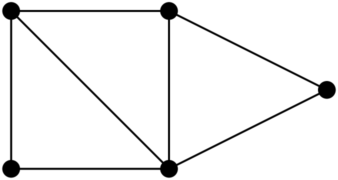
Graph theory



Context

Graph theory ➡ **Graph algorithms**



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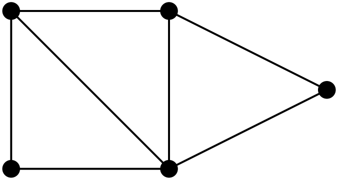
Graph theory ➡ **Graph algorithms** ➡ **Parameterized complexity**



Outline of the talk

1. Introduction
    - Parameterized complexity
    - Treewidth
2. FPT algorithms parameterized by treewidth
3. The $\mathcal{F}$-Deletion problem
4. Further research

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Crucial notion in complexity theory: NP-completeness

- Cook-Levin Theorem (1971): the SAT problem is NP-complete.
- Karp (1972): list of 21 *important* NP-complete problems.
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- Nowadays, literally thousands of problems are known to be NP-hard: unless $\mathsf{P} = \mathsf{NP}$, they cannot be solved in polynomial time.
- But what does it mean for a problem to be NP-hard?

  No algorithm solves all instances optimally in polynomial time.

Are all instances really hard to solve?

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- VLSI design: the number of circuit layers is usually ≤ 10.
- Computational biology: Real instances of DNA chain reconstruction usually have treewidth ≤ 11.
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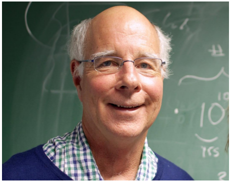
$\boxed{\text{Message}}$ In many applications, not only the total size of the instance matters, but also the value of an additional parameter.

The area of parameterized complexity

$\boxed{\text{Idea}}$ Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established area with hundreds of articles published every year in the most prestigious TCS journals and conferences.

Parameterized problems

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These three problems are NP-hard, but are they $\boxed{\text{equally}}$ hard?

They behave quite differently...

- k-VERTEX COVER: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- k-CLIQUE: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
- VERTEX k-COLORING: NP-hard for fixed $k = 3$.

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- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n)) = f(k) \cdot n^{\mathcal{O}(1)}$.

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Working hypothesis of parameterized complexity: $\boxed{k\text{-CLIQUE is not FPT}}$

(in classical complexity: 3-SAT cannot be solved in poly-time)

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Do all FPT problems admit polynomial kernels? $\boxed{\text{NO!}}$

Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)

Deciding whether a graph has a PATH *with $\geq k$ vertices is* FPT *but does not admit a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

Typical approach to deal with a parameterized problem

Parameterized problem L

- k-Clique
- k-Vertex Cover
- k-Path
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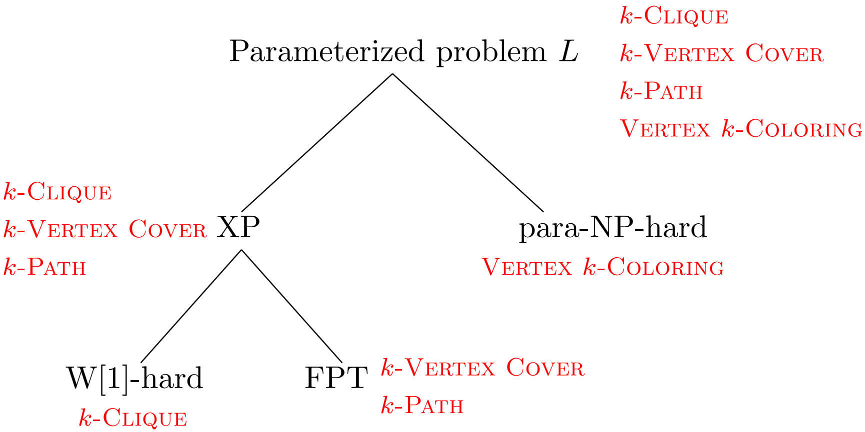

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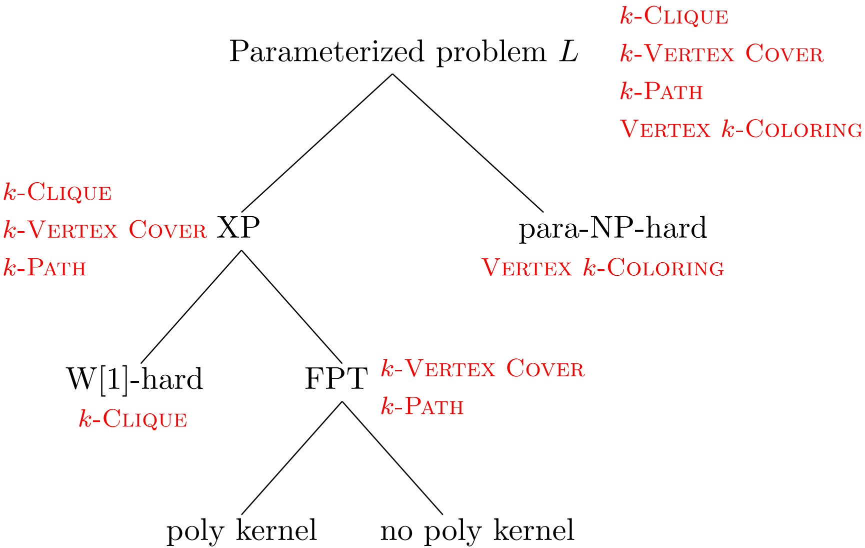

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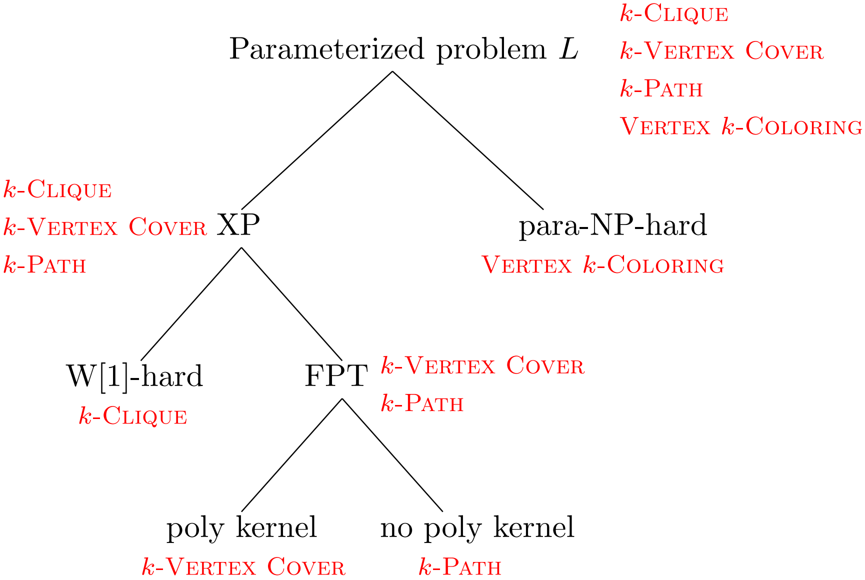
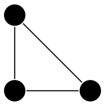


Next subsection is...

Treewidth via k-trees

Example of a 2-tree:



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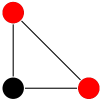

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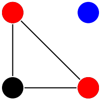

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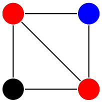

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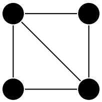

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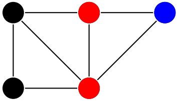

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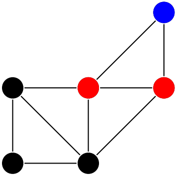

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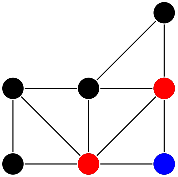

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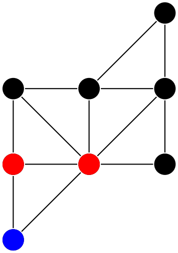

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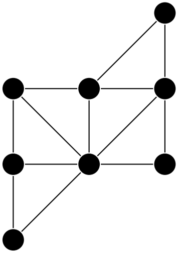

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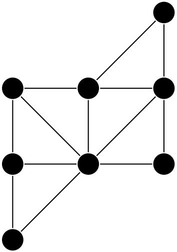

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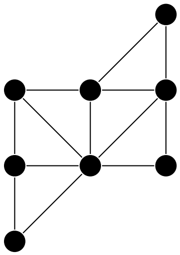

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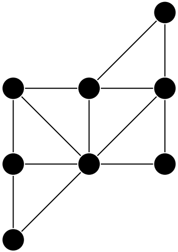

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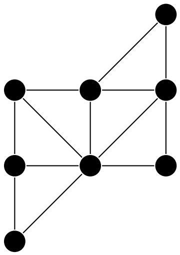

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Construction suggests the notion of tree decomposition: small separators.

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3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

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Graph logic that allows quantification over sets of vertices and edges.

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   Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

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Remark: Algorithms parameterized by treewidth appear very often as a "black box" in all kinds of parameterized algorithms.

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Typical statements:

ETH $\Rightarrow$ k-Vertex Cover cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$.

ETH $\Rightarrow$ Planar k-Vertex Cover cannot in time $2^{o(\sqrt{k})} \cdot n^{O(1)}$.

Dynamic programming on tree decompositions

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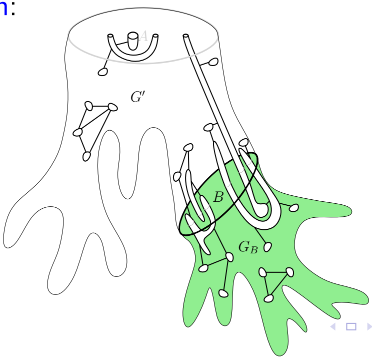
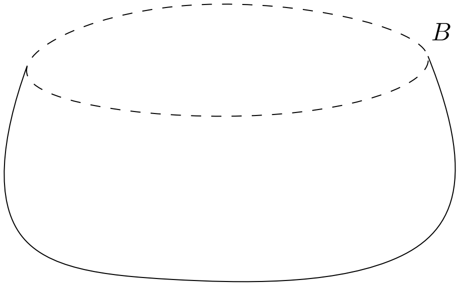
- Typically, FPT algorithms parameterized by treewidth are based on dynamic programming (DP) over a tree decomposition.
- Starting from the leaves of the tree decomposition, a set of appropriately defined partial solutions is computed recursively until the root, where a global solution is obtained.
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Two behaviors for problems parameterized by treewidth

Local problems: Vertex Cover, Dominating Set, Clique, Independent Set, q-Coloring for fixed q.



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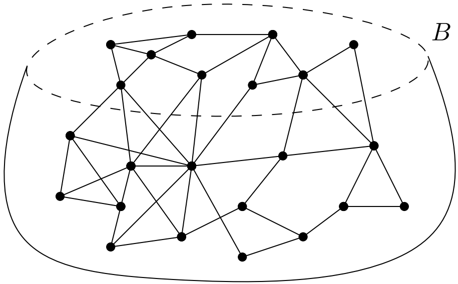

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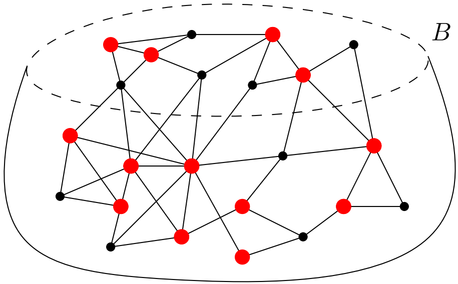

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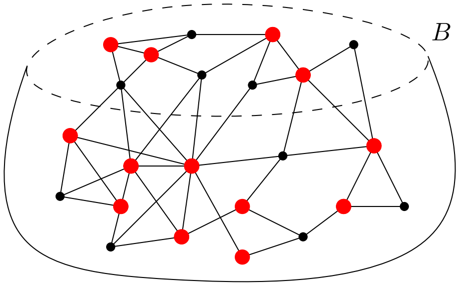

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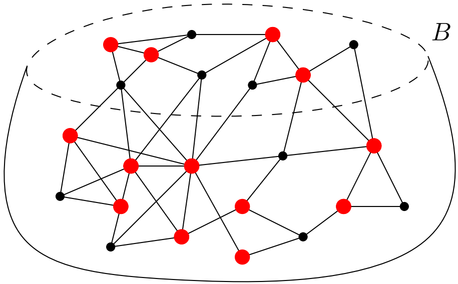

- It is sufficient to store, for each bag B, the subset of vertices of B that belong to a partial solution: $\boxed{2^{\text{tw}} \text{ choices}}$
- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

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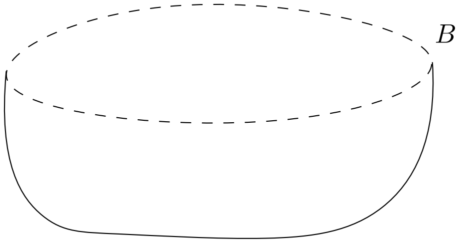
Connectivity problems seem to be more complicated...

Connectivity problems HAMILTONIAN CYCLE, LONGEST PATH, STEINER TREE, CONNECTED VERTEX COVER.



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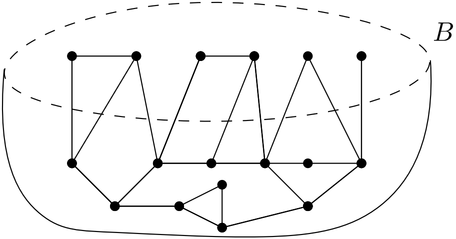

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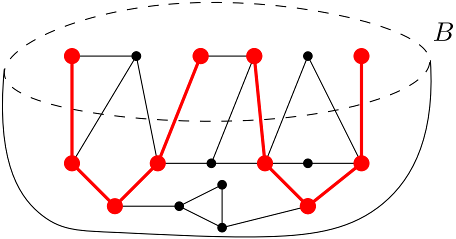

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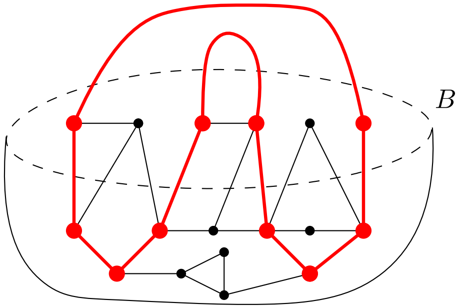

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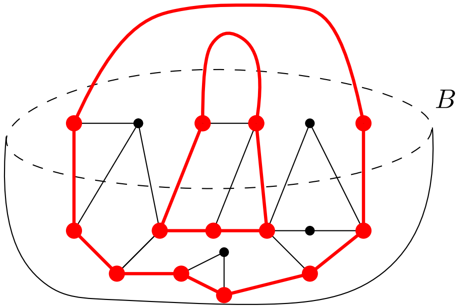

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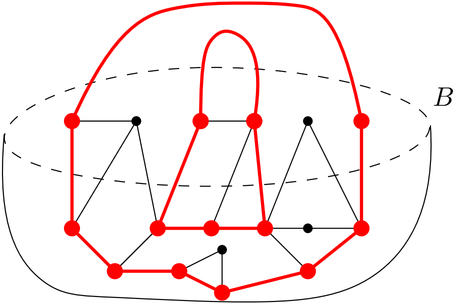

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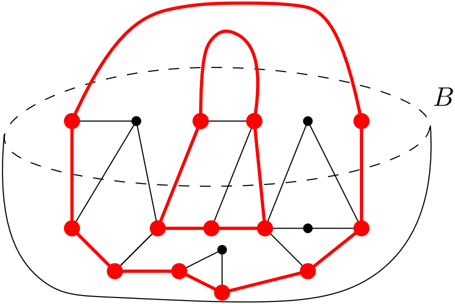

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- The "natural" DP algorithms provide only time $2^{\mathcal{O}(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$.

Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

$$\boxed{2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

Vertex Cover, Dominating Set, ...

- Connectivity problems:

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Longest Path, Steiner Tree, ...

The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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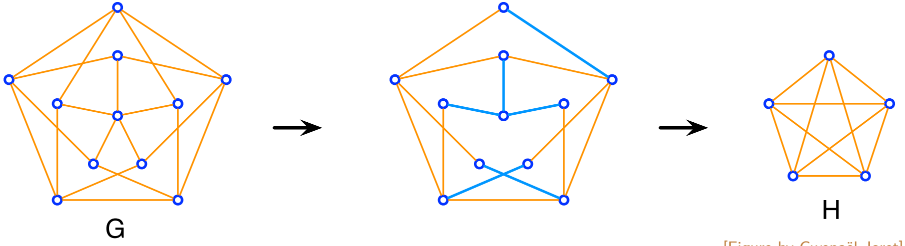
There are other examples of such problems...

Next section is...

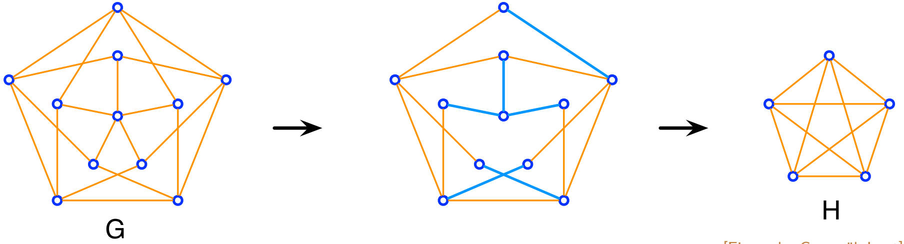
Minors and topological minors



[Figure by Gwenaël Joret]

- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

Minors and topological minors



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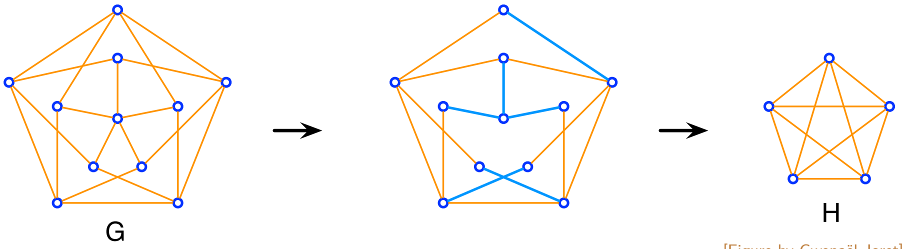

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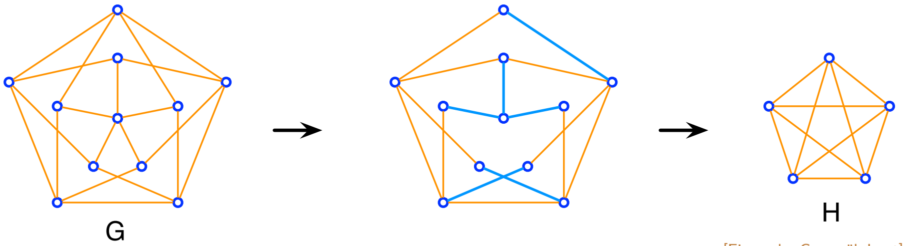

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  "Hardly" solvable in time $2^{\Theta(\text{tw})} \cdot n^{\mathcal{O}(1)}$. [Cut&Count. 2011]

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- $\mathcal{F} = \{K_2\}$: Vertex Cover.
  Easily solvable in time $2^{\Theta(\text{tw})} \cdot n^{\mathcal{O}(1)}$.
- $\mathcal{F} = \{C_3\}$: Feedback Vertex Set.
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The $\mathcal{F}$-M-Deletion problem

Let $\mathcal{F}$ be a fixed finite collection of graphs.

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FPT by Courcelle's Theorem.

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-Deletion/$\mathcal{F}$-TM-Deletion can be solved in time

$$f_{\mathcal{F}}(\mathsf{tw}) \cdot n^{\mathcal{O}(1)}$$

on n-vertex graphs.

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- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

Summary of our results

[1] Connected collection $\mathcal{F}$: all the graphs are connected.

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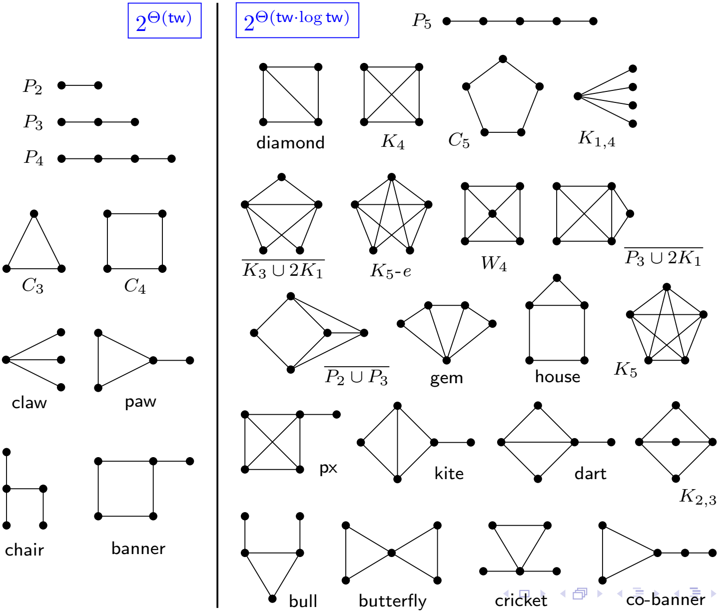
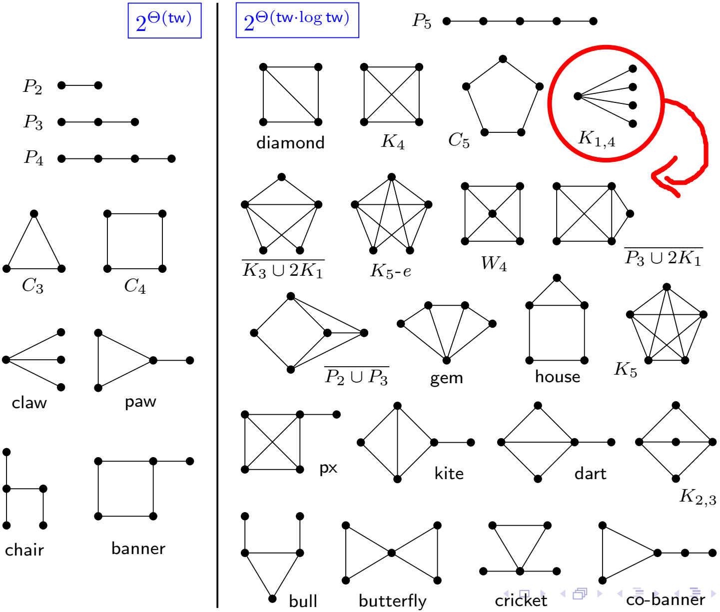
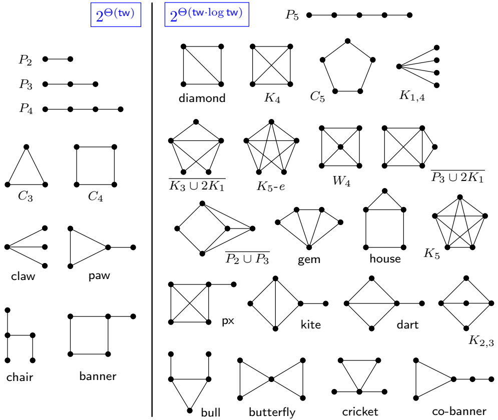
Complexity of hitting a single connected minor H



For topological minors, there is (at least) one change



A compact statement for a single connected graph



All these cases can be succinctly described as follows:

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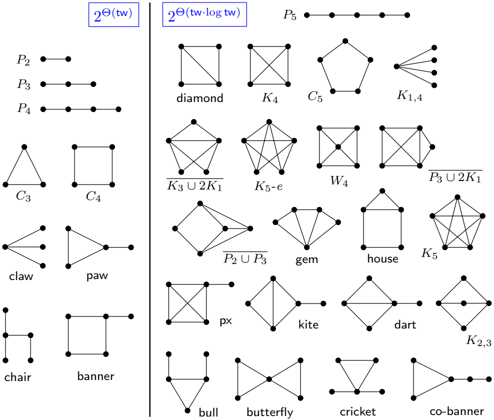
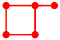
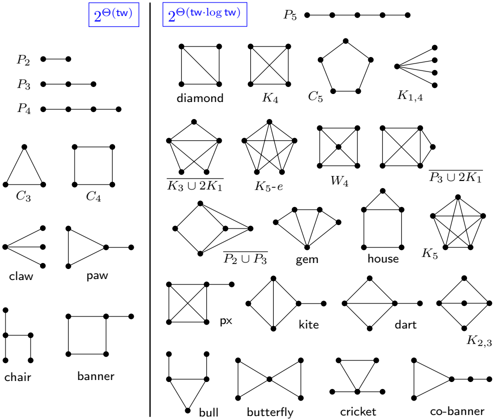


All these cases can be succinctly described as follows:

- All the graphs on the left are minors of  (called the banner)

A compact statement for a single connected graph



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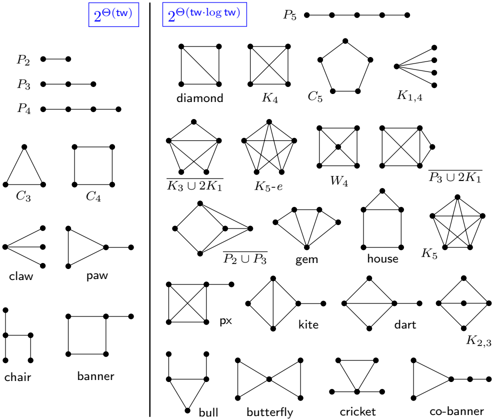
A compact statement for a single connected graph



All these cases can be succinctly described as follows:

- All the graphs on the left are minors of  (called the banner)
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A dichotomy for hitting connected minors

We can prove that any connected H with $|V(H)| \geq 6$ is $\boxed{\text{hard}}$:
$\{H\}$-M-Deletion cannot be solved in time $2^{o(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ under the ETH.

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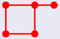
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{m}}$  *and* $H \neq P_5$.

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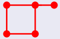
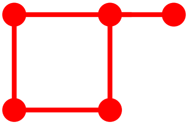
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_m$  *and* $H \neq P_5$.
- $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

In both cases, the running time is asymptotically optimal under the ETH.

skip

Why the banner??



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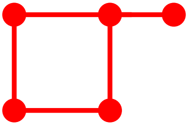

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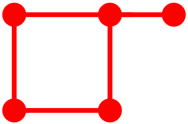

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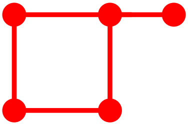

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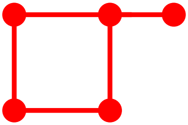

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- Both such types of components can be maintained by a dynamic programming algorithm in single-exponential time.
- If the characterization of the allowed connected components is enriched in some way, such as restricting the length of the allowed cycles or forbidding certain degrees, the problem becomes harder.

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1. $\boxed{\text{General algorithms}}$

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3. Lower bounds under the ETH

   - $2^{o(\mathrm{tw})}$ is "easy".
   - $2^{o(\mathrm{tw} \cdot \log \mathrm{tw})}$ is much more involved and we get ideas from:

     [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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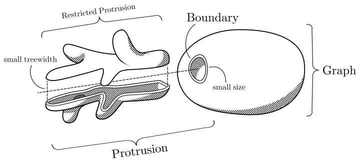

[Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh, Thilikos. 2009]

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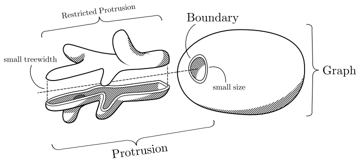

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  Extra: Bidimensionality, irrelevant vertices, protrusion decomposition...

skip

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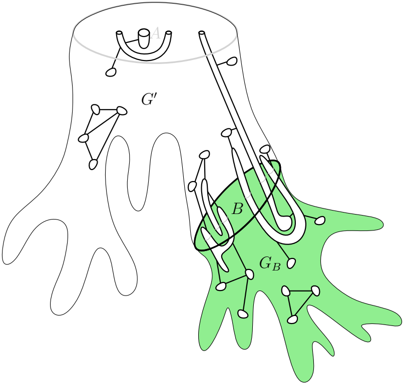
- We see G as a t-boundaried graph.



Algorithm for a general collection $\mathcal{F}$

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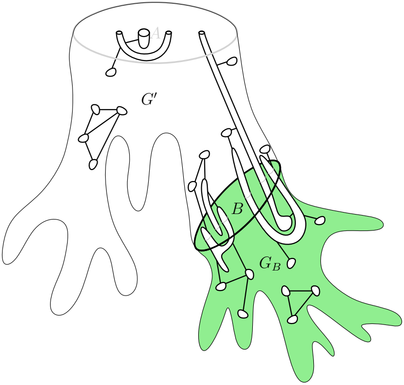

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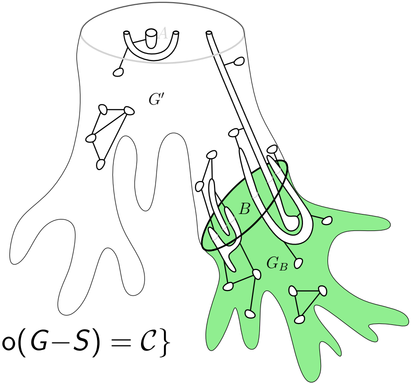

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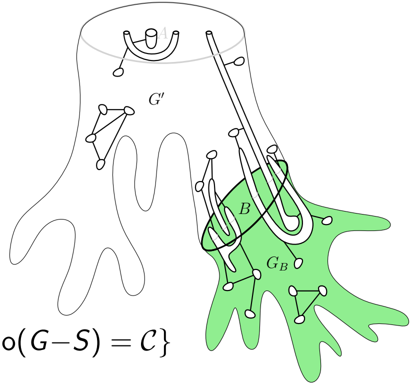

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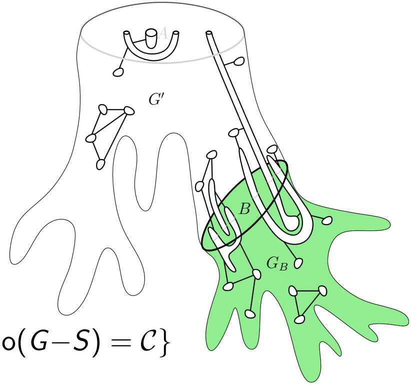

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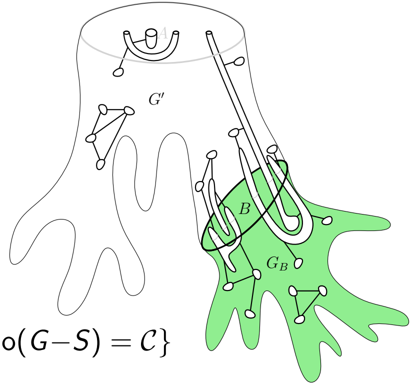

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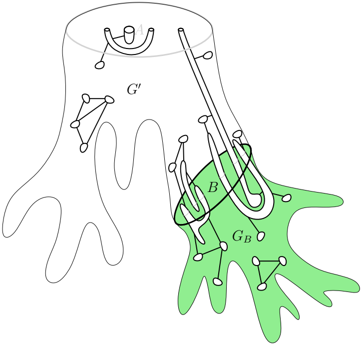

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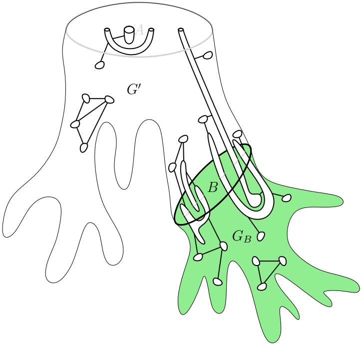

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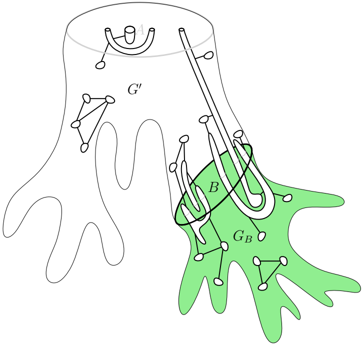

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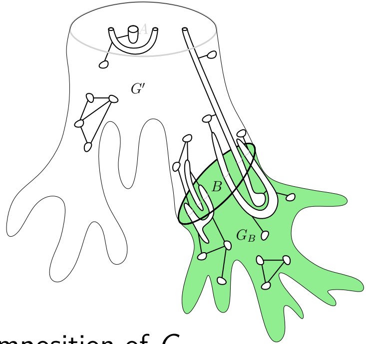

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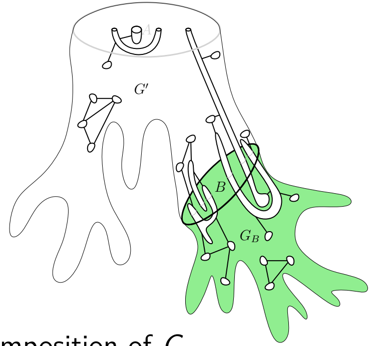

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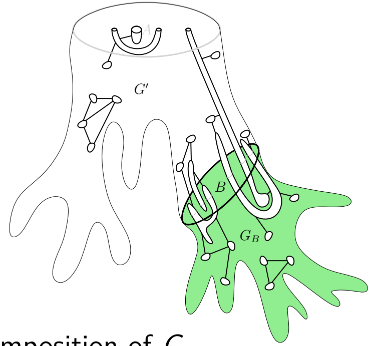

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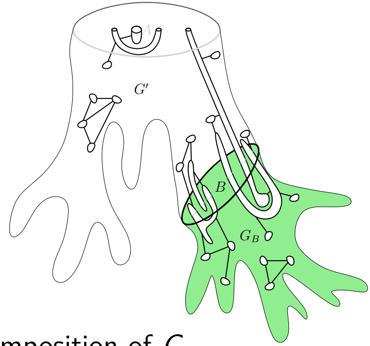

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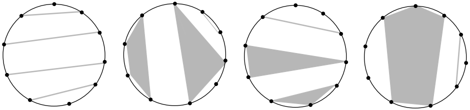

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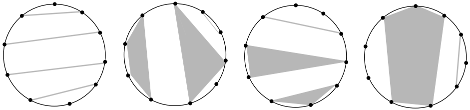

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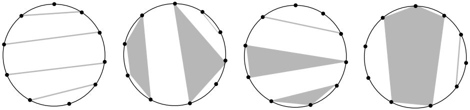

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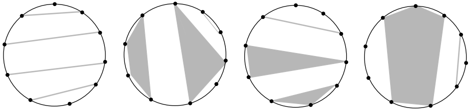

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. [Rué, S., Thilikos. 2014]

Next section is...

1. Introduction
   - Parameterized complexity
   - Treewidth
2. FPT algorithms parameterized by treewidth
3. The $\mathcal{F}$-Deletion problem
4. Further research

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Gràcies!