Optimal Erdős-Pósa property for pumpkins

Samuel Fiorini[1] Gwenaël Joret[1] Ignasi Sau[2]

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CSASC 2013. *Koper, Slovenia*

Outline of the talk

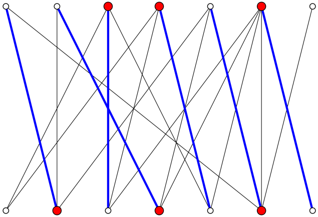
1. Motivation
2. Our result
3. Sketch of proof
4. Further research

Next section is...

Packing and covering

König's min-max theorem in bipartite graphs:

MIN VERTEX COVER = MAX MATCHING



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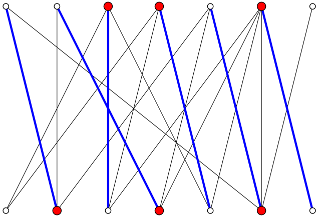

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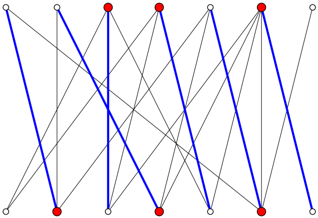

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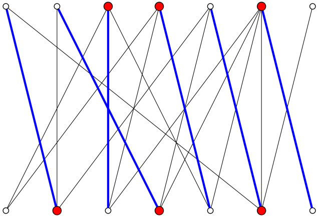

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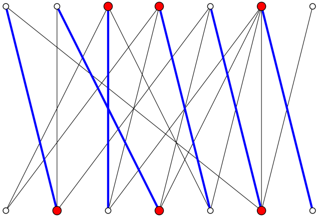

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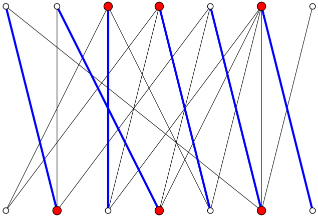

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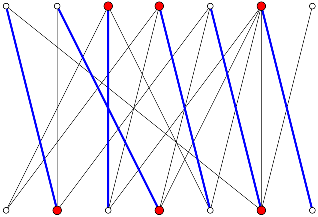

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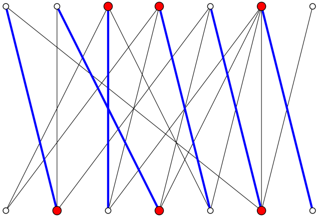

If there exists such f for all G, then $\mathcal{H}$ satisfies the **Erdős-Pósa property**.

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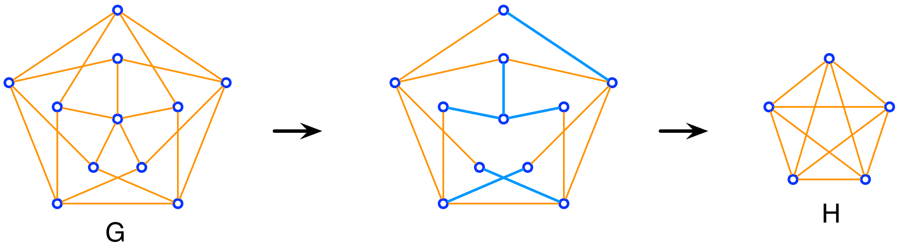
Minors and models in graphs



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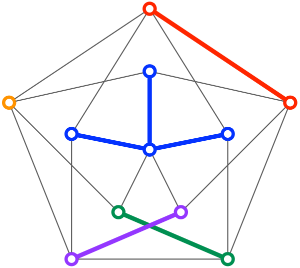
H-model in G: collection $\{S_u : u \in V(H)\}$ s.t.

- the S_u's are vertex-disjoint connected subgraphs of G, and
- there is an edge between S_u and S_v in G for every edge $uv \in E(H)$.



A K_5-model

The S_u's are called vertex images.

Packing and covering H-models

Let H be a **fixed** graph. For a graph G, we define:

$\nu_H(G)$:= packing number
$= $ max. number of vertex-disjoint H-models in G.

$\tau_H(G)$:= covering (or transversal) number
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Clearly, $\tau_H(G) \geqslant \nu_H(G) \quad \forall G.$

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This is called the **Erdős-Pósa property of H-minors**.

Erdős-Pósa property of H-minors

Fundamental result:

$$\tau_H(G) \leqslant f(\nu_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is } \textbf{planar}$$

[Robertson, Seymour ' 86]

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- The derived upper bounds $\tau_H \leqslant f(\nu_H)$ are **huge**: $f(\nu_H) = \Omega(2^{\nu_H{}^2})$.

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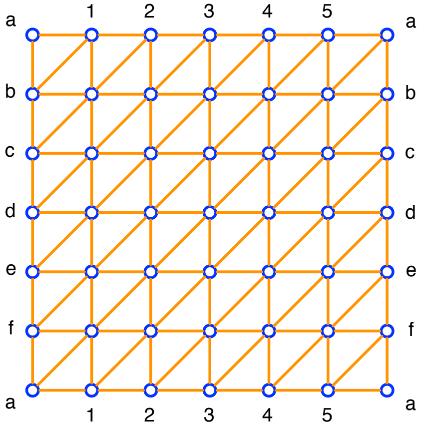
- **Natural objective**: optimize $f(\nu_H)$.

The property does NOT hold if H is not planar

$H = K_5$ ✗

Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid G:



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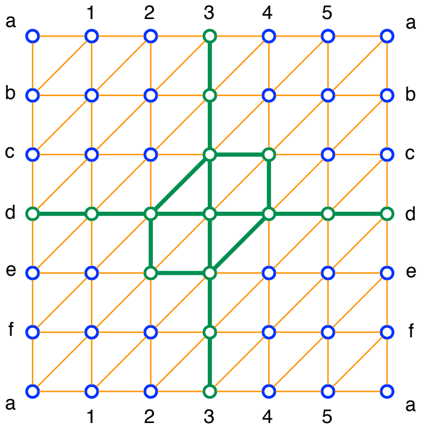

$\nu_H(G) = 1$ but $\tau_H(G) = \Theta(\sqrt{n})$

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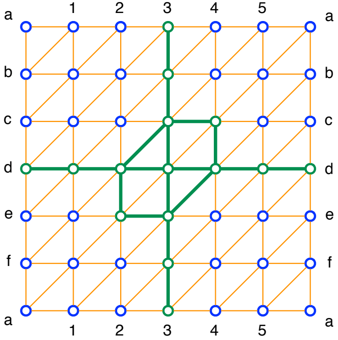
H not planar ✗



Therefore, the result of Robertson and Seymour is best possible.

Brief state of the art of Erdős-Pósa property for minors

- $\tau_H(G) \leqslant f(\nu_H(G)) \quad \forall G \quad \Leftrightarrow \quad H$ is **planar** [Robertson, Seymour ' 86]

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★ $f(k) = O(k \text{ poly} \log(k))$ for any planar graph H. [Chekuri, Chuzhoy '13]

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Theorem: $\exists f_H(k)$ s.t. $\forall G, k$, either $\nu_H(G) \geqslant k$ or $\tau_H(G) \leqslant f_H(k)$.

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- This implies that (easy to check) $\exists$ constant $b > 0$ such that $f_H(k) > b \cdot k \log k$ (i.e., $f_H(k) = \Omega(k \log k)$).

Summarizing...

For any planar graph H with a cycle and a general graph G:

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- Upper bound: $f_H(k) = O(k \operatorname{poly}\log(k))$. [Chekuri, Chuzhoy '13]

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For any planar graph H with a cycle and a general graph G:

- Lower bound: $f_H(k) = \Omega(k \log k)$.
- Upper bound: $f_H(k) = O(k \log^{35} k)$. [Chekuri, Chuzhoy '13]

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For any planar graph H with a cycle and a general graph G:

- Lower bound: $f_H(k) = \Omega(k \log k)$.
- Upper bound: $f_H(k) = O(k \log^{35} k)$. [Chekuri, Chuzhoy '13]
- Only graph H for which the lower bound is attained is the triangle:

$$f_{\triangle}(k) = O(k \log k).$$

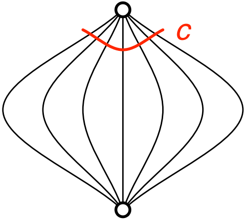
[Erdős, Pósa ' 65]

Next section is...

Pumpkins 

Pumpkins

c-pumpkin:



Can be seen as a natural generalization of a cycle.

(N.B: "graph" = multigraph)

Graphs with no c-pumpkin minor

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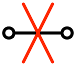

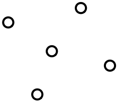

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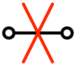
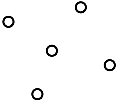
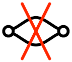
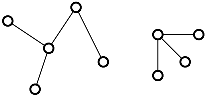




- $c = 2$: forests





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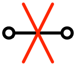

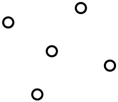

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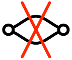
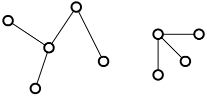
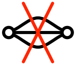
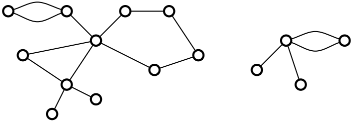




- $c = 3$: no two cycles share an edge

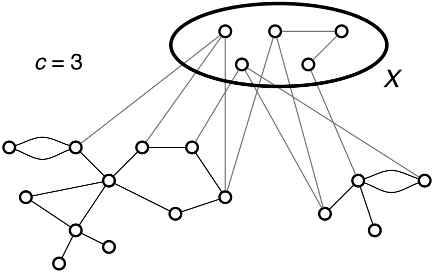




- etc.

Covering (or hitting) pumpkins

c-pumpkin hitting set:
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor



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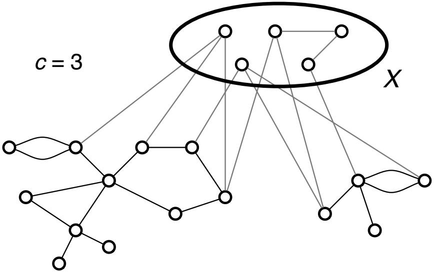
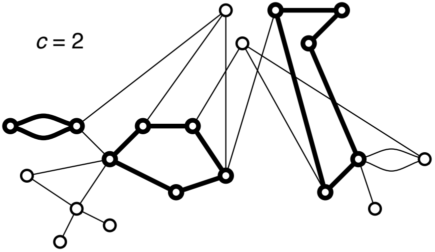


Hitting set number $\tau_c(G)$: min. size of a c-pumpkin hitting set

Packing pumpkins

c-pumpkin packing:
collection of vertex-disjoint subgraphs of G, each containing a c-pumpkin minor



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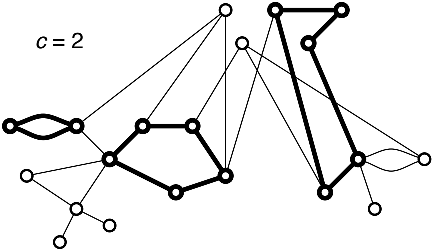

Packing number $\nu_c(G)$: max. cardinality of a c-pumpkin packing

A recent result and our main theorem

- A recent result on Erdős-Pósa property for pumpkins:

Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh '12)

For any fixed integer $c \geqslant 1$ *and given an integer* $k \geqslant 1$, *every graph* G *either contains* k *vertex-disjoint* c*-pumpkins-models, or has a* c*-pumpkin hitting set of size at most* $f(k) = O(k^2)$.

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- Our result:

Theorem (Fiorini, Joret, S. '13)

For any fixed integer $c \geqslant 1$ and given an integer $k \geqslant 1$, every graph G either contains k vertex-disjoint c-pumpkins-models, or has a c-pumpkin hitting set of size at most $f(k) = O(k \log k)$.

That is, $\tau_c \leqslant \nu_c \log \nu_c$

A recent result and our main theorem

- A recent result on Erdős-Pósa property for pumpkins:

Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh '12)

For any fixed integer $c \geqslant 1$ and given an integer $k \geqslant 1$, every graph G either contains k vertex-disjoint c-pumpkins-models, or has a c-pumpkin hitting set of size at most $f(k) = O(k^2)$. That is, $\tau_c \leqslant {\nu_c}^2$

⋆ Their proof uses tree decompositions and brambles.

- Our result:

Theorem (Fiorini, Joret, S. '13)

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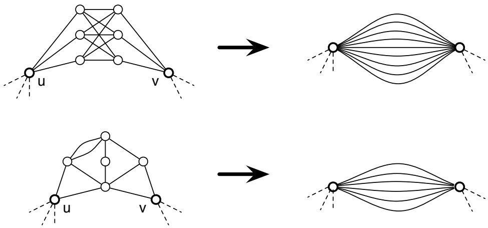
⋆ Our proof follows and generalizes Erdős-Pósa's proof for the case $c = 2$.

Next section is...

Useful reduction rules

We first need two reduction rules **R1** and **R2** dealing with 1-connected and 2-connected components without c-pumpkin minors, respectively, that preserve both $\nu_c(G)$ and $\tau_c(G)$:



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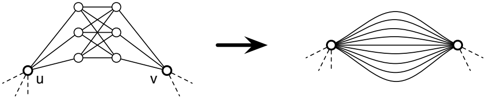

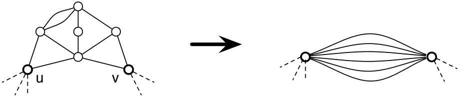

- For $c = 2$: **R1** $=$ deleting degree-1 vertices
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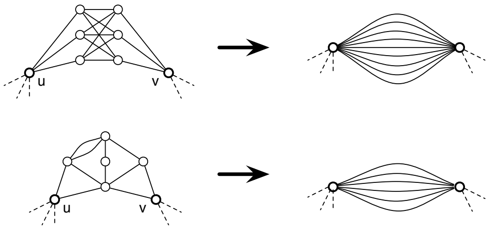

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Lemma

Let $c \geqslant 2$ be a fixed integer. Suppose that G^* results from the application of **R1** *or* **R2** *on a graph G. Then $\tau_c(G) = \tau_c(G^*)$ and $\nu_c(G) = \nu_c(G^*)$.*

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Main Lemma

If $|V(\overline{H})| \geqslant d \cdot k \log k$ for some constant d (depending only on c), then H contains k vertex-disjoint c-pumpkin-models.

Ingredients in the proof of the Main Lemma

- We prove it by induction on k, using that:

Lemma

Every n-vertex c-reduced graph G contains a c-pumpkin-model of size $O(\log n)$.

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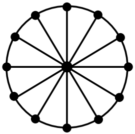
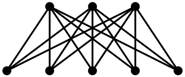
- Crucial: $\forall\ p \geqslant 0$, $\exists\ f(p)$ s.t. every 3-connected graph with $\geqslant f(p)$ vertices has a minor isomorphic to: [Oporowski, Oxley, Thomas '93]

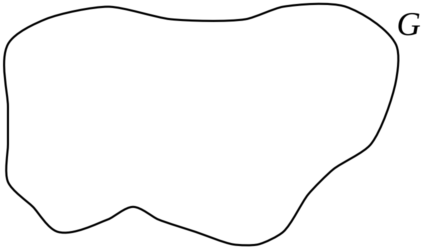
W_p



$K_{3,p}$

(Note that for $p \geqslant c$, both W_p and $K_{3,p}$ contain the c-pumpkin as a minor)

Outline of the overall proof

- Given G,



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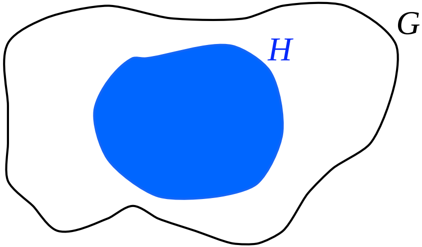

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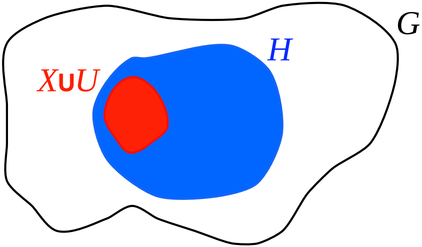

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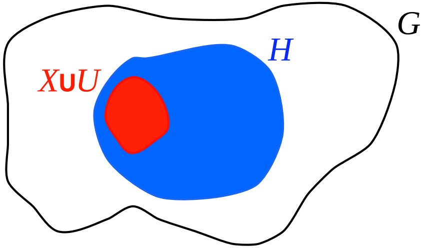

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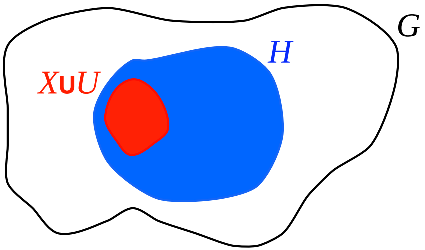

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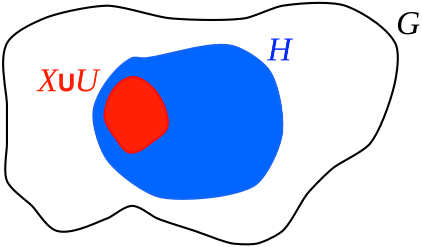

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- As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless H contains k disjoint c-pumpkin-models.
- This follows from the Main Lemma applied to the graph $\overline{H}$.

Next section is...

Further research

Main open problem: H non-acyclic planar, $f_H(k) = O(k \operatorname{poly}\log(k))$

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Approximation algorithms

Goal: Given a graph G, finding

- a c-pumpkin packing $\mathcal{M}$ and
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s.t. $|X| \leqslant f(c, n) \cdot |\mathcal{M}|$ for some approximation ratio $f(c, n)$

(these problems generalize Vertex Cover, Feedback Vertex Set, ...)

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★ constant-factor (**deterministic**) approximation for the hitting version? (so far, such an algorithm is only known for $c \leqslant 3$) [Fiorini, Joret, Pietropaoli '10]

Gràcies!