Optimal Erdős-Pósa property for pumpkins

Samuel Fiorini\textsuperscript{1} Gwenaël Joret\textsuperscript{1} Ignasi Sau\textsuperscript{2}

\textsuperscript{1}Université Libre de Bruxelles (Belgique)
\textsuperscript{2}CNRS, LIRMM, Montpellier (France)

CSASC 2013. \textit{Koper, Slovenia}
Outline of the talk

1. Motivation
2. Our result
3. Sketch of proof
4. Further research
1. Motivation

2. Our result

3. Sketch of proof

4. Further research
König’s min-max theorem in bipartite graphs:

\[ \text{Min Vertex Cover} = \text{Max Matching} \]
König’s min-max theorem in bipartite graphs:

\[
\text{Min Vertex Cover} = \text{Max Matching}
\]

\[
\min \text{ # vertices covering all edges} \geq \max \text{ # of disjoint edges}
\]
Packing and covering

König’s min-max theorem in bipartite graphs:

\[
\text{Min Vertex Cover} = \text{Max Matching}
\]

\[
\begin{align*}
\text{min} \ # \ \text{vertices covering all edges} & \geq \ \text{max} \ # \ \text{of disjoint edges} \\
\text{min} \ # \ \text{vertices covering all edges} & \leq \ \text{max} \ # \ \text{of disjoint edges}
\end{align*}
\]
Packing and covering

König’s min-max theorem in bipartite graphs:

\[
\text{Min Vertex Cover} = \text{Max Matching}
\]

\[
\text{min } \# \text{ vertices covering all edges} \geq \text{max } \# \text{ of disjoint edges}
\]
König's min-max theorem in bipartite graphs:

\[
\text{Min Vertex Cover} = \text{Max Matching}
\]

min \# vertices covering all \( H \in \mathcal{H} \) \( \geq \) max \# of disjoint \( H \in \mathcal{H} \)
König’s min-max theorem in bipartite graphs:

\[
\text{Min Vertex Cover} = \text{Max Matching}
\]

\[
\text{min } \text{# vertices covering all } H \in \mathcal{H} \geq \text{max } \text{# of disjoint } H \in \mathcal{H}
\]

\[
\text{min } \text{# vertices covering all } H \in \mathcal{H} \leq \text{max } \text{# of disjoint } H \in \mathcal{H}
\]
Packing and covering

König’s min-max theorem in bipartite graphs:

\[
\text{Min Vertex Cover} = \text{Max Matching}
\]

\[
\min \# \text{ vertices covering all } H \in \mathcal{H} \geq \max \# \text{ of disjoint } H \in \mathcal{H}
\]

\[
\min \# \text{ vertices covering all } H \in \mathcal{H} \leq f(\max \# \text{ of disjoint } H \in \mathcal{H})
\]
König’s min-max theorem in bipartite graphs:

\[ \text{Min Vertex Cover} = \text{Max Matching} \]

If there exists such \( f \) for all \( G \), then \( \mathcal{H} \) satisfies the Erdős-Pósa property.

\[ \min \# \text{ vertices covering all } H \in \mathcal{H} \leq f(\max \# \text{ of disjoint } H \in \mathcal{H}) \]
Minors and models in graphs

$H$ is a **minor** of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.

The $S_u$'s are called vertex images.
Minors and models in graphs

$H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.

$H$-model in $G$: collection $\{S_u : u \in V(H)\}$ s.t.
- the $S_u$’s are vertex-disjoint connected subgraphs of $G$, and
- there is an edge between $S_u$ and $S_v$ in $G$ for every edge $uv \in E(H)$.

A $K_5$-model

The $S_u$’s are called vertex images.
Let $H$ be a **fixed** graph. For a graph $G$, we define:

\[ \nu_H(G) := \text{packing number} \]
\[ = \text{max. number of vertex-disjoint } H\text{-models in } G. \]

\[ \tau_H(G) := \text{covering (or transversal) number} \]
\[ = \text{min. number of vertices hitting all } H\text{-models in } G. \]

Clearly, \[ \tau_H(G) \geq \nu_H(G) \quad \forall G. \]
Let $H$ be a fixed graph. For a graph $G$, we define:

$\nu_H(G) := \text{packing number}$

$= \max.$ number of vertex-disjoint $H$-models in $G$.

$\tau_H(G) := \text{covering (or transversal) number}$

$= \min.$ number of vertices hitting all $H$-models in $G$.

Clearly, $\tau_H(G) \geq \nu_H(G) \ \forall G$.

For which $H \quad \boxed{\tau_H(G) \leq f(\nu_H(G))} \quad \forall G$, for some function $f$ ?
Packing and covering $H$-models

Let $H$ be a fixed graph. For a graph $G$, we define:

$\nu_H(G) :=$ packing number
= max. number of vertex-disjoint $H$-models in $G$.

$\tau_H(G) :=$ covering (or transversal) number
= min. number of vertices hitting all $H$-models in $G$.

Clearly, $\tau_H(G) \geq \nu_H(G) \ \forall G$.

For which $H$ $\tau_H(G) \leq f(\nu_H(G)) \ \forall G$, for some function $f$?

This is called the Erdős-Pósa property of $H$-minors.
Fundamental result:

\[ \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \iff H \text{ is planar} \]

[Robertson, Seymour '86]

Is it the end of the story?

NO!

The derived upper bounds \( \tau_H(G) \leq f(\nu_H(G)) \) are huge:

\[ f(\nu_H(G)) = \Omega(2^{\nu_H(G)^2}) \]

This is because Robertson and Seymour's proof uses the excluded grid theorem from Graph Minors.

Natural objective: optimize \( f(\nu_H(G)) \).
Fundamental result:

\[ \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \quad \iff \quad H \text{ is planar} \]

[Robertson, Seymour '86]

Is it the end of the story?
Fundamental result:

\[ \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \iff H \text{ is planar} \]  

[Robertson, Seymour '86]

Is it the end of the story? NO!

- The derived upper bounds \( \tau_H \leq f(\nu_H) \) are huge: \( f(\nu_H) = \Omega(2^{\nu_H^2}) \).

  This is because Robertson and Seymour’s proof uses the excluded grid theorem from Graph Minors.
Erdős-Pósa property of $H$-minors

Fundamental result:

\[ \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \iff H \text{ is planar} \]

[Robertson, Seymour ’86]

Is it the end of the story? \textbf{NO!}

- The derived upper bounds $\tau_H \leq f(\nu_H)$ are \textbf{huge}: $f(\nu_H) = \Omega(2^{\nu_H^2})$.
  
  This is because Robertson and Seymour’s proof uses the \textit{excluded grid theorem} from Graph Minors.

- \textbf{Natural objective:} optimize $f(\nu_H)$. 
The property does NOT hold if $H$ is not planar

$H = K_5 \times$

Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $G$: 

![Diagram of a triangulated toroidal grid](image-url)
The property does NOT hold if $H$ is not planar

$H = K_5 \times 1$

Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $G$:

\[ \nu_H(G) = 1 \] but \[ \tau_H(G) = \Theta(\sqrt{n}) \]
The property does NOT hold if $H$ is not planar

$H = K_5 \times \not\!

H$ not planar \not\!

Therefore, the result of Robertson and Seymour is best possible.
Brief state of the art of Erdős-Pósa property for minors

\[ \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \quad \iff \quad H \text{ is planar} \]  

[Robertson, Seymour '86]
Brief state of the art of Erdős-Pósa property for minors

- \( \tau_H(G) \leq f(\nu_H(G)) \) \( \forall G \) \( \iff \) \( H \) is planar \[\text{[Robertson, Seymour '86]}\]

- Erdős and Pósa seminal result for \( H = \text{triangle} \) (optimal):
  \( f(k) = O(k \log k) \).
  \[\text{[Erdős, Pósa '65]}\]
Brief state of the art of Erdős-Pósa property for minors

- $\tau_H(G) \leq f(\nu_H(G)) \quad \forall G \iff H$ is planar \hfill [Robertson, Seymour '86]

- Erdős and Pósa seminal result for $H = \text{triangle}$ (optimal):
  
  $f(k) = O(k \log k)$.
  \hfill [Erdős, Pósa '65]

- $f(k) = O(k)$ when $H$ is a forest (optimal).
  \hfill [Fiorini, Joret, Wood '12]
Brief state of the art of Erdős-Pósa property for minors

- \( T_H(G) \leq f(\nu_H(G)) \ \forall G \iff H \text{ is planar} \)  
  [Robertson, Seymour '86]

- Erdős and Pósa seminal result for \( H = \text{triangle} \) (optimal): \( f(k) = O(k \log k) \).  
  [Erdős, Pósa '65]

- \( f(k) = O(k) \) when \( H \) is a forest (optimal).  
  [Fiorini, Joret, Wood '12]

- \( f(k) = O(k) \) when \( H \) is planar and \( G \) belongs to a minor-closed graph class (optimal).  
  [Fomin, Saurabh, Thilikos '10]
Brief state of the art of Erdős-Pósa property for minors

- \( \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \iff H \text{ is planar} \) \[\text{[Robertson, Seymour '86]}\]

- Erdős and Pósa seminal result for \( H = \text{triangle (optimal):} \)
  \( f(k) = O(k \log k). \) \[\text{[Erdős, Pósa '65]}\]

- \( f(k) = O(k) \) when \( H \) is a forest (optimal). \[\text{[Fiorini, Joret, Wood '12]}\]

- \( f(k) = O(k) \) when \( H \) is planar and \( G \) belongs to a minor-closed graph class (optimal). \[\text{[Fomin, Saurabh, Thilikos '10]}\]

- ★ \( f(k) = O(k \text{ poly log}(k)) \) for any planar graph \( H \). \[\text{[Chekuri, Chuzhoy '13]}\]
Lower bound for any planar graph $H$ with a cycle

**Theorem:** $\exists f_H(k)$ s.t. $\forall G, k$, either $\nu_H(G) \geq k$ or $\tau_H(G) \leq f_H(k)$.
Lower bound for any planar graph $H$ with a cycle

**Theorem:** $\exists f_H(k)$ s.t. $\forall G, k$, either $\nu_H(G) \geq k$ or $\tau_H(G) \leq f_H(k)$.

We have $f_H(k) = \Omega(k \log k)$ for any planar graph $H$ containing a cycle:
Lower bound for any planar graph $H$ with a cycle

**Theorem:** $\exists f_H(k) \text{ s.t. } \forall G, k, \text{ either } \nu_H(G) \geq k \text{ or } \tau_H(G) \leq f_H(k).$

We have $f_H(k) = \Omega(k \log k)$ for any planar graph $H$ containing a cycle:

- Let $G$ be an $n$-vertex cubic graph with $\text{tw}(G) = \Omega(n)$ and $\text{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)
Theorem: \( \exists f_H(k) \) s.t. \( \forall G, k \), either \( \nu_H(G) \geq k \) or \( \tau_H(G) \leq f_H(k) \).

We have \( f_H(k) = \Omega(k \log k) \) for any planar graph \( H \) containing a cycle:

- Let \( G \) be an \( n \)-vertex cubic graph with \( \text{tw}(G) = \Omega(n) \) and \( \text{girth}(G) = \Omega(\log n) \).

- Any \( H \)-minor-free graph \( F \) satisfies \( \text{tw}(F) \leq d \) for some constant \( d \), as \( H \) is planar.

- Thus \( \text{tw}(G - X) \leq d \) for any \( H \)-hitting set \( X \), and therefore \( \tau_H(G) = \Omega(n) \).

(such graphs are well-known to exist)

[Robertson, Seymour ’86]
Lower bound for any planar graph $H$ with a cycle

Theorem: $\exists f_H(k)$ s.t. $\forall G, k$, either $\nu_H(G) \geq k$ or $\tau_H(G) \leq f_H(k)$.

We have $f_H(k) = \Omega(k \log k)$ for any planar graph $H$ containing a cycle:

- Let $G$ be an $n$-vertex cubic graph with $\text{tw}(G) = \Omega(n)$ and $\text{girth}(G) = \Omega(\log n)$.
  (such graphs are well-known to exist)
- Any $H$-minor-free graph $F$ satisfies $\text{tw}(F) \leq d$ for some constant $d$, as $H$ is planar.
- Thus $\text{tw}(G - X) \leq d$ for any $H$-hitting set $X$, and therefore $\tau_H(G) = \Omega(n)$.
- [Robertson, Seymour '86]
- On the other hand, every subgraph $S$ of $G$ containing an $H$-model has a cycle, so $|V(S)| = O(\log n)$, and therefore $\nu_c(G) = O(n/\log n)$. 
Lower bound for any planar graph $H$ with a cycle

**Theorem:** $\exists f_H(k)$ s.t. $\forall G, k$, either $\nu_H(G) \geq k$ or $\tau_H(G) \leq f_H(k)$.

We have $f_H(k) = \Omega(k \log k)$ for any planar graph $H$ containing a cycle:

- Let $G$ be an $n$-vertex cubic graph with $\text{tw}(G) = \Omega(n)$ and $\text{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)

- Any $H$-minor-free graph $F$ satisfies $\text{tw}(F) \leq d$ for some constant $d$, as $H$ is planar.

- Thus $\text{tw}(G - X) \leq d$ for any $H$-hitting set $X$, and therefore $\tau_H(G) = \Omega(n)$.

- On the other hand, every subgraph $S$ of $G$ containing an $H$-model has a cycle, so $|V(S)| = O(\log n)$, and therefore $\nu_c(G) = O(n/\log n)$.

- This implies that (easy to check) $\exists$ constant $b > 0$ such that $f_H(k) > b \cdot k \log k$ (i.e., $f_H(k) = \Omega(k \log k)$).
For any planar graph $H$ with a cycle and a general graph $G$:

- **Lower bound:** $f_H(k) = \Omega(k \log k)$.

[Chekuri, Chuzhoy '13]

Only graph $H$ for which the lower bound is attained is the triangle: $f_\triangle(k) = O(k \log k)$. [Erd˝os, P´osa '65]
For any planar graph $H$ with a cycle and a general graph $G$:

- **Lower bound**: $f_H(k) = \Omega(k \log k)$.
- **Upper bound**: $f_H(k) = O(k \text{ poly log}(k))$. [Chekuri, Chuzhoy '13]
For any planar graph $H$ with a cycle and a general graph $G$:

- **Lower bound**: $f_H(k) = \Omega(k \log k)$.
- **Upper bound**: $f_H(k) = O(k \log^{35} k)$. 

[Chekuri, Chuzhoy '13]
For any planar graph $H$ with a cycle and a general graph $G$:

- **Lower bound:** $f_H(k) = \Omega(k \log k)$.
- **Upper bound:** $f_H(k) = O(k \log^{35} k)$. \[\text{[Chekuri, Chuzhoy '13]}\]

Only graph $H$ for which the lower bound is attained is the triangle:

$$f_\Delta(k) = O(k \log k).$$ \[\text{[Erdős, Pósa '65]}\]
1 Motivation

2 Our result

3 Sketch of proof

4 Further research
Pumpkins
A $c$-pumpkin:

Can be seen as a natural generalization of a cycle.

(N.B: “graph” = multigraph)
Graphs with no $c$-pumpkin minor

- $c = 1$: empty graphs
- $c = 2$: forests
- $c = 3$: no two cycles share an edge

etc.
Graphs with no $c$-pumpkin minor

- $c = 1$: empty graphs
Graphs with no $c$-pumpkin minor

- $c = 1$: empty graphs

- $c = 2$: forests
Graphs with no $c$-pumpkin minor

- $c = 1$: empty graphs

- $c = 2$: forests

- $c = 3$: no two cycles share an edge

- etc.
**c-pumpkin hitting set:**
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor
Covering (or hitting) pumpkins

c-pumpkin hitting set:
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor

Hitting set number $\tau_c(G)$: min. size of a c-pumpkin hitting set
c-pumpkin packing:
collection of vertex-disjoint subgraphs of $G$, each containing a $c$-pumpkin minor

$c = 2$

$\nu_c(G)$: max. cardinality of a $c$-pumpkin packing
c-pumpkin packing:
collection of vertex-disjoint subgraphs of $G$, each containing a c-pumpkin minor

Packing number $\nu_c(G)$: max. cardinality of a c-pumpkin packing
A recent result and our main theorem

- A recent result on Erdős-Pósa property for pumpkins:

**Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh ’12)**

For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin hitting set of size at most $f(k) = O(k^2)$.

- Our result:

**Theorem (Fiorini, Joret, S. ’13)**

For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin hitting set of size at most $f(k) = O(k \log k)$.
A recent result and our main theorem

- A recent result on Erdős-Pósa property for pumpkins:

**Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh ’12)**

For any fixed integer \( c \geq 1 \) and given an integer \( k \geq 1 \), every graph \( G \) either contains \( k \) vertex-disjoint \( c \)-pumpkins-models, or has a \( c \)-pumpkin hitting set of size at most \( f(k) = O(k^2) \).

That is, \( \tau_c \leq \nu_c^2 \).
A recent result and our main theorem

- A recent result on Erdős-Pósa property for pumpkins:

**Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh ’12)**

For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin hitting set of size at most $f(k) = O(k^2)$.

That is, $\tau_c \leq \nu_c^2$.

- Our result:

**Theorem (Fiorini, Joret, S. ’13)**

For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin hitting set of size at most $f(k) = O(k \log k)$.

That is, $\tau_c \leq \nu_c \log \nu_c$.
A recent result and our main theorem

- A recent result on Erdős-Pósa property for pumpkins:

**Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh ’12)**

For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin hitting set of size at most $f(k) = O(k^2)$. That is, $\tau_c \leq \nu_c^2$

- Their proof uses tree decompositions and brambles.

- Our result:

**Theorem (Fiorini, Joret, S. ’13)**

For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin hitting set of size at most $f(k) = O(k \log k)$. That is, $\tau_c \leq \nu_c \log \nu_c$

- Our proof follows and generalizes Erdős-Pósa’s proof for the case $c = 2$. 
Next section is...

1 Motivation

2 Our result

3 Sketch of proof

4 Further research
Useful reduction rules

We first need two reduction rules $\textbf{R1}$ and $\textbf{R2}$ dealing with 1-connected and 2-connected components without $c$-pumpkin minors, respectively, that preserve both $\nu_c(G)$ and $\tau_c(G)$:

For $c = 2$:
- $\textbf{R1}$: deleting degree-1 vertices
- $\textbf{R2}$: suppressing degree-2 vertices

Lemma

Let $c \geq 2$ be a fixed integer. Suppose that $G^*$ results from the application of $\textbf{R1}$ or $\textbf{R2}$ on a graph $G$. Then $\tau_c(G) = \tau_c(G^*)$ and $\nu_c(G) = \nu_c(G^*)$. 


Useful reduction rules

We first need two reduction rules \( R1 \) and \( R2 \) dealing with 1-connected and 2-connected components without \( c \)-pumpkin minors, respectively, that preserve both \( \nu_c(G) \) and \( \tau_c(G) \):

- **For** \( c = 2 \): \( R1 \) = deleting degree-1 vertices
- **For** \( c = 2 \): \( R2 \) = suppressing degree-2 vertices
Useful reduction rules

We first need two reduction rules \( \textbf{R1} \) and \( \textbf{R2} \) dealing with 1-connected and 2-connected components without c-pumpkin minors, respectively, that preserve both \( \nu_c(G) \) and \( \tau_c(G) \):

For \( c = 2 \):
- \( \textbf{R1} \) = deleting degree-1 vertices
- \( \textbf{R2} \) = suppressing degree-2 vertices

Lemma

Let \( c \geq 2 \) be a fixed integer. Suppose that \( G^* \) results from the application of \( \textbf{R1} \) or \( \textbf{R2} \) on a graph \( G \). Then \( \tau_c(G) = \tau_c(G^*) \) and \( \nu_c(G) = \nu_c(G^*) \).
We look at a subgraph $H$ with nice properties

- A graph is $c$-reduced if rules $R1$ or $R2$ cannot be applied anymore.
We look at a subgraph $H$ with nice properties

- A graph is $c$-reduced if rules $R1$ or $R2$ cannot be applied anymore.
- For a graph $G$, we denote by $\overline{G}$ a $c$-reduced graph obtained from $G$ by applying reduction rules $R1$ and $R2$. 

Main Lemma

If $|V(H)| \geq d \cdot k \log k$ for some constant $d$ (depending only on $c$), then $H$ contains $k$ vertex-disjoint $c$-pumpkin-models.
We look at a subgraph $H$ with nice properties

- A graph is \textit{c-reduced} if rules \textbf{R1} or \textbf{R2} cannot be applied anymore.

- For a graph $G$, we denote by $\overline{G}$ a \textit{c-reduced} graph obtained from $G$ by applying reduction rules \textbf{R1} and \textbf{R2}.

- Given $G$, let $H$ be a \textbf{maximal subgraph} of $G$ (w.r.t. \# vertices and \# edges) such that

\[
\Delta(\overline{H}) \leq 3,
\]

where $\Delta$ denotes the \textbf{maximum degree}.
We look at a subgraph $H$ with nice properties

- A graph is $c$-reduced if rules $R1$ or $R2$ cannot be applied anymore.
- For a graph $G$, we denote by $\overline{G}$ a $c$-reduced graph obtained from $G$ by applying reduction rules $R1$ and $R2$.
- Given $G$, let $H$ be a maximal subgraph of $G$ (w.r.t. # vertices and # edges) such that

$$\Delta(\overline{H}) \leq 3,$$

where $\Delta$ denotes the maximum degree.

**Main Lemma**

*If $|V(\overline{H})| \geq d \cdot k \log k$ for some constant $d$ (depending only on $c$), then $H$ contains $k$ vertex-disjoint $c$-pumpkin-models.*
We prove it by induction on $k$, using that:

**Lemma**

*Every $n$-vertex $c$-reduced graph $G$ contains a $c$-pumpkin-model of size $O(\log n)$.***

(Generalization of: If $\delta(G) \geq 3$, then $\text{girth}(G) < 2\log n$)
Ingredients in the proof of the Main Lemma

- We prove it by **induction on** \( k \), using that:

**Lemma**

*Every n-vertex c-reduced graph* \( G \) *contains a c-pumpkin-model of size* \( O(\log n) \).

(Generalization of: If \( \delta(G) \geq 3 \), then \( \text{girth}(G) < 2 \log n \))

- We choose a **smallest c-pumpkin-model** \( C \), and to apply induction we need to prove that \( H - C \) contains a **subgraph** \( F \) such that
  \[
  |V(F)| \geq d \cdot (k - 1) \log(k - 1).
  \]
Ingredients in the proof of the Main Lemma

- We prove it by induction on $k$, using that:

**Lemma**

*Every* $n$*-vertex* $c$*-reduced* graph $G$ contains a $c$*-pumpkin-model* of size $O(\log n)$.

(Generalization of: If $\delta(G) \geq 3$, then $\text{girth}(G) < 2\log n$)

- We choose a smallest $c$*-pumpkin-model* $C$, and to apply induction we need to prove that $\overline{H} - C$ contains a subgraph $F$ such that
  \[ |V(F)| \geq d \cdot (k - 1) \log(k - 1). \]

- **Crucial:** $\forall \ p \geq 0$, $\exists f(p)$ s.t. every 3-connected graph with $\geq f(p)$ vertices has a minor isomorphic to:

  [Oporowski, Oxley, Thomas '93]

\[ W_p \quad K_{3,p} \]

(Note that for $p \geq c$, both $W_p$ and $K_{3,p}$ contain the $c$*-pumpkin* as a minor)
Outline of the overall proof

- Given $G$, 

\[
\exists \text{ set } X \cup U \subseteq V(H), \ \text{with } |X| = O(k),
\]

meeting every $c$-pumpkin-model in $G$.

As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless $H$ contains $k$ disjoint $c$-pumpkin-models. This follows from the Main Lemma applied to the graph $H$. 

\[22/25\]
Outline of the overall proof

- Given $G$, we consider the subgraph $H$ defined before:

\[ H \subseteq G \]

We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, meeting every $c$-pumpkin-model in $G$.

As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless $H$ contains $k$ disjoint $c$-pumpkin-models.

This follows from the Main Lemma applied to the graph $H$. 

Outline of the overall proof

- Given $G$, we consider the subgraph $H$ defined before:

- We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, meeting every $c$-pumpkin-model in $G$. 

![Diagram showing the subgraph $H$ as a subset of the graph $G$, with a set $X \cup U$ highlighted within $H$.](image)
Outline of the overall proof

- Given $G$, we consider the subgraph $H$ defined before:

  We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, meeting every $c$-pumpkin-model in $G$.

- As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$. 
Outline of the overall proof

- Given $G$, we consider the subgraph $H$ defined before:

- We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, meeting every $c$-pumpkin-model in $G$.

- As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless $H$ contains $k$ disjoint $c$-pumpkin-models.
Outline of the overall proof

- Given $G$, we consider the subgraph $H$ defined before:

We can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, meeting every $c$-pumpkin-model in $G$.

As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless $H$ contains $k$ disjoint $c$-pumpkin-models.

This follows from the Main Lemma applied to the graph $\overline{H}$. 
Next section is...

1. Motivation

2. Our result

3. Sketch of proof

4. Further research
Further research

Main open problem: $H$ non-acyclic planar, $f_H(k) = O(k \text{ poly log}(k))$
Further research

Main open problem: $H$ non-acyclic planar, $f_H(k) = \Omega(k \log k)$
Further research

Main open problem: $H$ non-acyclic planar, $f_H(k) = \Omega(k \log k)$

Conjecture

For all non-acyclic planar $H$, we have $f_H(k) = O(k \log k)$. (optimal)
Further research

Main open problem: $H$ non-acyclic planar, $f_H(k) = \Omega(k \log k)$

Conjecture

For all non-acyclic planar $H$, we have $f_H(k) = O(k \log k)$. \hspace{1cm} (optimal)

Approximation algorithms

**Goal**: Given a graph $G$, finding

- a $c$-pumpkin packing $\mathcal{M}$ and
- a $c$-pumpkin hitting set $X$

s.t. $|X| \leq f(c, n) \cdot |\mathcal{M}|$ \hspace{1cm} for some approximation ratio $f(c, n)$

(these problems generalize VERTEX COVER, FEEDBACK VERTEX SET, \ldots)
Further research

Main open problem: $H$ non-acyclic planar, $f_H(k) = \Omega(k \log k)$

**Conjecture**

For all non-acyclic planar $H$, we have $f_H(k) = O(k \log k)$. (optimal)

**Approximation algorithms**

**Goal**: Given a graph $G$, finding
- a $c$-pumpkin packing $\mathcal{M}$ and
- a $c$-pumpkin hitting set $X$

s.t. $|X| \leq f(c, n) \cdot |\mathcal{M}|$ for some approximation ratio $f(c, n)$

(These problems generalize Vertex Cover, Feedback Vertex Set, …)

★ we provided an $O_c(\log n)$-approximation algorithm for $c$-Pumpkin Hitting Set and $c$-Pumpkin Packing. [Joret, Paul, S., Saurabh, Thomassé '11]
Further research

Main open problem: \( H \) non-acyclic planar, \( f_H(k) = \Omega(k \log k) \)

Conjecture

For all non-acyclic planar \( H \), we have \( f_H(k) = O(k \log k) \). (optimal)

Approximation algorithms

**Goal**: Given a graph \( G \), finding
- a \( c \)-pumpkin packing \( M \) and
- a \( c \)-pumpkin hitting set \( X \)
  
  \[
  |X| \leq f(c, n) \cdot |M|
  \]
  
  for some approximation ratio \( f(c, n) \)

(These problems generalize Vertex Cover, Feedback Vertex Set, . . .)

★ we provided an \( O_c(\log n) \)-approximation algorithm for \( c \)-Pumpkin Hitting Set and \( c \)-Pumpkin Packing. [Joret, Paul, S., Saurabh, Thomassé '11]

★ constant-factor (deterministic) approximation for the hitting version? (so far, such an algorithm is only known for \( c \leq 3 \)) [Fiorini, Joret, Pietropaoli '10]
Gràcies!