On Self-Duality of Branchwidth in Graphs of Bounded Genus

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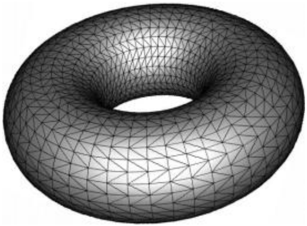
Outline of the talk

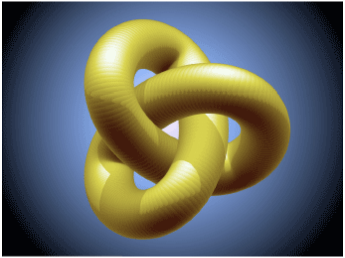
1. Preliminaries
   - Surfaces
   - Graphs on surfaces
   - Branchwidth
   - Minors
   - Clique-sums
2. Motivation
3. The result
   - Main idea
   - The algorithm
4. Conclusions

1. Preliminaries

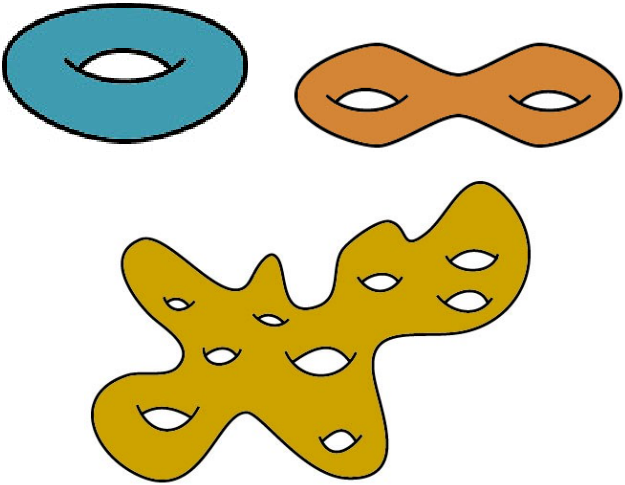
Surfaces

- Surface: connected compact 2-manifold without boundaries.

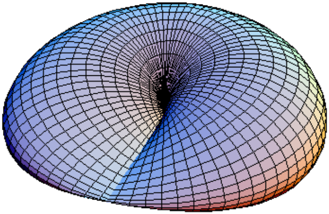


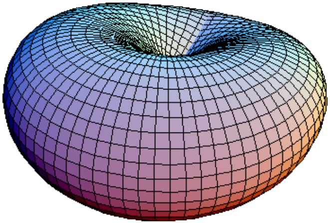


Handles



Cross-caps





Genus of a surface

- The surface classification Theorem: any compact, connected and without boundary surface can be obtained from the sphere $\mathbb{S}^2$ by adding *handles* and *cross-caps*.
- Orientable surfaces: obtained by adding $g \geq 0$ *handles* to the sphere $\mathbb{S}^2$, obtaining the g-torus $\mathbb{T}_g$ with Euler genus $\mathbf{eg}(\mathbb{T}_g) = 2g$.
- Non-orientable surfaces: obtained by adding $h > 0$ *cross-caps* to the sphere $\mathbb{S}^2$, obtaining a non-orientable surface $\mathbb{P}_h$ with Euler genus $\mathbf{eg}(\mathbb{P}_h) = h$.

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Graphs on Surfaces

- An embedding of a graph G on a surface Σ is a drawing of G on Σ without edge crossings.
- An embedding defines vertices, edges, and faces.
- The Euler genus of a graph G, $\mathbf{eg}(G)$, is the least Euler genus of the surfaces in which G can be embedded (NP-hard problem).

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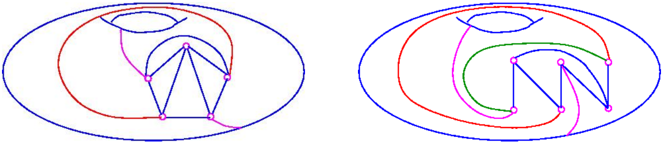
Example: K_5 and $K_{3,3}$

Theorem (Kuratowski, 1930)

A graph G is planar if and only if contains neither K_5 nor $K_{3,3}$ as a topological minor.

But K_5 and $K_{3,3}$ can be embedded in the torus $\mathbb{T}_1$:



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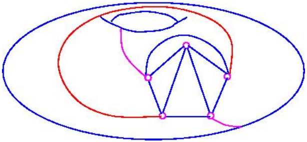

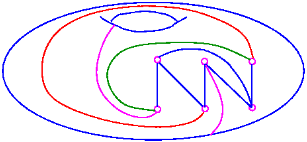

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Polyhedral embeddings

- A subset of Σ meeting the drawing only at vertices of G is called G-normal. An O-arc is a subset of Σ homeomorphic to $\mathbb{S}^1$.
- If an O-arc is G-normal, then we call it a noose. The **length** of a noose is the number of its vertices.
- A noose is contractible if it is homotopically equivalent to a point.
- A noose is surface separating it its removal disconnects Σ.
- The representativity **rep**(G, Σ) of a graph embedding (G, Σ) is the smallest length of a non-contractible noose in Σ.
- An embedding (G, Σ) is polyhedral if G is 3-connected and **rep**$(G, \Sigma) \geq 3$.

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Dual embedding

- A 2-cell embedding is an embedding in which every face is homeomorphic to an open disk.
- For 2-cell embeddings, the Euler's formula applies:

$$|V| - |E| + |F| = \mathbf{eg}(\Sigma).$$

- For a given embedding (G, Σ), we denote by (G^*, Σ) its dual embedding (it is the *geometric dual*).

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Branchwidth

- A branch decomposition of a graph $G = (V, E)$ is a tuple (T, μ), where:
  - T is a tree where all the internal nodes have degree 3.
  - μ is a bijection between the leaves of T and $E(G)$.
- Each edge $e \in T$ partitions $E(G)$ into two sets A_e and B_e.
- For each $e \in E(T)$, we define $\mathrm{mid}(e) = V(A_e) \cap V(B_e)$.
- The width of a branch decomposition is $\max_{e \in E(T)} |\mathrm{mid}(e)|$.
- The branchwidth of a graph G (denoted $\mathbf{bw}(G)$) is the minimum width over all branch decompositions of G:

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Graph minors

- H is a contraction of G ($H \preceq_c G$) if H occurs from G after applying a series of edge contractions.
- H is a minor of G ($H \preceq_m G$) if H is the contraction of some subgraph of G.
- A parameter **p** is any function mapping graphs to non-negative integers:

$$\mathbf{p} : \mathcal{G} \rightarrow \mathbb{N}^+$$

- We say that a parameter **p** is minor closed if for every graph H,

$$H \preceq_m G \ \Rightarrow \ \mathbf{p}(H) \leq \mathbf{p}(G).$$

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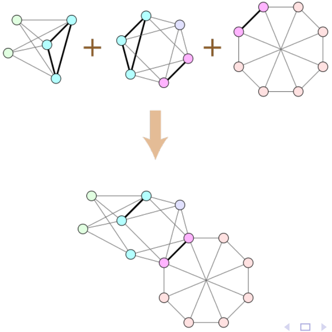
Clique-sums

- Suppose G_1 and G_2 are graphs with disjoint vertex-sets and $k \geq 0$ is an integer.
- For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k.
- A clique-sum $G_1 \oplus G_2$ of G_1 and G_2 is the graph obtained by gluing G_1 and G_2 at the cliques, and possibly deleting some edges.



2. Motivation

Motivation

- Let $\mathcal{G}$ be a class of graphs embeddable in a surface Σ.
- A graph parameter **p** is *(c, d)-self-dual* on $\mathcal{G}$ if for every graph $G \in \mathcal{G}$ and for its geometric dual G^*,

$$\mathbf{p}(G^*) \leq c \cdot \mathbf{p}(G) + d.$$

- Main motivation: **Graph Minors project** [Robertson and Seymour, 1982–].
- Branchwidth is $(1, 0)$-self-dual in planar graphs that are not forests. [Seymour and Thomas, Combinatorica 1994].
- We prove that branchwidth is $(6, 2g - 4)$-self-dual in graphs of Euler genus at most g.

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3. The result:

Theorem

Let (G, Σ) *be an embedding with* $g = \mathbf{eg}(\Sigma)$*. Then*

$$\mathbf{bw}(G^*) \leq 6 \cdot \mathbf{bw}(G) + 2g - 4.$$

Main idea

The result holds for polyhedral embeddings:

Proposition (With the idea of Fomin and Thilikos, Journal of Graph Theory 2007)

Let (G, Σ) and (G^*, Σ) be dual polyhedral embeddings in a surface of Euler genus g. Then

$$\mathbf{bw}(G^*) \leq 6 \cdot \mathbf{bw}(G) + 2g - 4.$$

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The algorithm

1. Set $\mathcal{B} = \{G\}$, and $\mathcal{B}^* = \{G^*\}$
   (we call the members of $\mathcal{B}$ and $\mathcal{B}^*$ *blocks*).
2. If (G, Σ) has a minimal separator S with $|S| \leq 2$:
   - Let $C_1, \ldots, C_\rho$ be the connected components of $G[V(G) \setminus S]$ and, for $i = 1, \ldots, \rho$, let G_i be the graph obtained by $G[V(C_i) \cup S]$ by adding an edge with both endpoints in S in the case where $|S| = 2$ and such an edge does not already exist.
   - Notice that a (non-empty) separator S of size at most 2 corresponds to a non-empty separator S^* of G^*.
   - Let G_i^*, $i = 1, \ldots, \rho$, be the graphs obtained by cutting G^* along S^*.
   - We say that each G_i (resp G_i^*) is a block of G (resp. G^*).

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The algorithm (II)

- Notice that each G and G^* is the clique-sum of its blocks.
- Using the following lemma:

Lemma (Fomin and Thilikos, SIAM J. Comp. 2006)

Let G_1 and G_2 be graphs with one edge or one vertex in common. Then $\mathbf{bw}(G_1 \cup G_2) \leq \max\{\mathbf{bw}(G_1), \mathbf{bw}(G_2), 2\}$.

Yields $\mathbf{bw}(G^*) \leq \max\{2, \max\{\mathbf{bw}(G_i^*) \mid i = 1, \ldots, \rho\}\}$.

- Observe that we may assume that for each $i = 1, \ldots, \rho$, G_i and G_i^* are embedded in a surface Σ_i such that G_i is the dual of G_i^* and $\mathbf{eg}(\Sigma) = \sum_{i=1,\ldots,\rho} \mathbf{eg}(\Sigma_i)$.
- Notice that $\mathbf{bw}(G_i) \leq \mathbf{bw}(G)$, $i = 1, \ldots, \rho$, as the possible edge addition does not increase the branchwidth, since each block of G is a minor of G.
- We set $\mathcal{B} \leftarrow \mathcal{B} \setminus \{G\} \cup \{G_1, \ldots, G_\rho\}$ and $\mathcal{B}^* \leftarrow \mathcal{B}^* \setminus \{G^*\} \cup \{G_1^*, \ldots, G_\rho^*\}$.

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- Observe that we may assume that for each $i = 1, \ldots, \rho$, G_i and G_i^* are embedded in a surface Σ_i such that G_i is the dual of G_i^* and $\mathbf{eg}(\Sigma) = \sum_{i=1,\ldots,\rho} \mathbf{eg}(\Sigma_i)$.
- Notice that $\mathbf{bw}(G_i) \leq \mathbf{bw}(G)$, $i = 1, \ldots, \rho$, as the possible edge addition does not increase the branchwidth, since each block of G is a minor of G.
- We set $\mathcal{B} \leftarrow \mathcal{B} \setminus \{G\} \cup \{G_1, \ldots, G_\rho\}$ and $\mathcal{B}^* \leftarrow \mathcal{B}^* \setminus \{G^*\} \cup \{G_1^*, \ldots, G_\rho^*\}$.

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- Notice that each G and G^* is the clique-sum of its blocks.
- Using the following lemma:

Lemma (Fomin and Thilikos, SIAM J. Comp. 2006)

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3. If (G, Σ) has a non-contractible and non-surface-separating noose meeting a set $S \subseteq V(G)$ with $|S| \leq 2$:
   - Let $G' = G[V(G) \setminus S]$ and let F be the set of of faces in G^* corresponding to the vertices in S.
   - Observe that the obtained graph G' has an embedding to some surface Σ' of Euler genus *strictly* smaller than Σ that, in turn, has some dual G'^* in Σ'. Therefore $\mathbf{eg}(\Sigma') < \mathbf{eg}(\Sigma)$.
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Lemma

The removal of a vertex or the contraction of a face from an embedded graph decreases its branchwidth by at most 1.

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4. Apply Steps 2–4 for each block $G \in \mathcal{B}$ and its dual.

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Sketch of proof

- Idea: induction on the distance from the root of the recursion tree of the algorithm.

- **Base case of the induction**: All embeddings of graphs in the collections $\mathcal{B}$ and $\mathcal{B}^*$ constructed by the algorithm are polyhedral (except from the trivial case that they are just cliques of size 2). Therefore, the result holds.

- **Induction step (case 1)**: Suppose that G (resp. G^*) is the clique-sum of its blocks $G_1, \ldots, G_\rho$ (resp. $G_1^*, \ldots, G_\rho^*$) embedded in the surfaces $\Sigma_1, \ldots, \Sigma_\rho$ (Step 2).
  - By induction, we have that $\mathbf{bw}(G_i^*) \leq 6 \cdot \mathbf{bw}(G_i) + 2\mathbf{eg}(\Sigma_i) - 4, i = 1, \ldots, \rho.$
  - Then, the claim follows from
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