Reducing graph transversals via edge contractions

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Outline of the talk

1. Introduction
2. Our results
3. Some proofs
4. Further research

Next section is...

Graph modification problems

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$\mathcal{M}$-Modification to $\mathcal{C}$	
Input:	A graph G and an integer k.
Question:	Can we transform G to a graph in $\mathcal{C}$ by applying at most k operations from $\mathcal{M}$?

This meta-problem has a huge expressive power.

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- π = chromatic/independence/clique/matching/domination number
  [Bentz et al. 2010] [Costa et al. 2011] [Bazgan et al. 2011, 2015] [Diner et al. 2018] [Paulusma et al. 2019] [Fomin et al. 2020]

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Proposition (Galby, Lima, Ries. 2019)

Let π be a graph parameter such that

(i) *it is NP-hard to compute the π-number of a graph and*

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Unless P=NP, there exists no polynomial-time algorithm deciding whether contracting one given edge decreases the π-number of a graph.

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These three parameters satisfy the conditions of the previous Proposition.

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CONTRACTION(fvs) *and* CONTRACTION(oct) co-NP-*hard for* $k = d = 1$.

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- CONNECTED VERTEX COVER is NP-hard even if vc is polynomial (bipartite graphs). [Escoffier et al. 2010]

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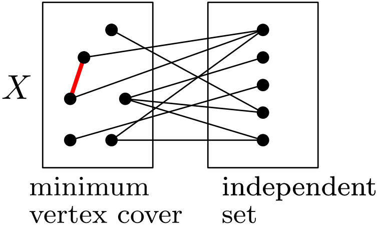

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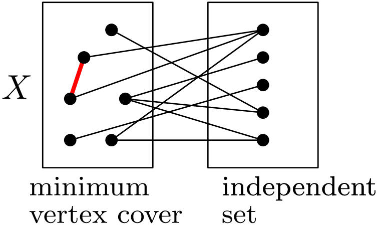

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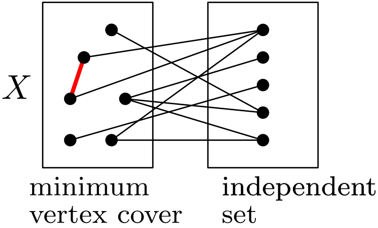

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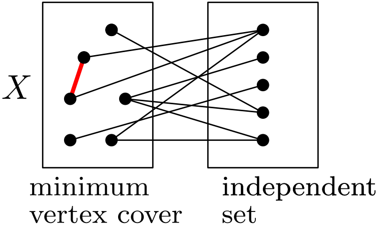

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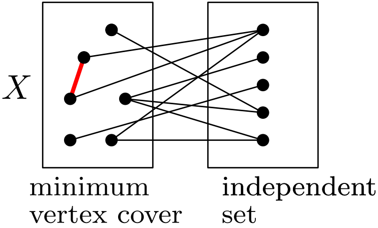

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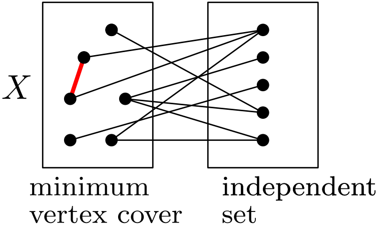

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  - For every edge $e \in E(G)$, we compute $\mathrm{vc}(G/e)$ in polynomial time.

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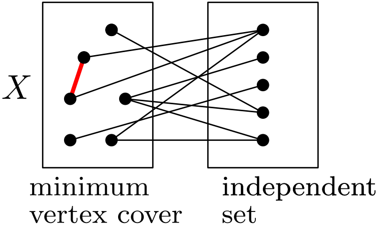

- Given G, consider a minimum vertex cover X of G.
- If G is not bipartite, then $G[X]$ contains some edge $\Rightarrow$ YES-instance!
- Otherwise, G is bipartite:
  - We first compute $\mathrm{vc}(G)$ in polynomial time.
  - For every edge $e \in E(G)$, we compute $\mathrm{vc}(G/e)$ in polynomial time.
  - We check whether $\mathrm{vc}(G/e) < \mathrm{vc}(G)$ for some edge $e \in E(G)$.

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          We solve the problem in time $f(d) \cdot n^{\mathcal{O}(1)}$ by expressing it with an MSO formula (Courcelle), and then using a simple dynamic programming algorithm to combine the solutions for the connected components.

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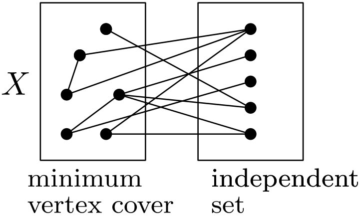

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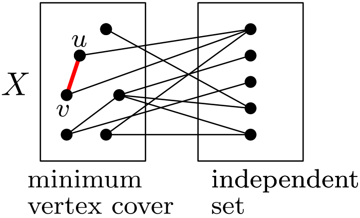

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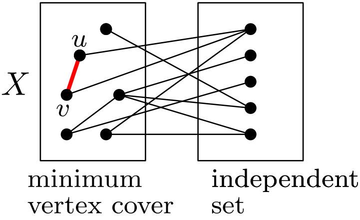

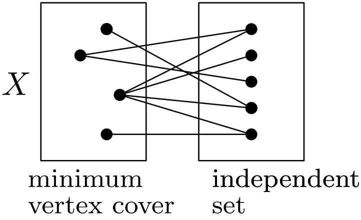

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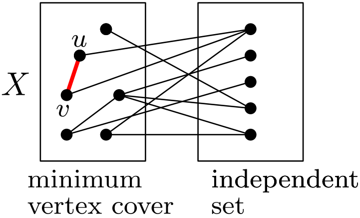

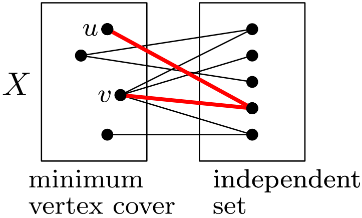

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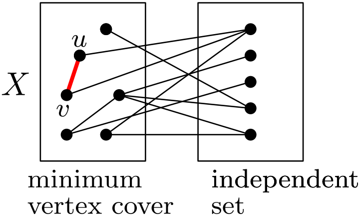

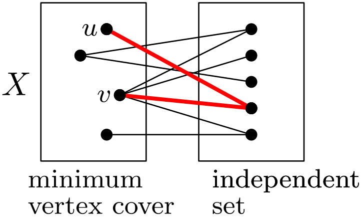

Since $\mathrm{vc}(C) \geq d+1$, iteratively contracting such pairs of vertices $u, v \in X$ gives the desired set $F \subseteq E(G)$ with $|F| \leq 2d$ s.t. $\mathrm{vc}(G/F) \leq \mathrm{vc}(G) - d$.

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      Set $B := V(L) \cup V_F$ (vertices resulting from the contraction of F).
  - Finally, check whether $\mathrm{vc}(G/F) < \mathrm{vc}(G) - d$ for some set $F \subseteq E(G)$.

Theorem

Let $\mathcal{H}$*: collection of 2-connected graphs containing a non-complete graph.*
Let $\prec \in \{$*subgraph, induced subgraph, minor, topological minor*$\}$.

The $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ *problem is* co-NP-*hard for fixed* $k = d = 1$.

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Our reduction is inspired by the classical NP-hardness reduction from 3-$\textsc{Sat}$ to $\textsc{Vertex Cover}$:

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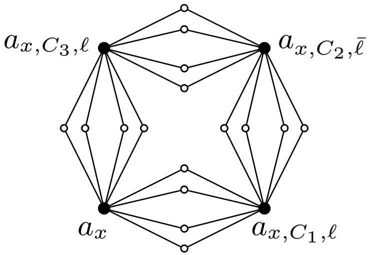
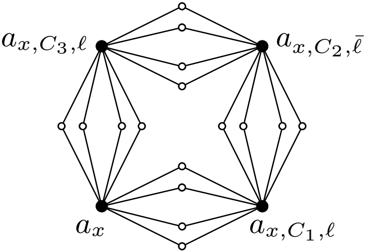
For each variable x and each clause C_i containing x in a literal $\ell \in \{x, \bar{x}\}$:

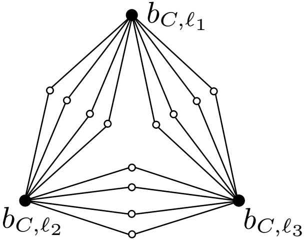


We present the reduction for $\mathcal{H} = \{C_4\}$ and $\prec\, = \{\text{subgraph}\}$.

$\boxed{\text{Goal:}}$ given a clean formula φ, construct in poly time a graph G_φ such that φ is satisfiable $\Leftrightarrow$ G_φ is a No-instance of 1-Contraction$(\tau^{\prec}_{C_4}, 1)$.

For each clause C of φ and for each literal ℓ_j in C:





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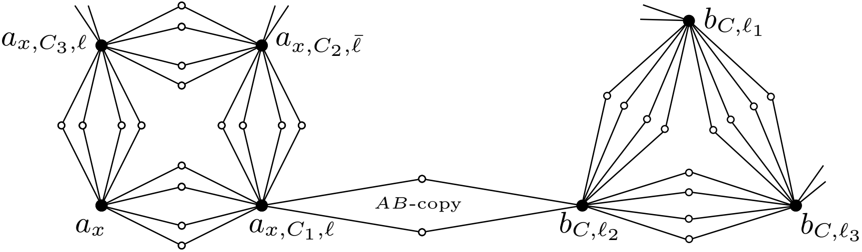
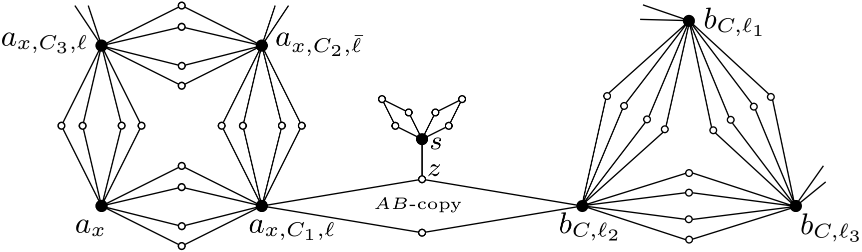


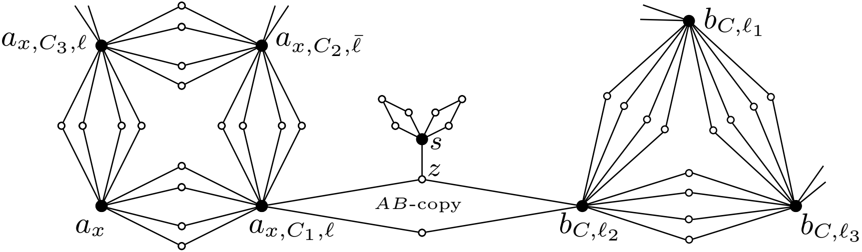
We present the reduction for $\mathcal{H} = \{C_4\}$ and $\prec = \{\text{subgraph}\}$.

$\boxed{\text{Goal:}}$ given a clean formula φ, construct in poly time a graph G_φ such that φ is satisfiable $\Leftrightarrow$ G_φ is a No-instance of 1-Contraction$(\tau^{\prec}_{C_4}, 1)$.

Finally, we add this pendent gadget to each AB-copy of C_4:



$\boxed{\text{Goal:}}$ given a clean formula φ, construct in poly time a graph G_φ such that φ is satisfiable $\Leftrightarrow$ G_φ is a No-instance of 1-Contraction$(\tau_{C_4}^{\preceq}, 1)$.



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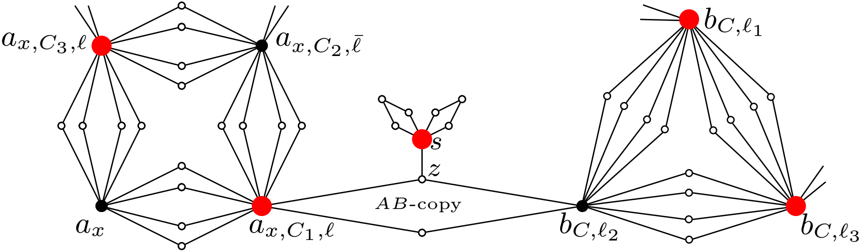
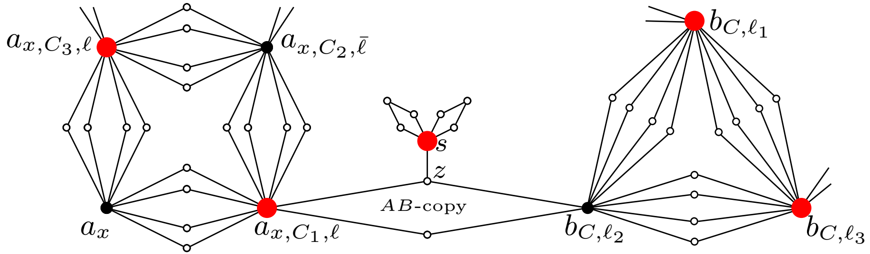


Budget: 2 per variable, $|C|-1$ per clause C, 1 per literal: $8n-m$.

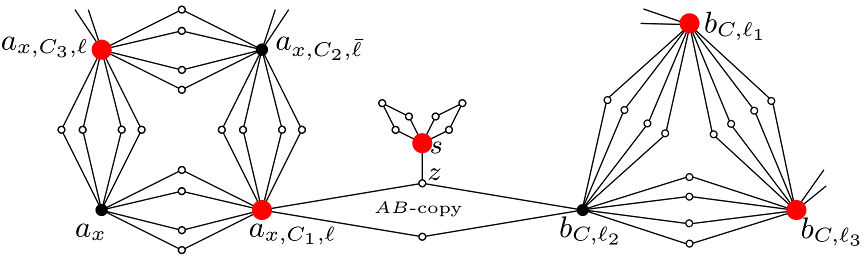
$\boxed{\text{Goal:}}$ given a clean formula φ, construct in poly time a graph G_φ such that φ is satisfiable $\Leftrightarrow$ G_φ is a No-instance of 1-Contraction$(\tau^{\preceq}_{C_4}, 1)$.



Budget: 2 per variable, $|C| - 1$ per clause C, 1 per literal: $8n - m$.

Claim 1: $\tau_{C_4}(G_\varphi) = 8n - m \;\Leftrightarrow\; \varphi$ is satisfiable.

$\boxed{\text{Goal:}}$ given a clean formula φ, construct in poly time a graph G_φ such that φ is satisfiable $\Leftrightarrow$ G_φ is a No-instance of 1-Contraction$(\tau^{\preceq}_{C_4}, 1)$.

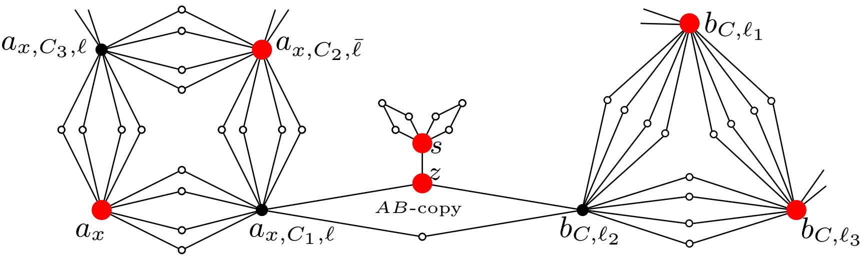


Budget: 2 per variable, $|C| - 1$ per clause C, 1 per literal: $8n - m$.

Claim 1: $\tau_{C_4}(G_\varphi) = 8n - m \;\Leftrightarrow\; \varphi$ is satisfiable.

Claim 2: If $\tau_{C_4}(G_\varphi) = 8n - m$, $\nexists$ edge e such that $\tau_{C_4}(G_\varphi/e) < \tau_{C_4}(G_\varphi)$.

$\boxed{\text{Goal:}}$ given a clean formula φ, construct in poly time a graph G_φ such that φ is satisfiable $\Leftrightarrow$ G_φ is a No-instance of 1-Contraction$(\tau_{C_4}^{\prec}, 1)$.



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Claim 3: If $\tau_{C_4}(G_\varphi) > 8n - m$, $\exists$ edge e s.t. $\tau_{C_4}(G_\varphi/e) < \tau_{C_4}(G_\varphi)$.

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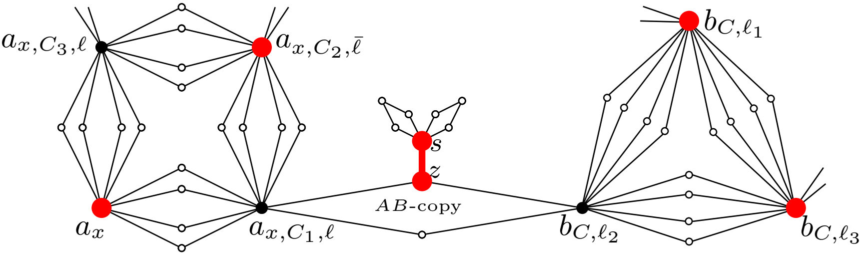

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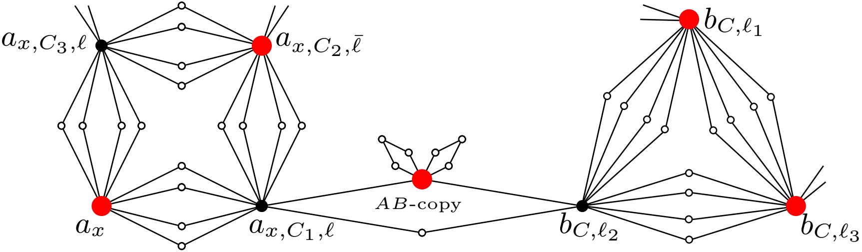
$\boxed{\text{Goal:}}$ given a clean formula φ, construct in poly time a graph G_φ such that φ is satisfiable $\Leftrightarrow$ G_φ is a No-instance of 1-CONTRACTION$(\tau_{C_4}^{\prec}, 1)$.



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Next section is...

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

- $\mathcal{H} =$ 2-connected graphs containing at least one non-complete graph, $\prec =$ (induced) subgraph or (topological) minor.
- $\mathcal{H} =$ cliques with at least three vertices, $\prec =$ (topological) minor.
- $\mathcal{H} = \{P_i\}$ with $i \geq 4$, $\prec =$ (induced) subgraph or (topological) minor.

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- $\mathcal{H} = \{P_i\}$ with $i \geq 4$, $\prec$ = (induced) subgraph or (topological) minor.

Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

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- $\mathcal{H} = \{P_i\}$ with $i \geq 4$, $\prec$ = (induced) subgraph or (topological) minor.

Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

Open cases:

- $\mathcal{H} = \{K_h\}$ with $h \geq 3$ for $\prec$ = (induced) subgraph.

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Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

Open cases:

- $\mathcal{H} = \{K_h\}$ with $h \geq 3$ for $\prec$ = (induced) subgraph.
- $\mathcal{H} = \{P_3\}$ for any $\prec$.
- $\mathcal{H} = \{T\}$ for a tree T, for any $\prec$.

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- $\mathcal{H} = \{P_3\}$ for any $\prec$.
- $\mathcal{H} = \{T\}$ for a tree T, for any $\prec$.
- What about if $\mathcal{H}$ contains disconnected graphs?

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

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$\textsc{Contraction}(\tau_{K_2}^{\prec})$ in time $f(d) \cdot n^{2d}$.

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

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$\textsc{Contraction}(\tau_{K_2}^{\prec})$ in time $f(d) \cdot n^{2d}$. FPT or W[1]-hard by d?

We proved that $\text{CONTRACTION}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

- $\mathcal{H}$ = 2-connected graphs containing at least one non-complete graph, $\prec$ = (induced) subgraph or (topological) minor.
- $\mathcal{H}$ = cliques with at least three vertices, $\prec$ = (topological) minor.
- $\mathcal{H} = \{P_i\}$ with $i \geq 4$, $\prec$ = (induced) subgraph or (topological) minor.

Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

Open cases:

- $\mathcal{H} = \{K_h\}$ with $h \geq 3$ for $\prec$ = (induced) subgraph.
- $\mathcal{H} = \{P_3\}$ for any $\prec$.
- $\mathcal{H} = \{T\}$ for a tree T, for any $\prec$.
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$\text{CONTRACTION}(\tau_{K_2}^{\prec})$ in time $f(d) \cdot n^{2d}$. FPT or W[1]-hard by d?

co-NP-hard cases: natural to parameterize $\text{CONTRACTION}(\tau_{\mathcal{H}}^{\prec})$ by $\tau_{\mathcal{H}}^{\prec}$.

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

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Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

Open cases:

- $\mathcal{H} = \{K_h\}$ with $h \geq 3$ for $\prec$ = (induced) subgraph.
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$\textsc{Contraction}(\tau_{K_2}^{\prec})$ in time $f(d) \cdot n^{2d}$. FPT or W[1]-hard by d?

co-NP-hard cases: natural to parameterize $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ by $\tau_{\mathcal{H}}^{\prec}$.

- If $\prec$ = minor and $\mathcal{H}$ contains a planar graph, FPT param. by $\tau_{\mathcal{H}}^{\prec} + k$.

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

- $\mathcal{H}$ = 2-connected graphs containing at least one non-complete graph, $\prec$ = (induced) subgraph or (topological) minor.
- $\mathcal{H}$ = cliques with at least three vertices, $\prec$ = (topological) minor.
- $\mathcal{H} = \{P_i\}$ with $i \geq 4$, $\prec$ = (induced) subgraph or (topological) minor.

Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

Open cases:

- $\mathcal{H} = \{K_h\}$ with $h \geq 3$ for $\prec$ = (induced) subgraph.
- $\mathcal{H} = \{P_3\}$ for any $\prec$.
- $\mathcal{H} = \{T\}$ for a tree T, for any $\prec$.
- What about if $\mathcal{H}$ contains disconnected graphs?

$\textsc{Contraction}(\tau_{K_2}^{\prec})$ in time $f(d) \cdot n^{2d}$. FPT or W[1]-hard by d?

co-NP-hard cases: natural to parameterize $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ by $\tau_{\mathcal{H}}^{\prec}$.

- If $\prec$ = minor and $\mathcal{H}$ contains a planar graph, FPT param. by $\tau_{\mathcal{H}}^{\prec} + k$.
- In particular, $\textsc{Contraction}(\text{fvs})$ is FPT param. by $\text{fvs} + k$.

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

- $\mathcal{H}$ = 2-connected graphs containing at least one non-complete graph, $\prec$ = (induced) subgraph or (topological) minor.
- $\mathcal{H}$ = cliques with at least three vertices, $\prec$ = (topological) minor.
- $\mathcal{H} = \{P_i\}$ with $i \geq 4$, $\prec$ = (induced) subgraph or (topological) minor.

Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

Open cases:

- $\mathcal{H} = \{K_h\}$ with $h \geq 3$ for $\prec$ = (induced) subgraph.
- $\mathcal{H} = \{P_3\}$ for any $\prec$.
- $\mathcal{H} = \{T\}$ for a tree T, for any $\prec$.
- What about if $\mathcal{H}$ contains disconnected graphs?

$\textsc{Contraction}(\tau_{K_2}^{\prec})$ in time $f(d) \cdot n^{2d}$. FPT or W[1]-hard by d?

co-NP-hard cases: natural to parameterize $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ by $\tau_{\mathcal{H}}^{\prec}$.

- If $\prec$ = minor and $\mathcal{H}$ contains a planar graph, FPT param. by $\tau_{\mathcal{H}}^{\prec} + k$.
- In particular, $\textsc{Contraction}(\text{fvs})$ is FPT param. by fvs $+ k$.
- What about non-planar collections $\mathcal{H}$?

Gràcies!