Algorithmic aspects of minor-closed graph classes

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Outline of this course

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
4. Bidimensionality
5. Irrelevant vertex technique
6. Application to hitting minors
7. Kernelization (?)

Outline of this course (more precise)

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   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
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6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

Evaluación de este curso

- En los slides, hay $\sim$ 20 preguntas, indicadas con (why?)
- El último día de curso, voy a elegir 12 o 13 de ellas, y podréis elegir 10 entre ellas para responderlas por escrito.
- Todos los slides están disponibles en `www.lirmm.fr/~sau/talks/ECI-2023-Ignasi.pdf`.
- Se podrán traer los slides en un ordenador, y apuntes.

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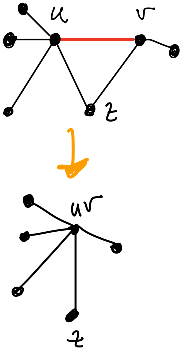
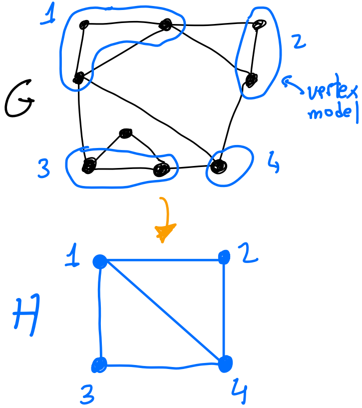
Graph minors

A graph H is a minor of a graph G, denoted by $H \leqslant_m G$, if H can be obtained by a subgraph of G by contracting edges.





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Note that, in general, this list $\mathcal{F}_{\mathcal{C}} = \{G_1, G_2, \dots\}$ may be infinite.

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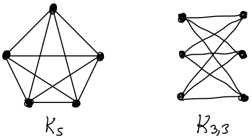

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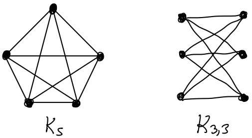

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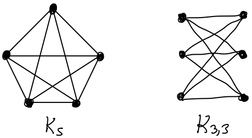

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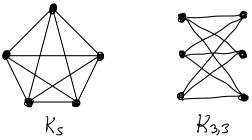

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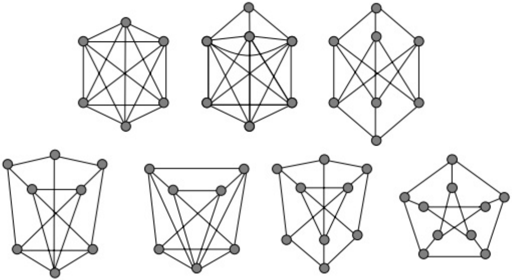

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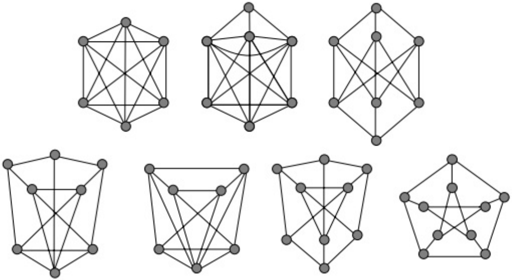

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$\mathcal{F}_{\mathcal{C}}$ seems to get complicated... but always finite!

Wagner's conjecture

Conjecture (Wagner. 1970)

For every minor-closed graph class $\mathcal{C}$, there exists a finite set of graphs $\mathcal{F}_{\mathcal{C}}$ such that $\mathcal{C} = \mathrm{exc}(\mathcal{F}_{\mathcal{C}})$.

Wagner's conjecture... now Robertson-Seymour's theorem

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Equivalent: For every minor-closed graph class $\mathcal{C}$, $\text{obs}(\mathcal{C})$ is finite.

Yet equivalent: Every infinite set $\{G_1, G_2, \ldots\}$ of finite graphs contains two graphs such that one is a minor of the other (there is no infinite antichain).

Well-quasi orders

A partially ordered set (poset) is a set P with a partial binary relation $\leq$:

1. Reflexive: $a \leq a$.
2. Antisymmetric: if $a \leq b$ and $b \leq a$, then $a = b$.
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A poset $(P, \leq)$ is well-quasi-ordered (wqo) if every infinite sequence $(x_1, x_2, \ldots)$ has two elements x_i and x_j such that $i < j$ and $x_i \leq x_j$.

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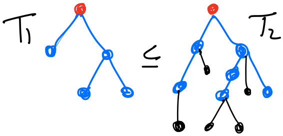
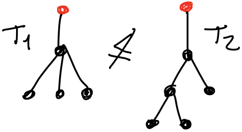
R&S theorem: Finite graphs are wqo with respect to the minor relation.

Illustrative example: rooted trees

Let T_1 and T_2 be two finite rooted trees.

Def: $T_1 \leq T_2$ if there is a subdivision of T_1 that occurs as a rooted subgraph of T_2 (the root of T_1 is not necessarily mapped to the root of T_2).





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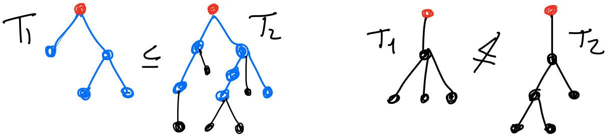

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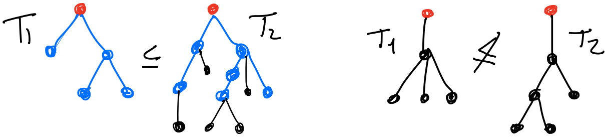

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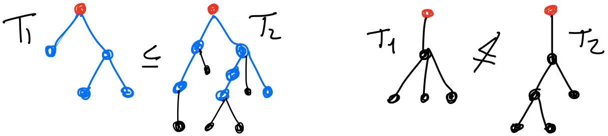

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We will now see a simple proof by [Nash-Williams. 1963]

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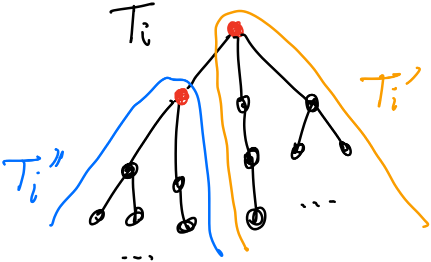
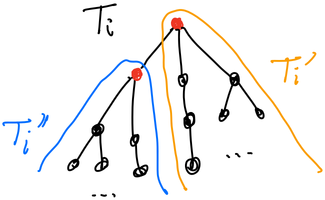
For $k \geq 1$:
Let T_i' be the tree obtained from T_i by deleting any branch from the root.
Let T_i'' be the deleted branch (rooted at a child of the root of T_i).





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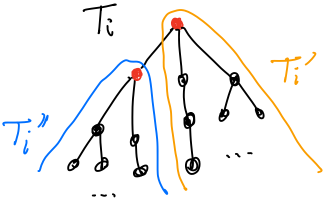

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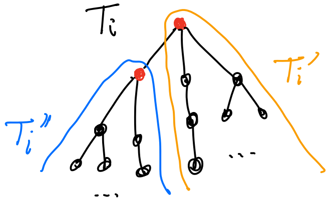
Proof: Suppose it does, and let $(T_{i_1}', T_{i_2}', \ldots)$ be a bad subsequence.



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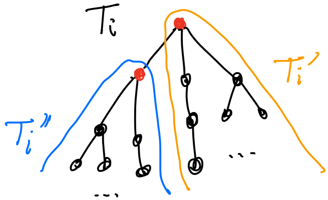

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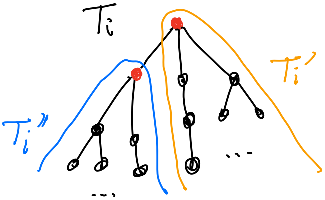

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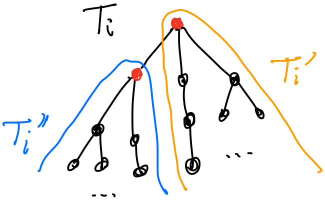

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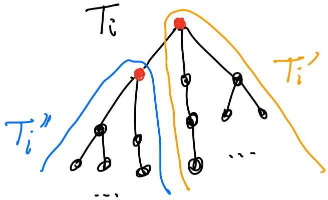

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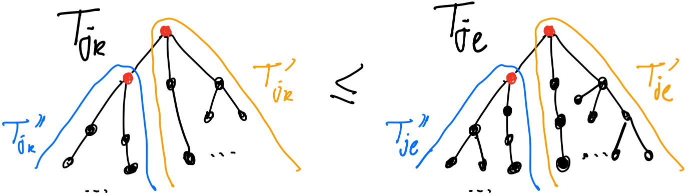

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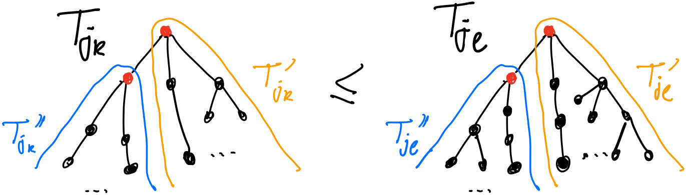

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A notion strongly linked to graph minors

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DISJOINT PATHS

Input: a graph G and $2k$ vertices $s_1, \dots, s_k, t_1, \dots, t_k$.

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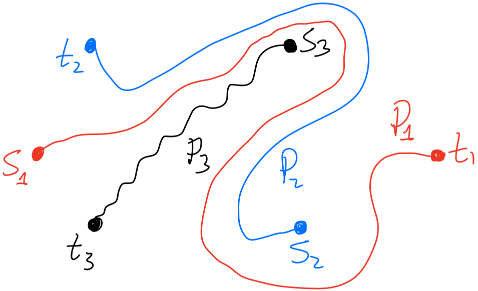

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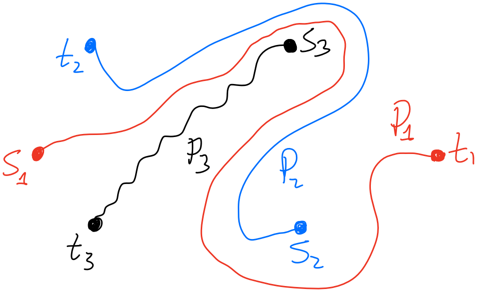

Much stronger than k vertex-disjoint paths from $s_1, \ldots, s_k$ to $t_1, \ldots, t_k$.

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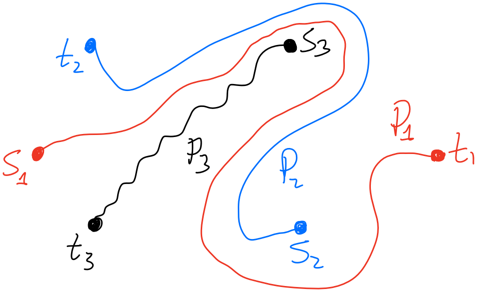

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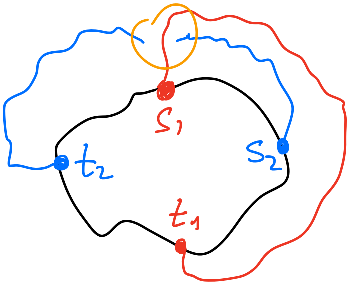
A graph G is k-linked if every instance of Disjoint Paths in G with k pairs is positive.

Topology appears naturally in linkages

Theorem (Thomassen and Seymour. 1980)

Let G be a 4-connected graph and $s_1, s_2, t_1, t_2 \in V(G)$. Then (s_1, s_2) and (t_1, t_2) are linked unless G is planar and s_1, s_2, t_1, t_2 are on the boundary of the same face, in this cyclic order.



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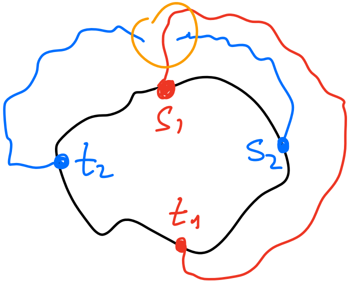

A combinatorial condition (linkage) is translated to a purely topological one (embedding).

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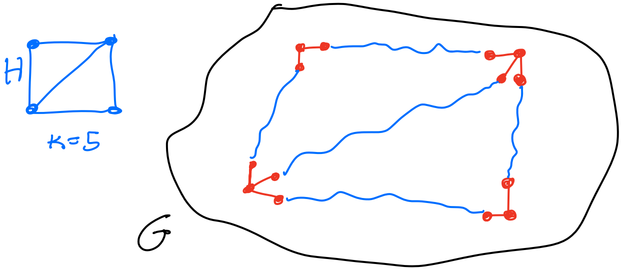

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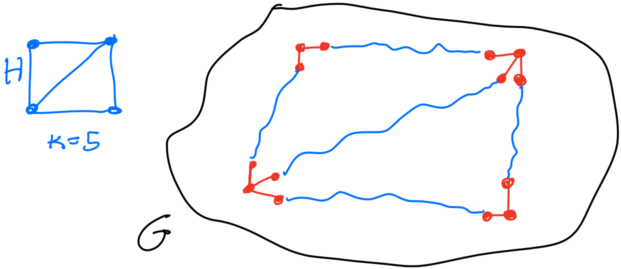

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Idea: if the goal is to decide whether $H \leq_m G$, if G is k-linked, then "yes".
Otherwise, we may exploit a topological obstruction to k-linkedness...

Another crucial notion: treewidth

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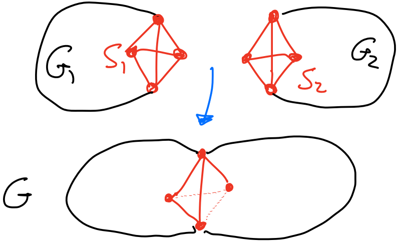

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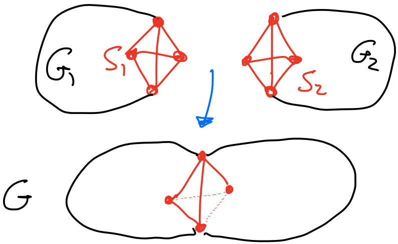

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We say that a graph G has treewidth at most k if it can be obtained by repeatedly taking a k-clique-sum with a graph on at most $k+1$ vertices.

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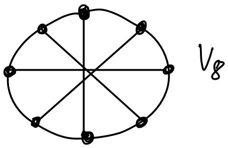
Theorem (Wagner. 1937)

A graph $G \in \mathrm{exc}(K_5)$ if and only if it can be obtained by 0-, 1-, 2- and 3-clique-sums from planar graphs and V_8.



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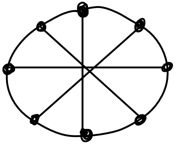

V_8

Paradigm: we find "pieces" that exclude K_5 for topological reasons (planarity), add some exceptions (V_8), and then define rules (clique-sums) that preserve being K_5-minor-free.

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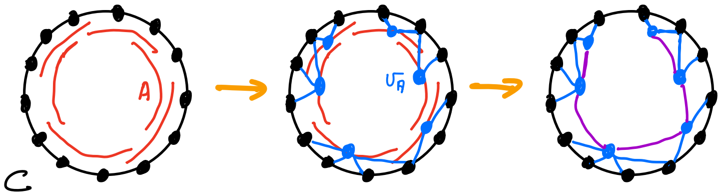
Note: this is an approximate characterization (i.e., not "iff").

Vortices



Adding a vortex of depth h to a cycle C:

- Select arcs on C so that each vertex is contained in at most h arcs.
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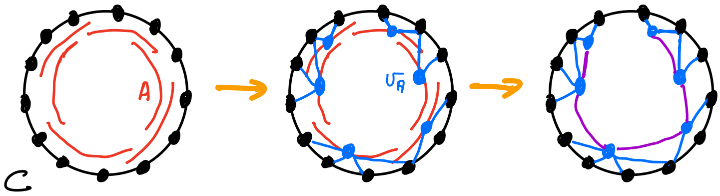

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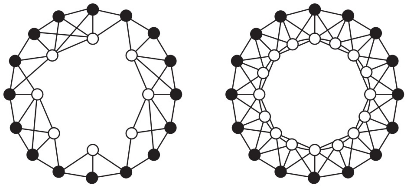

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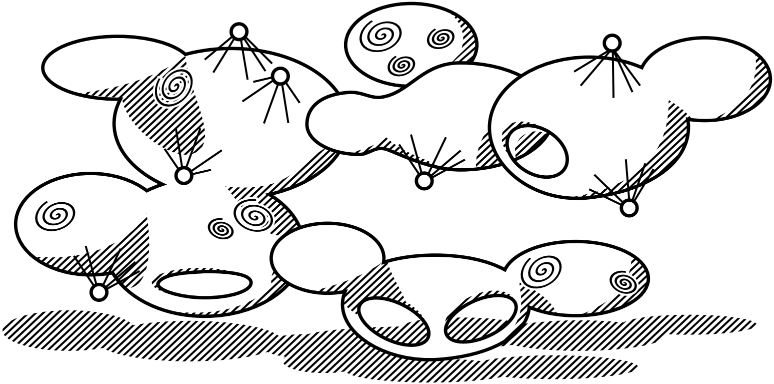
4. *Repeatedly construct the h-clique-sum of the current graph with another graph constructed using steps 1-2-3 above.*

A visualization of an H-minor-free graph



[Figure by Felix Riedl]

Sketch of sketch of sketch of proof of Wagner's conjecture

Let's try to mimic the proof for rooted trees by Nash-Williams:

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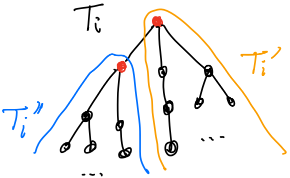

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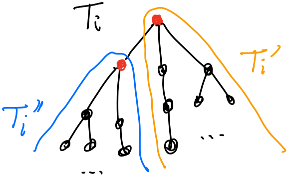

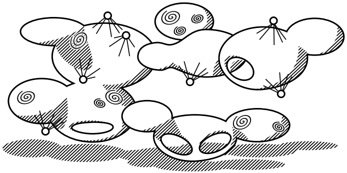

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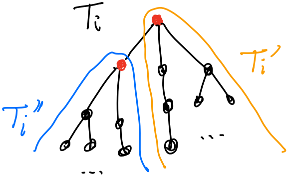

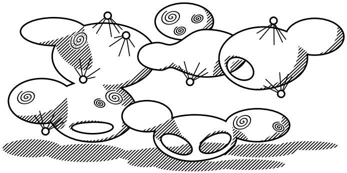

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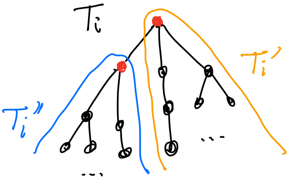

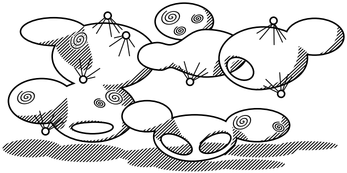

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- If G_1 is planar, every G_i has bounded treewidth: similar to trees.
- Otherwise, by the structure theorem: similar to "extended" surfaces (with apices and vortices), glued in a tree-like way.

Some algorithmic consequences

DISJOINT PATHS

Input: an n-vertex graph G and vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$.

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Theorem (Robertson, Seymour. 1995)

The DISJOINT PATHS *problem can be solved in time* $f(k) \cdot n^3$.

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The DISJOINT PATHS *problem can be solved in time* $f(k) \cdot n^3$.

Improved to $f(k) \cdot n^2$. [Kawarabayash, Kobayashi, Reed. 2012]

Some algorithmic consequences

DISJOINT PATHS
Input: an n-vertex graph G and vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$.
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Corollary

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This says that there $\boxed{\text{exists}}$ an algorithm... no idea how to $\boxed{\text{construct}}$ it!!

A few words on other containment relations

★ **Minor:** $H \preceq_m G$ if H can be obtained from a subgraph of G by contracting edges.

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Structure of sparse graphs

H-topological-minor-free

$\cup$

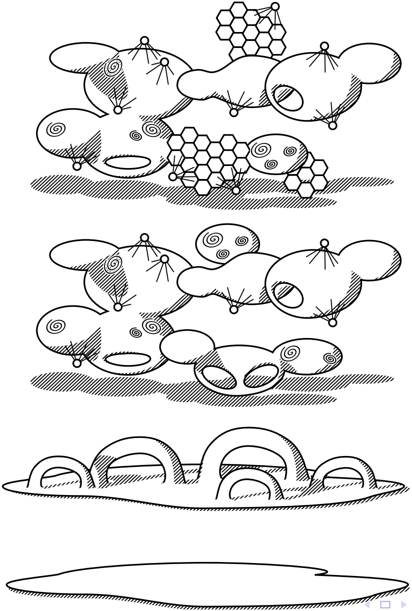
H-minor-free

$\cup$

bounded genus

$\cup$

planar



[Figure by Felix Riedl]

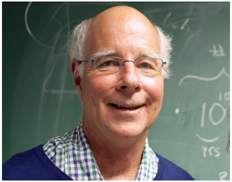
Next section is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

Parameterized complexity in a nutshell

$\boxed{\text{Idea}}$ Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established and very active area.

Parameterized problems

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed, finite alphabet.

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- k-Vertex Cover: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, containing at least an endpoint of every edge?
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These three problems are NP-hard, but are they $\boxed{\text{equally}}$ hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- k-Clique: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
- Vertex k-Coloring: NP-hard for fixed $k = 3$.

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skip

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Working hypothesis of parameterized complexity: $\boxed{k\text{-Clique is not FPT}}$

(in classical complexity: 3-SAT cannot be solved in poly-time)

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Fact: $\boxed{\text{A problem is FPT} \Leftrightarrow \text{it admits a kernel}}$

Do all FPT problems admit polynomial kernels? $\boxed{\text{NO!}}$

Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)

Deciding whether a graph has a PATH *with* $\geq k$ *vertices is* FPT *but does not admit a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

Typical approach to deal with a parameterized problem

Parameterized problem L

k-CLIQUE
k-VERTEX COVER
k-PATH
VERTEX k-COLORING

Typical approach to deal with a parameterized problem

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XP

para-NP-hard

k-CLIQUE

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Typical approach to deal with a parameterized problem

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W[1]-hard

FPT

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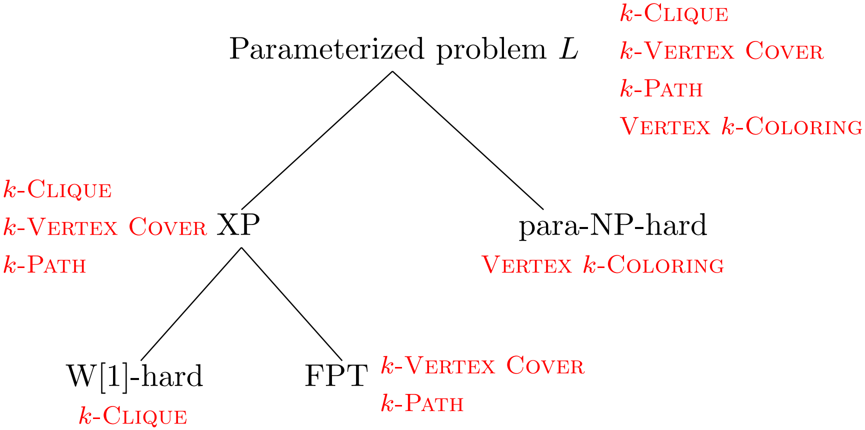

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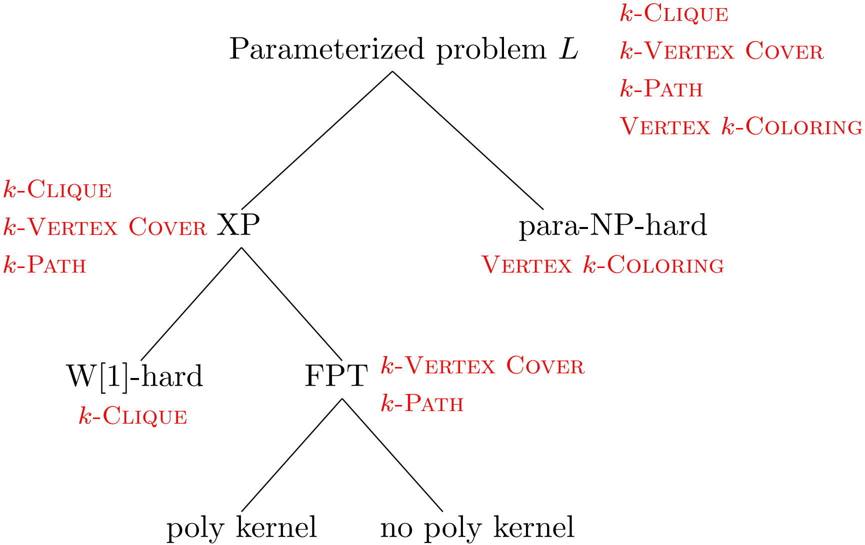

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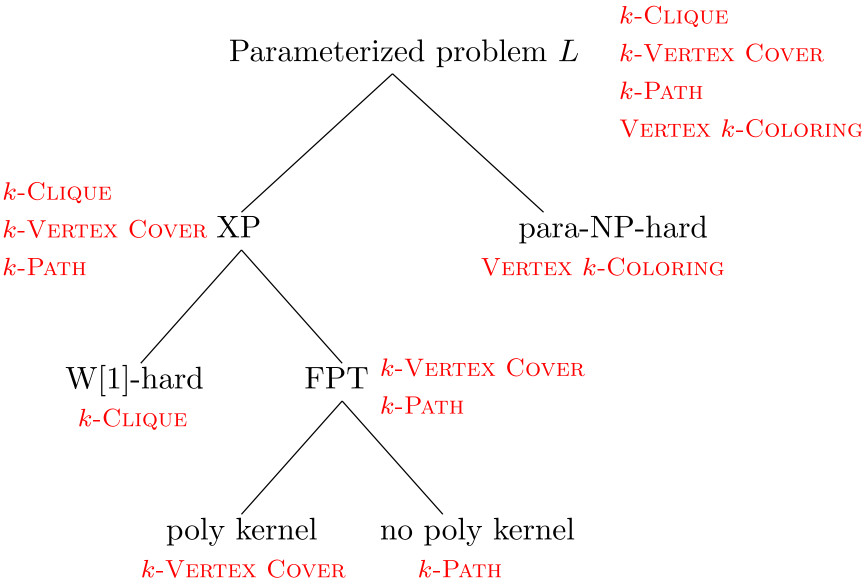

Next section is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
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   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

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The multiples origins of treewidth

- 1972: Bertelè and Brioschi (dimension).
- 1976: Halin (S-functions of graphs).
- 1984: Arnborg and Proskurowski (partial k-trees).
- 1984: Robertson and Seymour (treewidth).

A measure of the similarity with a tree

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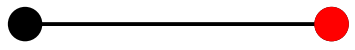

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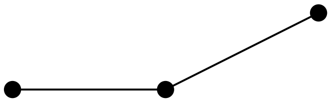

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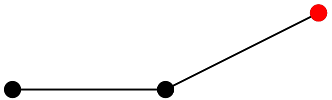

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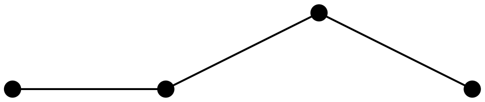

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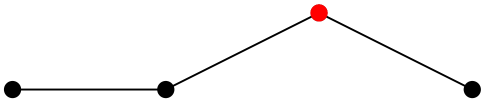

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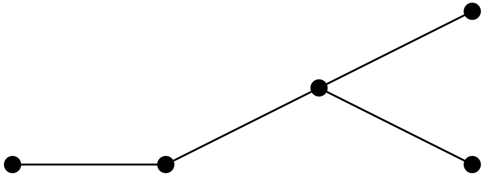

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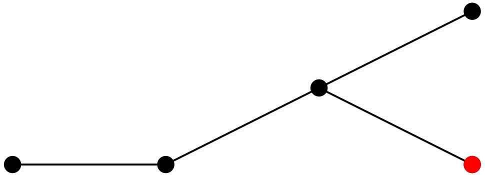

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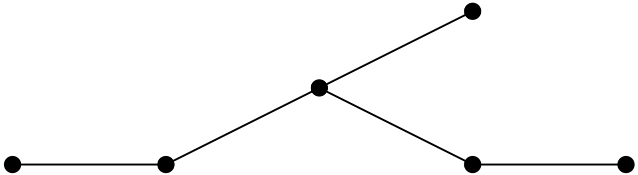
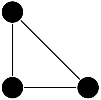


Treewidth via k-trees

Example of a 2-tree:



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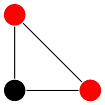

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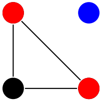

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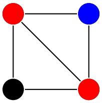

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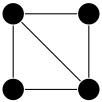

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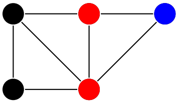

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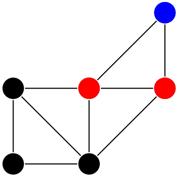

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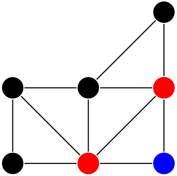

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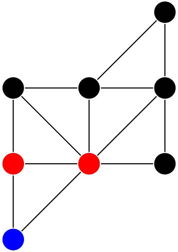

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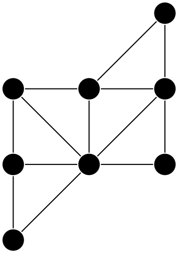

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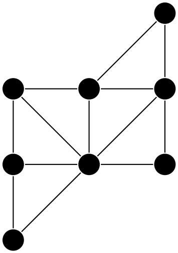

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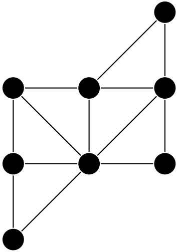

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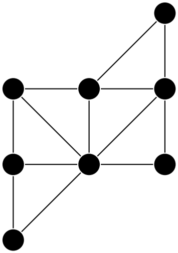

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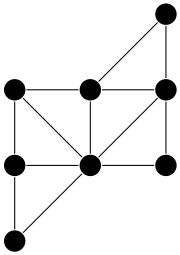

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{X_t \mid t \in V(T)\})$, where
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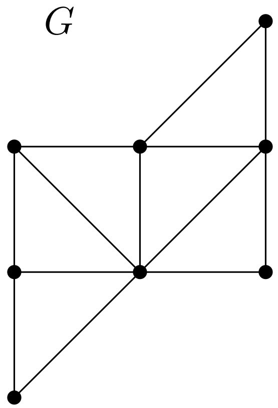

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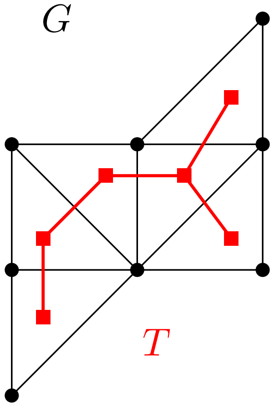

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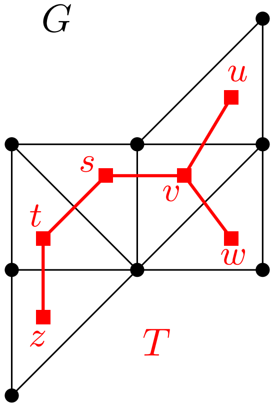

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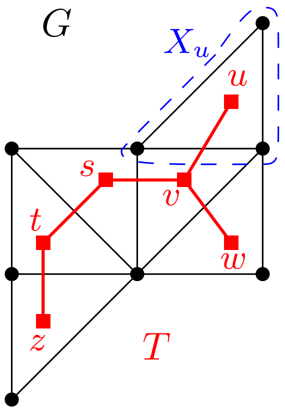

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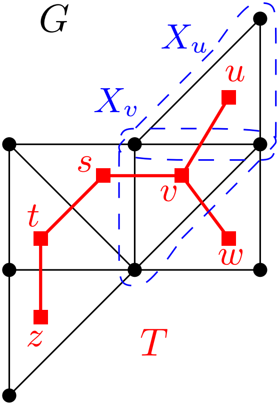

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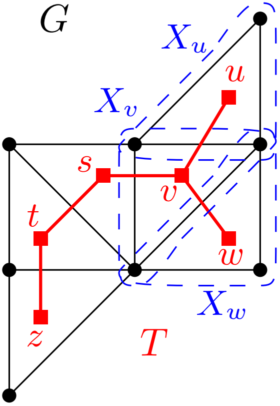

An equivalent (and more common) definition of treewidth

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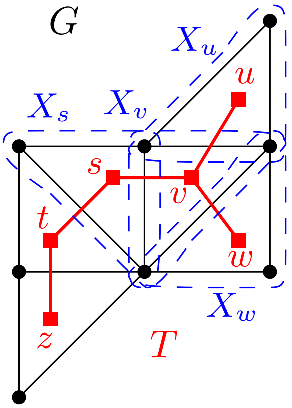

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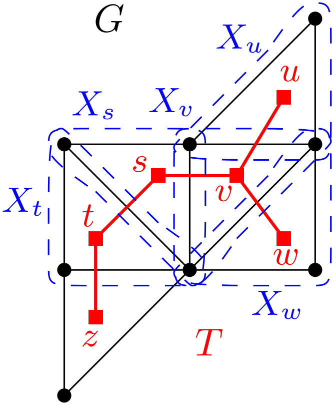

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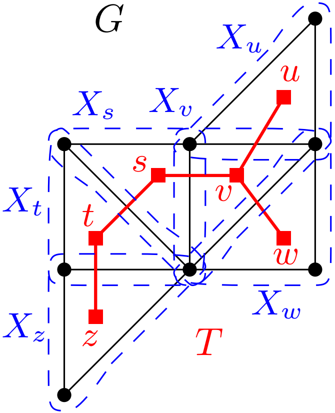

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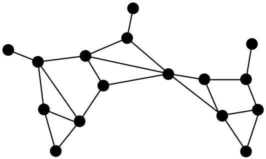

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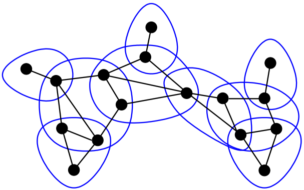

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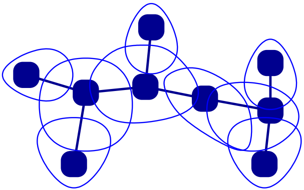

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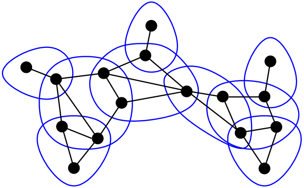

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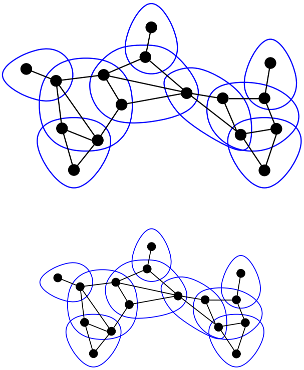

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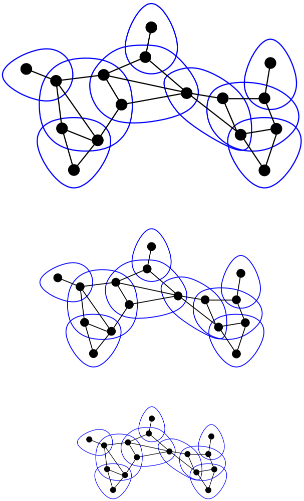

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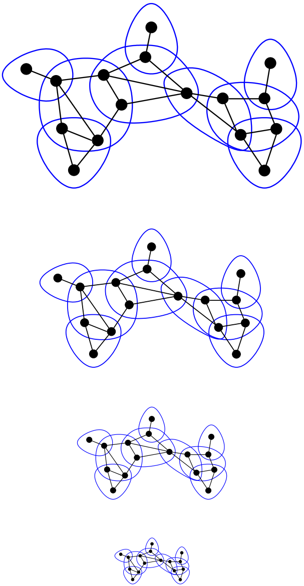

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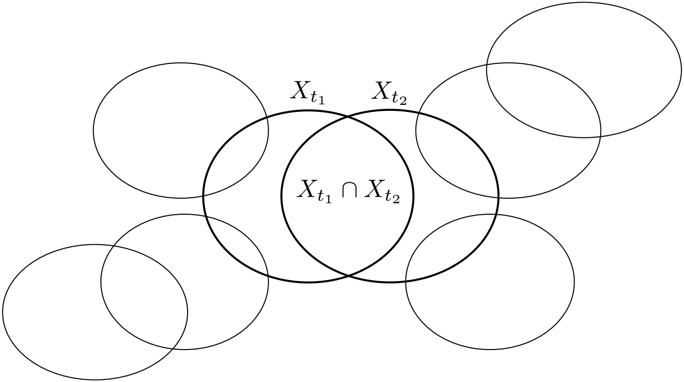

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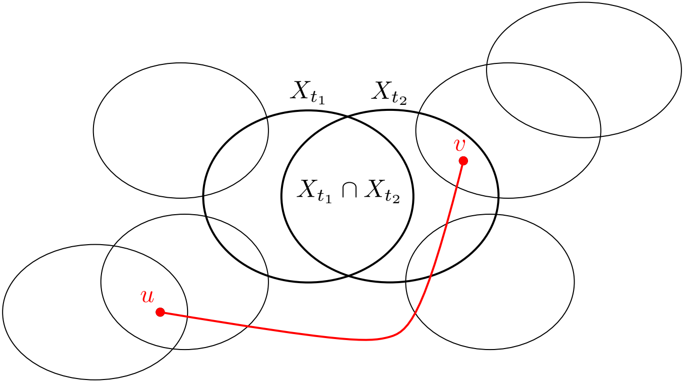

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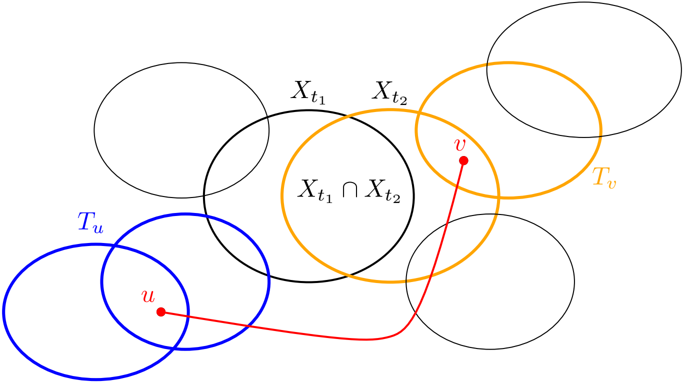

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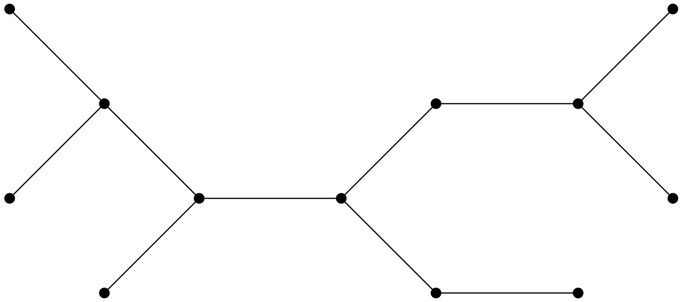

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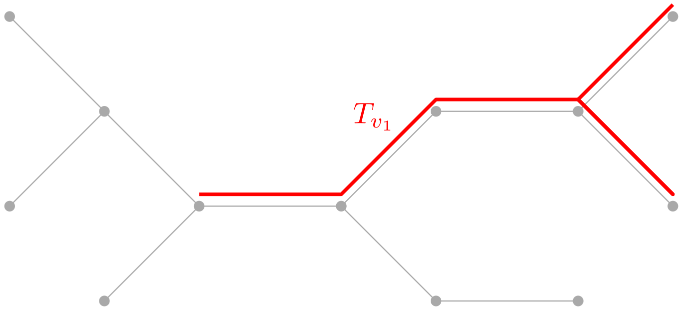

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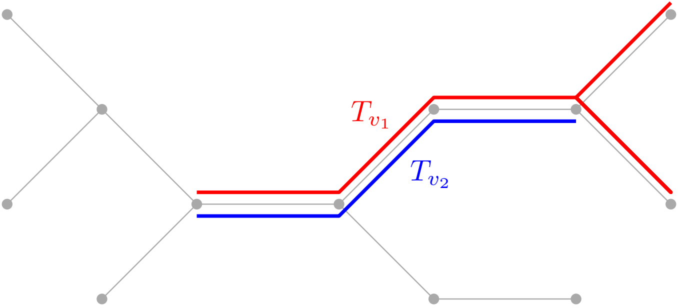

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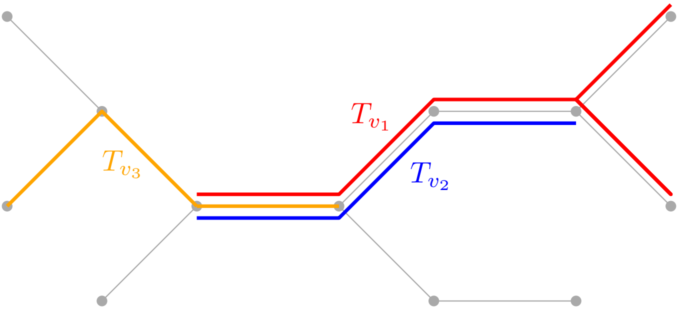

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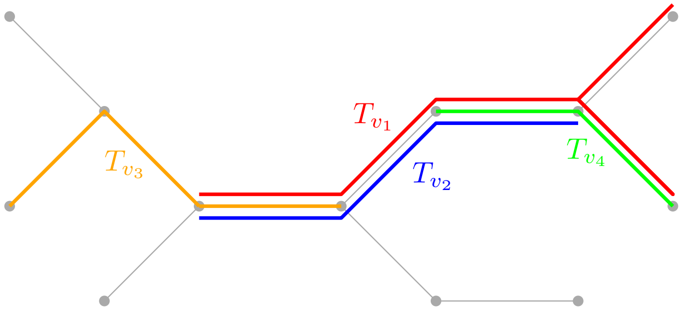

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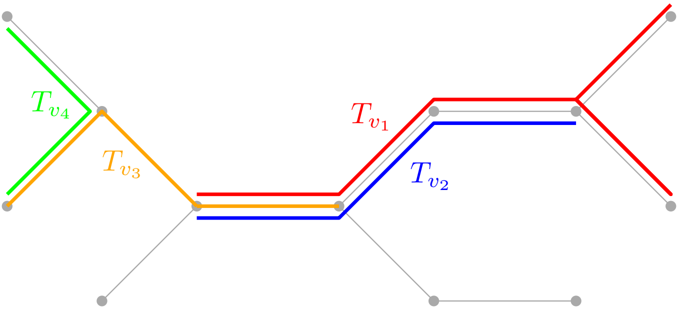

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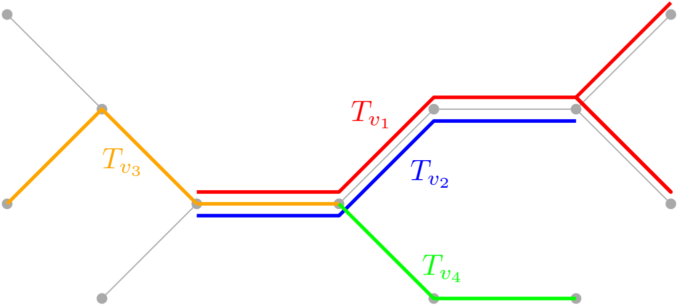

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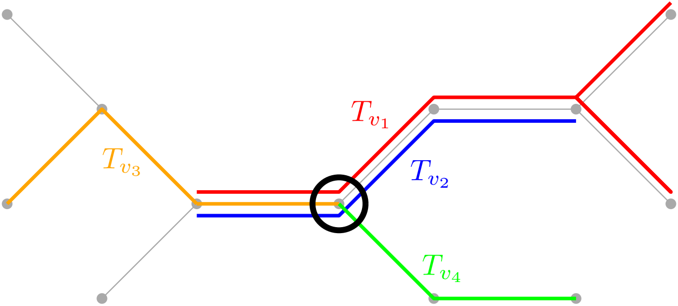

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3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

Next subsection is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

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Theorem (Robertson and Seymour. 1993)

For every $k \geq 0$ and graph G, the treewidth of G is at least k if and only if G contains a bramble of order at least $k+1$.

Another dual notion to treewidth: linkedness

[slides borrowed from Christophe Paul]

- Two sets $Y, Z \subseteq V(G)$, with $|Y| = |Z|$, are separable if there is a set $S \subseteq V(G)$ with $|S| < |Y|$ and such that $G - S$ contains no path between $Y \setminus S$ and $Z \setminus S$.

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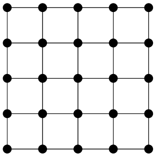

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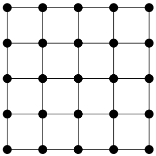

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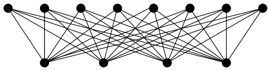

$K_{2k,k}$ is k-well-linked (why?)

Highly linked graphs have large treewidth

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If G contains a $(k+1)$-well-linked set X with $|X| \geqslant 3k$, then $\mathrm{tw}(G) \geq k$.

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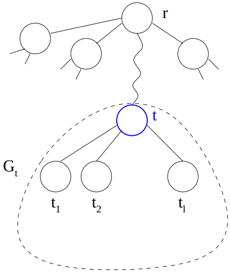

Let t be a "lowest" node with $|V_t \cap X| > 2k$.

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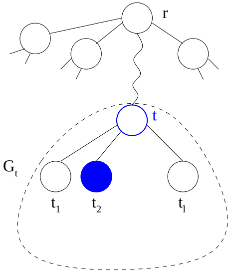

Let t be a "lowest" node with $|V_t \cap X| > 2k$.

If $\exists i \in [\ell]$ such that $|V_{t_i} \cap X| \geqslant k$, then we can choose $Y \subseteq V_{t_i} \cap X$, $|Y| = k$ and $Z \subseteq (V \setminus V_{t_i}) \cap X$, $|Z| = k$.

But $S = X_{t_i} \cap X_t$ separates Y and Z and $|S| \leq k - 1$.

Highly linked graphs have large treewidth

Lemma

If G contains a $(k+1)$-well-linked set X with $|X| \geqslant 3k$, then $\mathrm{tw}(G) \geq k$.

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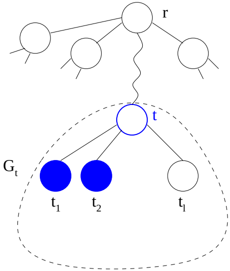

Let t be a "lowest" node with $|V_t \cap X| > 2k$.

Otherwise, let $W = V_{t_1} \cup \cdots \cup V_{t_i}$ with $|W \cap X| > k$ and $|(W \setminus V_{t_j}) \cap X| \leq k$ for $1 \leq j \leq i$.

$Y \subseteq W \cap X$, $|Y| = k+1$ and
$Z \subseteq (V \setminus W) \cap X$, $|Z| = k+1$ (why?).

But $S = X_t$ separates Y from Z and $|S| \leqslant k$.

Deciding linkedness is FPT

Lemma

Given a vertex set X of a graph G and $k \leq |X| \leq ck$ for some constant c, it is possible to decide whether X is k-well-linked in time $f(k) \cdot n^{\mathcal{O}(1)}$.

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Remark: If X is not k-well-linked we can find, within the same running time, two separable subsets $Y, Z \subseteq X$.

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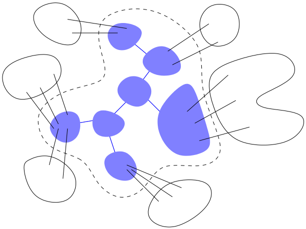
4-approximation of Robertson and Seymour

[slides borrowed from Christophe Paul]



Idea

- We add vertices to a set U in a greedy way, until $U = V(G)$.

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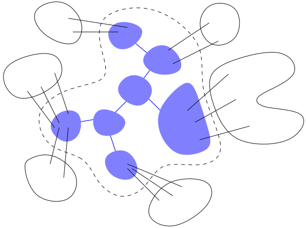

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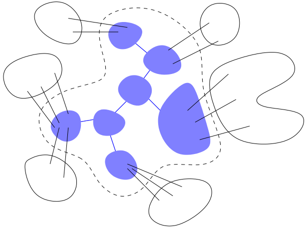

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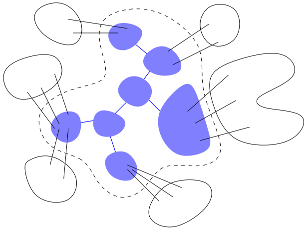

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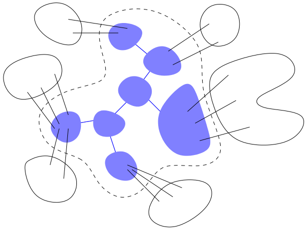

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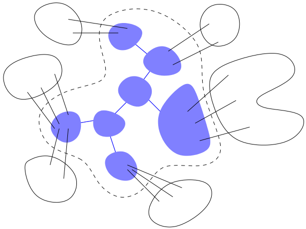

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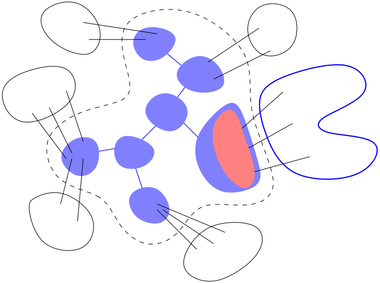
Initially, we start with U being any set of $3k$ vertices.

4-approximation of Robertson and Seymour (2)



Let X be the neighbors of a component C and t be the node s.t. $X \subseteq X_t$.

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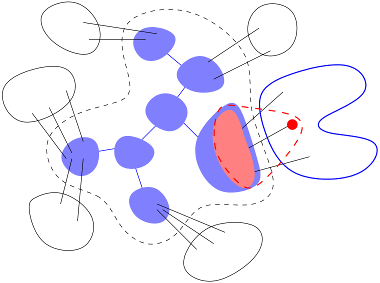

Let X be the neighbors of a component C and t be the node s.t. $X \subseteq X_t$.

- If $|X| < 3k$: we add a node t' neighbor of t such that $X_{t'} = \{x\} \cup X$, with $x \in C$ being a neighbor of X_t. The invariant is respected (why?).

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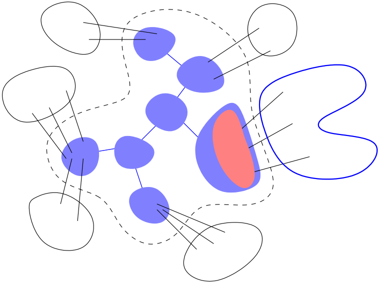

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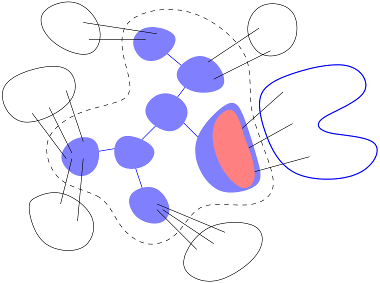

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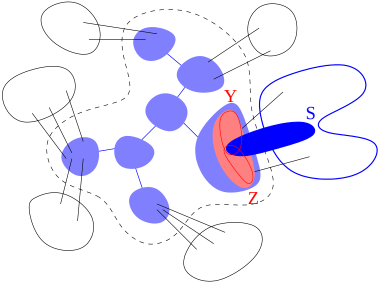

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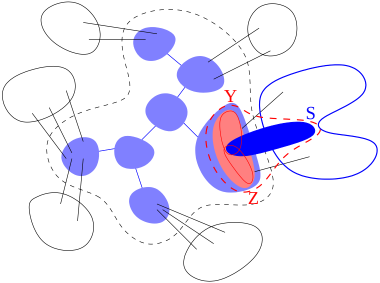

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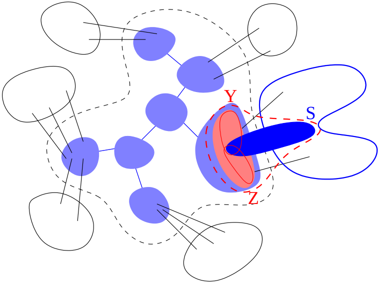

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     Obs: the neighbors of every new component $C' \subseteq C$ are in $(X \setminus Z) \cup (S \cap C)$ or in $(X \setminus Y) \cup (S \cap C)$

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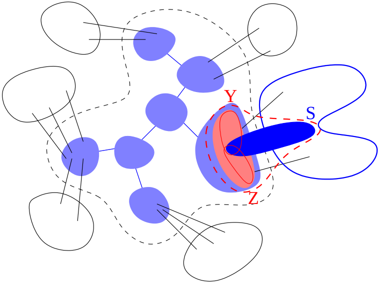

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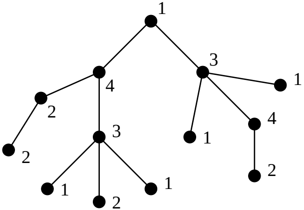
Weighted Independent Set on trees

[slides borrowed from Christophe Paul]



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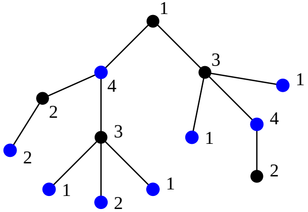

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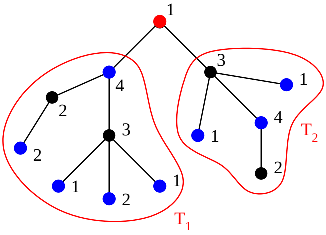

Observations:

1. Every vertex of a tree is a separator.
2. The union of independent sets of distinct connected components is an independent set.

Weighted Independent Set on trees

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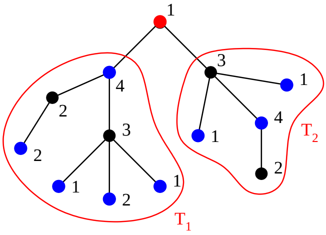

Let x be the root of T, $x_1 \dots x_\ell$ its children, $T_1, \dots T_\ell$ subtrees of $T - x$:

- $wIS(T, x)$: maximum weighted independent set containing x.
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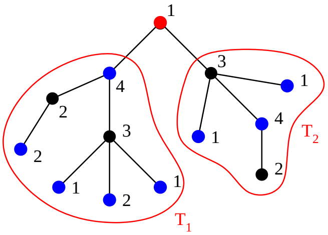

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$$\begin{cases} wIS(T,x) = \omega(x) + \sum_{i \in [\ell]} wIS(T_i, \overline{x_i}) \\ \end{cases}$$

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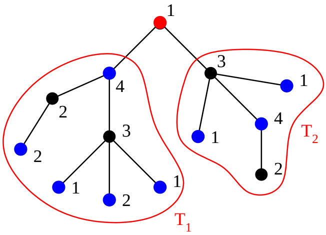

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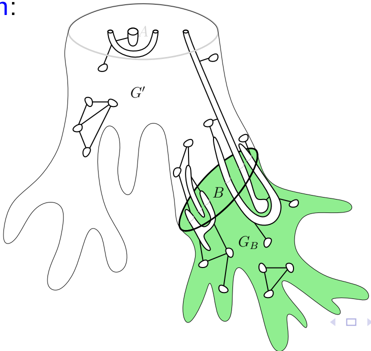
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Back to tree decompositions

Let $(T, \{X_t \mid t \in V(T)\})$ be a tree decomposition of a graph G.

- For every $t \in V(T)$, X_t is a separator in G.
- For every edge $\{t_1, t_2\} \in E(T)$, $X_{t_1} \cap X_{t_2}$ is a separator in G.

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- V_t: all vertices of G appearing in bags that are descendants of t.
- $G_t = G[V_t]$.

INDEPENDENT SET on tree decompositions

$\boxed{\forall S \subseteq X_t,\ IS(S,t) \ = \text{maximum independent set } I \text{ of } G_t \text{ s.t. } I \cap X_t = S}$

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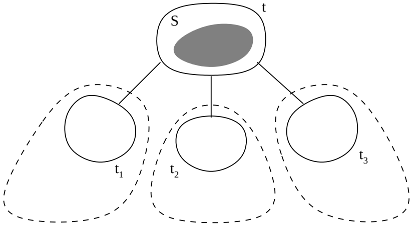

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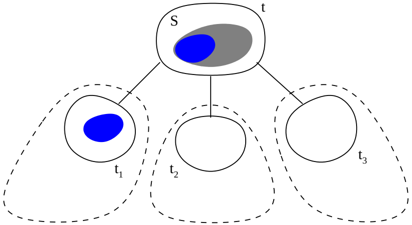

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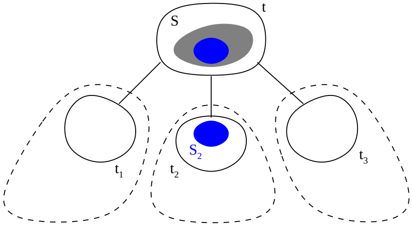

$\boxed{\text{Lemma}}$ If $S \subseteq X_t$ and $S_j = S \cap X_{t_j}$, then $|IS(S,t) \cap V_{t_j}| = |IS(S_j,t_j)|$.

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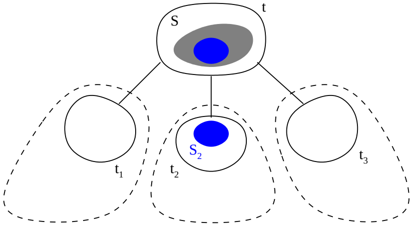

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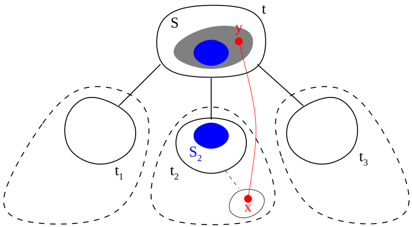

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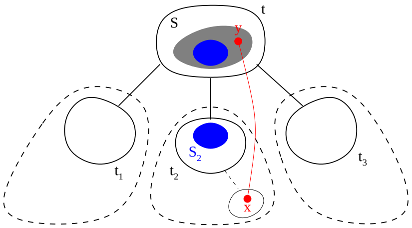

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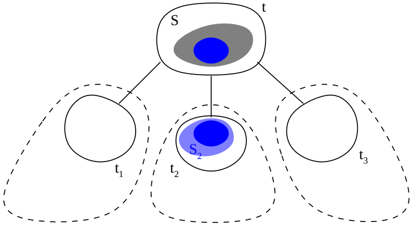
Contradiction! X_{t_j} is not a separator.

Idea of the dynamic programming algorithm:



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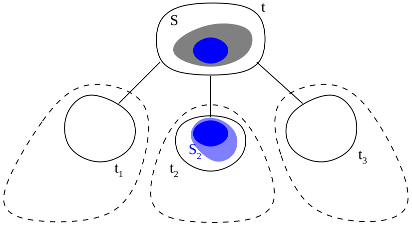

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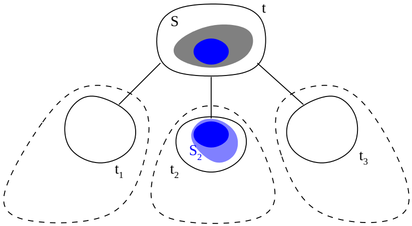

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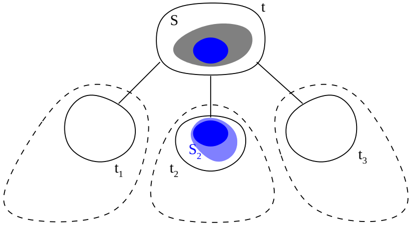

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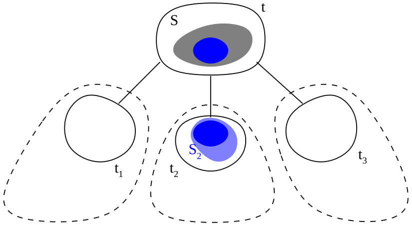

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$$|IS(S,t)| = \left\{ \sum_{i\in[\ell]} \max \begin{array}{l} |S| + \\ \{|IS(S_j^i, t_j)| - |S_j| : \\ S_j^i \cap X_t = S_j \ \wedge \ S_j \subseteq S_j^i \text{ independent}\} \end{array} \right.$$

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★ We have to add the time in order to compute a "good" tree decomposition of the input graph (as we have seen before).

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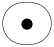

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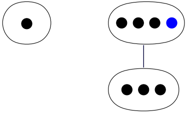

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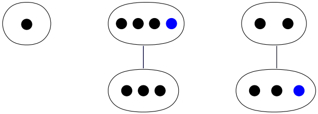

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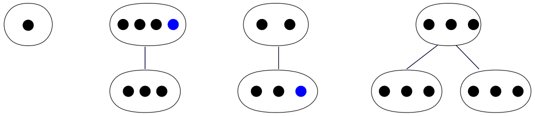

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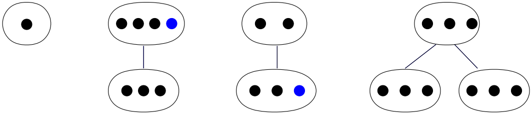

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Lemma

A tree decomposition $(T, \{X_t : t \in T\})$ of width k and x nodes of an n-vertex graph G can be transformed in time $\mathcal{O}(k^2 \cdot n)$ into a nice tree decomposition of G of width k and $\mathcal{O}(k \cdot x)$ nodes. (why?)

Simpler algorithm for INDEPENDENT SET

How to compute $IS(S, t)$ for every $S \subseteq X_t$:

Simpler algorithm for Independent Set

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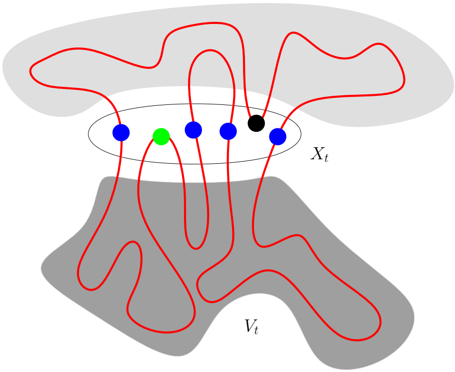
$$\boxed{\text{Complexity}: \mathcal{O}(2^k \cdot k^2 \cdot n)}$$

[slides borrowed from Christophe Paul]

Let $\mathcal{C}$ be a Hamiltonian cycle.

- Note that $\mathcal{C} \cap G[V_t]$ is a collection of paths.



Hamiltonian Cycle on tree decompositions

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Let $\mathcal{C}$ be a Hamiltonian cycle.

- Note that $\mathcal{C} \cap G[V_t]$ is a collection of paths.
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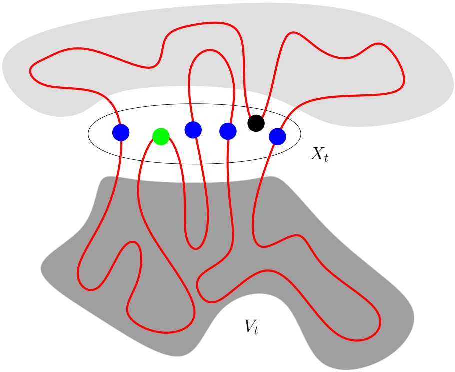

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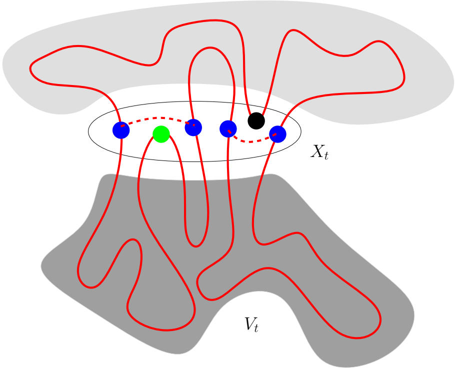


For every node t of the tree decomposition, we need to know if

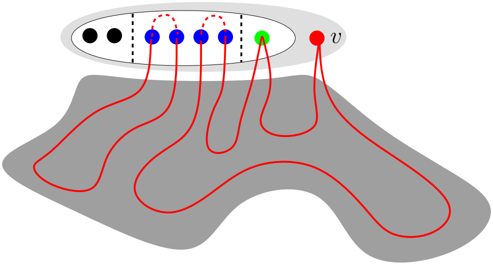
$$(X_t^0, X_t^1, X_t^2, M)$$

where M is a **matching** on X_t^1, corresponds to a partial solution.

skip

Forget node

Let t be a forget node and t' its child such that $X_t = X_{t'} \setminus \{v\}$.



Claim X_t is a separator $\Rightarrow$
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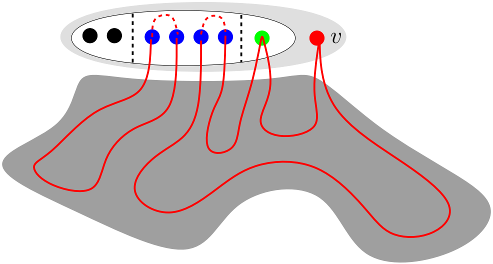

$\boxed{\text{Claim}}$ X_t is a separator $\Rightarrow$
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$$(X^0_{t'}, X^1_{t'}, X^2_{t'} \setminus \{v\}, M) \text{ is a partial solution for } t$$
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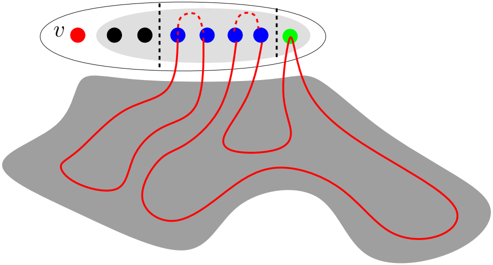

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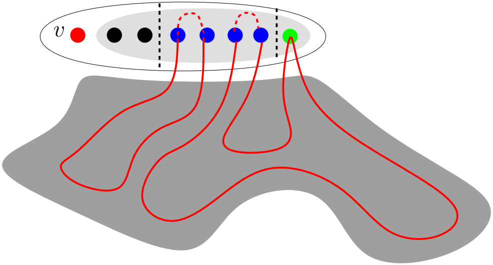

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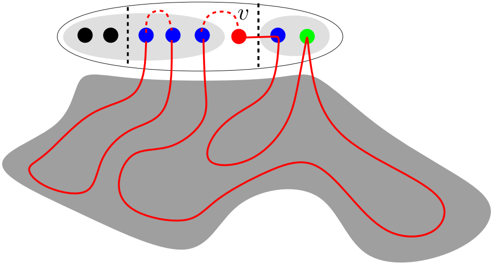

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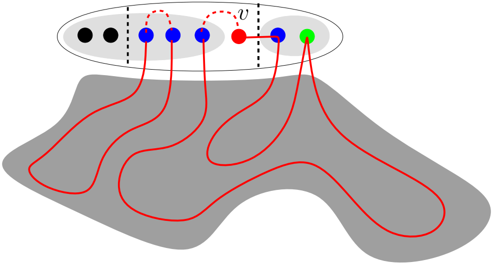

$\boxed{\text{Fact}}$ $X_{t'}$ is a separator $\Rightarrow$ $N(v) \cap V_t \subseteq X_t$.

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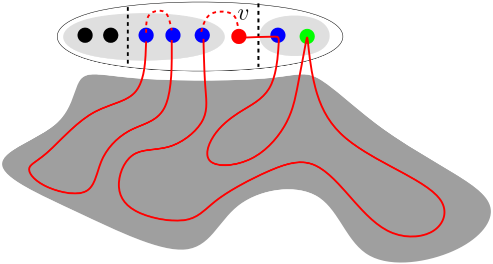

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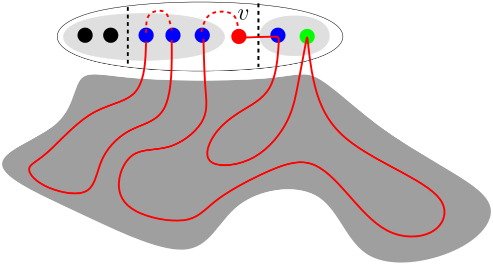

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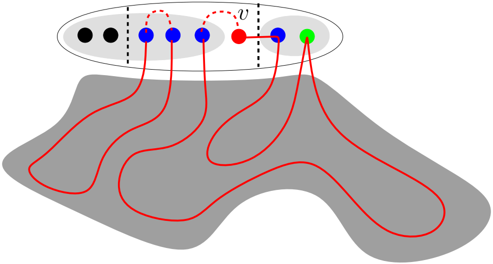

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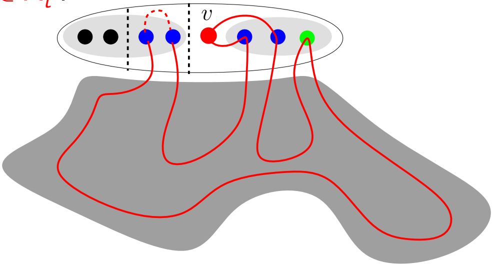

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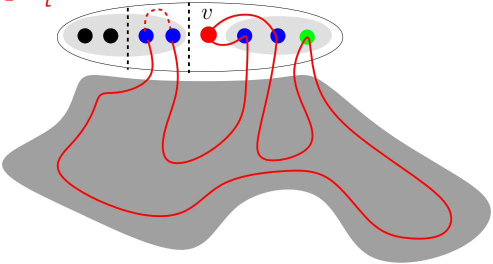

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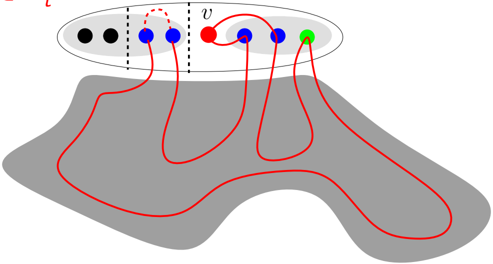

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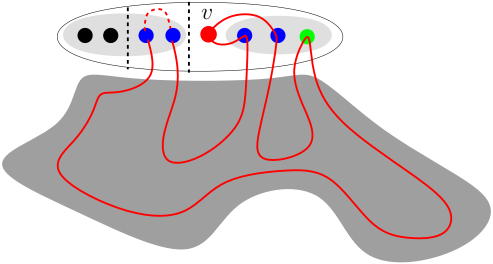

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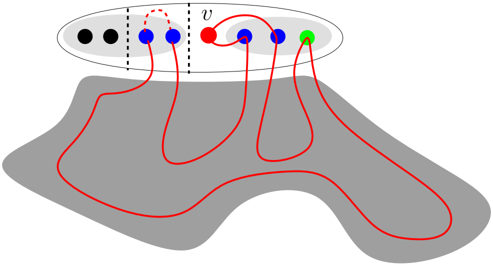

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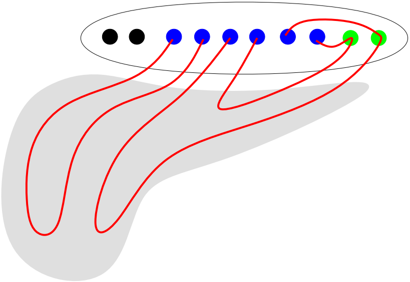

Fact For being compatible, partial solutions should verify:

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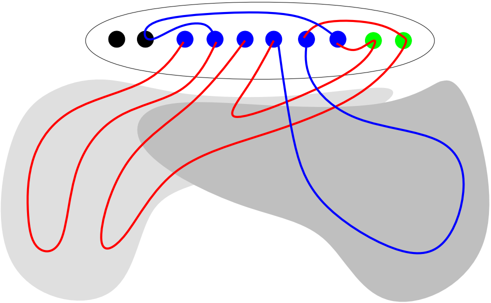

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Can this approach be generalized to more problems?

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- Quantifiers $\exists$, $\forall$ on vertex/edge variables or vertex/edge **sets**.

(MSO_1/MSO_2)

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2. Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

Next subsection is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

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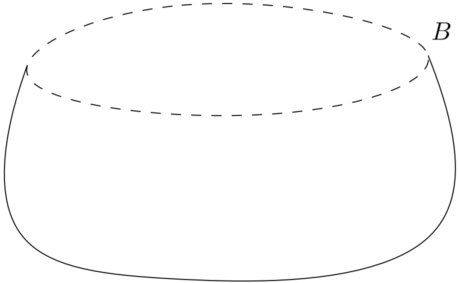
Remark: Algorithms parameterized by treewidth appear very often as a "black box" in all kinds of parameterized algorithms.

Two behaviors for problems parameterized by treewidth

$\boxed{\text{Local problems}}$ Vertex Cover, Dominating Set, Clique, Independent Set, q-Coloring for fixed q.



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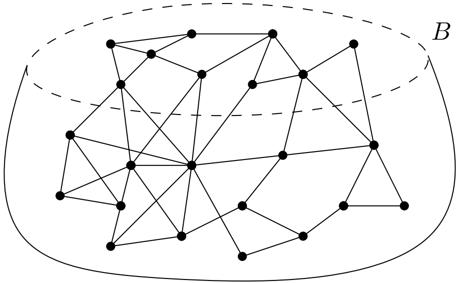

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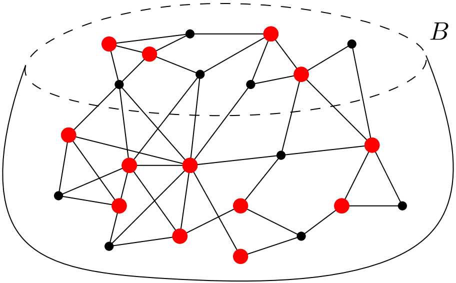

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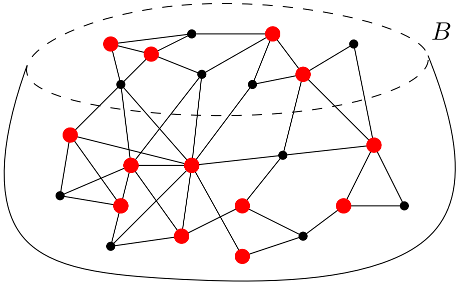

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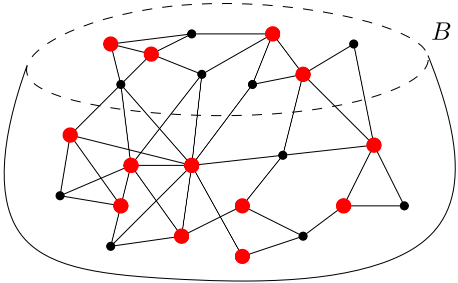

- It is sufficient to store, for each bag B, the subset of vertices of B that belong to a partial solution: 2^{tw} choices
- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

$$2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}.$$

Connectivity problems seem to be more complicated...

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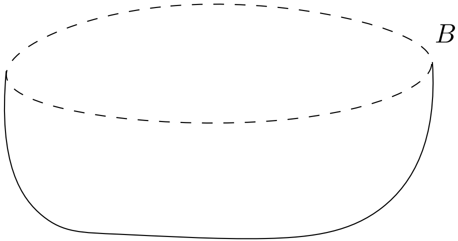

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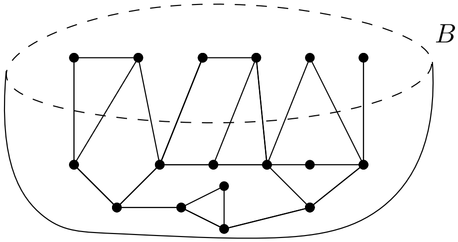

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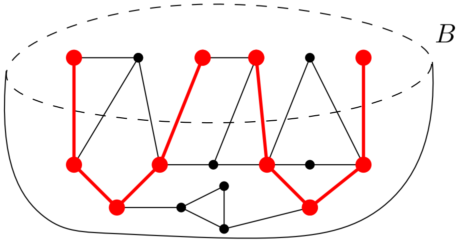

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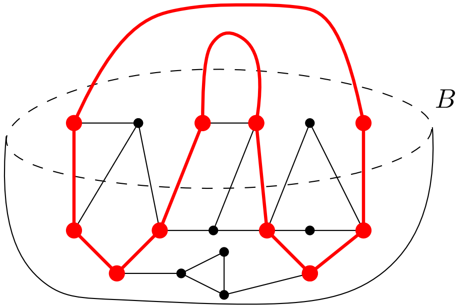

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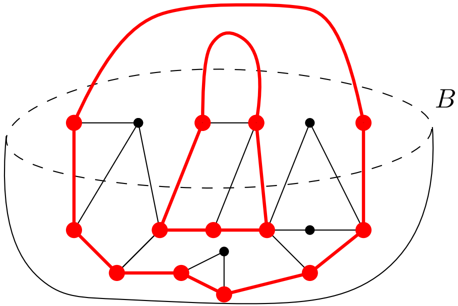

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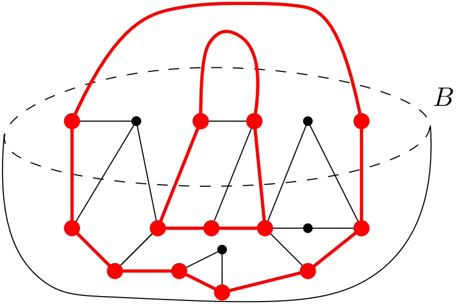

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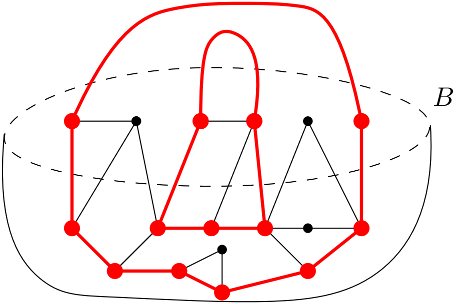

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- The "natural" DP algorithms provide only time $2^{\mathcal{O}(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$.

Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

$$\boxed{2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

Vertex Cover, Dominating Set, ...

- Connectivity problems:

$$\boxed{2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

Longest Path, Steiner Tree, ...

How topology helps for dynamic programming?

On $\boxed{\text{topologically structured}}$ graphs (planar, surfaces, minor-free), it is possible to solve connectivity problems in time $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$:

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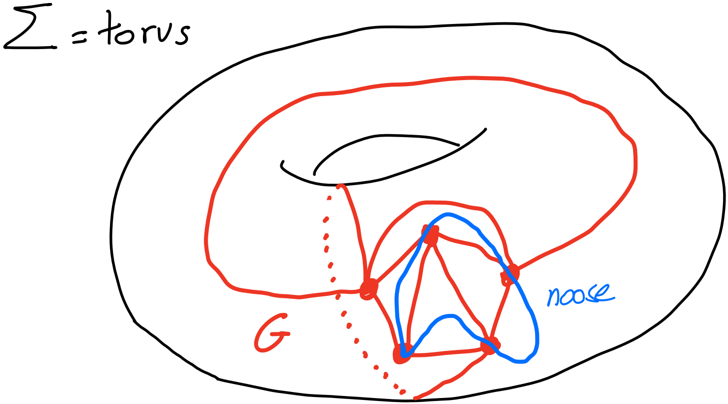
On $\boxed{\text{topologically structured}}$ graphs (planar, surfaces, minor-free), it is possible to solve connectivity problems in time $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$:

- We consider a special tree-decomposition of a sparse graph, and exploit the structure of the subgraph induced by the bags.
- More precisely, we use the existence of tree decompositions of small width and with nice topological properties.
- These nice properties do $\boxed{\text{not}}$ change the DP algorithms, but the analysis of their running time.

Nooses

Let G be a graph embedded in a surface Σ. A noose is a subset of Σ homeomorphic to $\mathbb{S}^1$ that meets G only at vertices.



- Let G be a planar graph. A sphere cut decomposition of G is a tree decomposition $(T, \{X_t : t \in V(T)\})$ of G such that the vertices in each bag X_t are situated around a noose in the plane.

[NB: several details are missing in this definition]

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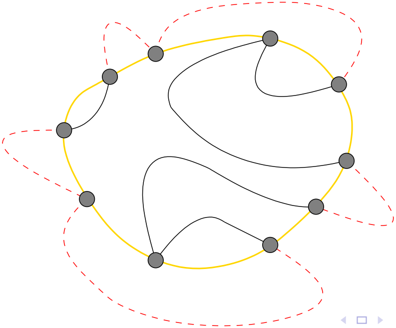

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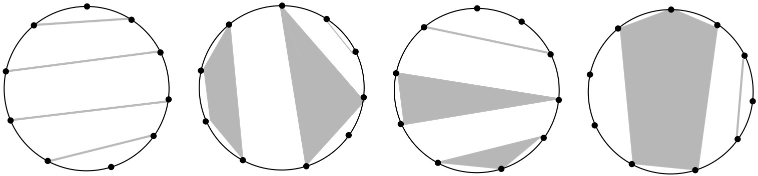

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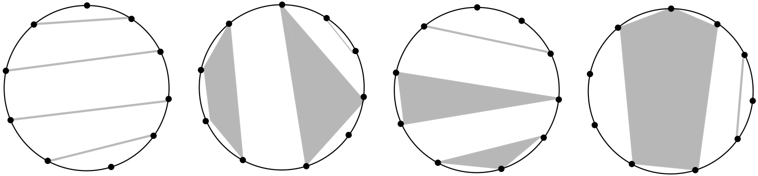

- Exactly the number of *non-crossing partitions* over k elements, which is given by the k-th Catalan number:

$$\mathrm{CN}(k) = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}} \approx 4^k.$$

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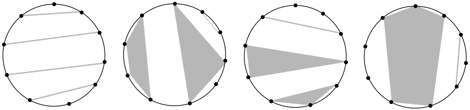
This idea was first used in [Dorn, Penninkx, Bodlaender, Fomin. 2005]

Generalizations to other sparse graph classes

Main idea special type of decomposition with nice topological properties:

partial solutions $\Longleftrightarrow$ non-crossing partitions



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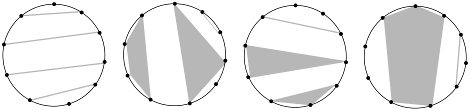

This idea has been generalized to other graph classes and problems:

- Graphs on surfaces: [Dorn, Fomin, Thilikos '06] [Rué, S., Thilikos '10]
- H-minor-free graphs: [Dorn, Fomin, Thilikos '08] [Rué, S., Thilikos '12]

The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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There are other examples of such problems (as we may see later)...

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2. Introduction to parameterized complexity
3. Treewidth
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   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
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A few representative problems

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Question: Does there exist a subset $C \subseteq V$ of size at most k such that $G[V \setminus C]$ is an independent set?

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LONG PATH
Input: A graph $G = (V, E)$ and a positive integer k.
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Question: Does there exist a path P in G of length at least k?

A few representative problems (II)

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Input: A graph $G = (V, E)$ and a positive integer k.
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DOMINATING SET
Input: A graph $G = (V, E)$ and a positive integers k.
Parameter: k.
Question: Does there exist a subset $D \subseteq V$ of size at most k such that for all $v \in V$, $N[v] \cap D \neq \emptyset$?

Minor-closed parameters

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- We say that a parameter P is *closed under taking of minors/contractions* (or, briefly, *minor/contraction-closed*) if for every graph H, $H \preceq_m G$ / $H \preceq_{cm} G$ implies that $P(H) \leq P(G)$.

Examples of minor/contraction closed parameters

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  VERTEX COVER, FEEDBACK VERTEX SET, LONG PATH, TREEWIDTH, . . . (why?)

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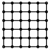
Grid Exclusion Theorem

- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  ℓ

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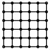
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_{\ell}$ We have $\text{tw}\,(H_{\ell,\ell}) = \ell$.

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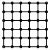
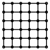
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_{\ell}$ We have $\text{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_{\ell} \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$.

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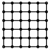
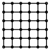
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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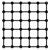
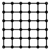
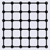
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\mathbf{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\mathbf{tw}(G) \geq \mathbf{tw}(H_{\ell,\ell}) = \ell$. $\boxed{\text{Does the reverse implication hold?}}$

Theorem (Robertson and Seymour. 1986)

For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_\ell$ *as a minor.*

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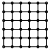
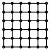
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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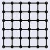
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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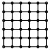
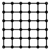
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\text{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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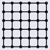
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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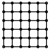
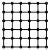
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\text{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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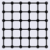
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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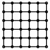
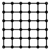
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. $\boxed{\text{Does the reverse implication hold?}}$

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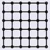
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_\ell$ *as a minor.*

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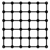
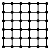
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\text{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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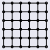
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_\ell$ *as a minor.*

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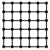
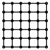
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\text{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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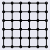
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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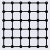
Important message grid-minors are the certificate of large treewidth.

Grid Exclusion Theorems on sparse graphs

Theorem (Robertson, Seymour, Thomas. 1994)

Every planar graph of treewidth $\geq 6 \cdot \ell$ *contains* $_{\ell}$ *as a minor.*

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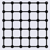
Every planar graph of treewidth $\geq 6 \cdot \ell$ contains $_\ell$ *as a minor.*

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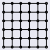
For every fixed g, there is a constant c_g such that every graph of genus g and of treewidth $\geq c_g \cdot \ell$ contains $_\ell$ *as a minor.*

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Every planar graph of treewidth $\geq 6 \cdot \ell$ contains $\boxplus_\ell$ as a minor.

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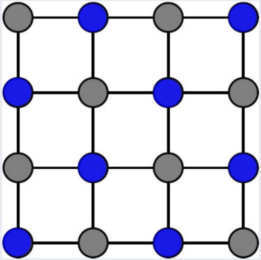
In sparse graphs: linear dependency between treewidth and grid-minors

How to use Grid Theorems algorithmically?

Example: FPT algorithm for Planar Vertex Cover

A vertex cover of a graph G is a set of vertices C such that every edge of G has at least one endpoint in C. Min size: $\mathbf{vc}(G)$.



Example: FPT algorithm for Planar Vertex Cover

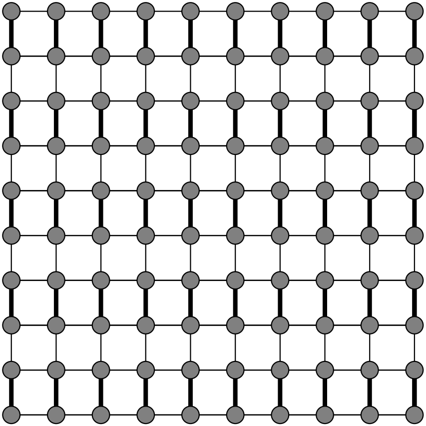
INPUT: Planar graph G on n vertices, and an integer k.

OUTPUT: Either a vertex cover of G of size $\leq k$, or a proof that G has no such a vertex cover.

RUNNING TIME: $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Objective subexponential FPT algorithm for PLANAR VERTEX COVER.

Example: FPT algorithm for Planar Vertex Cover



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- The size of any vertex cover of $H_{\ell,\ell}$ is at least $\ell^2/2$.
- Recall that Vertex Cover is a minor-closed parameter.
- Since $H_{\ell,\ell} \preceq_m G$, it holds that $\mathbf{vc}(G) \geq \mathbf{vc}(H_{\ell,\ell}) \geq \ell^2/2$.

We are already very close to an algorithm...

Recall:

- k is the parameter of the problem.
- We have that $\text{tw}(G) = 6 \cdot \ell$ and ℓ is the size of a grid-minor of G.
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This gives a subexponential FPT algorithm!

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$\boxed{\text{Where did we use planarity?}}$

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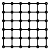
minor-closed $\quad + \quad$ **vc**($_k$) $= \Omega(k^2)$.

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Where did we use planarity?

★ Only the linear Grid Exclusion Theorem!

Arguments go through up to H-minor-free graphs.

Next subsection is...

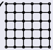
1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

Minor Bidimensionality:

[Demaine, Fomin, Hajiaghayi, Thilikos. 2005]

Definition

A parameter **p** is *minor bidimensional* if

1. **p** is closed under taking of minors (minor-closed), and
2. **p** $\Bigg($  $_{k}\Bigg) = \Omega(k^2)$.

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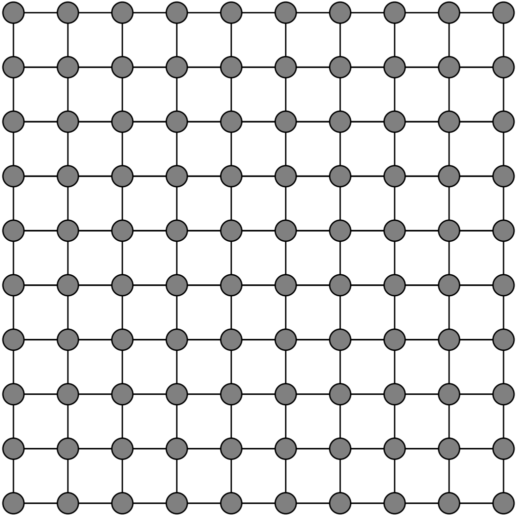

$H_{\ell,\ell}$ for $\ell = 10$

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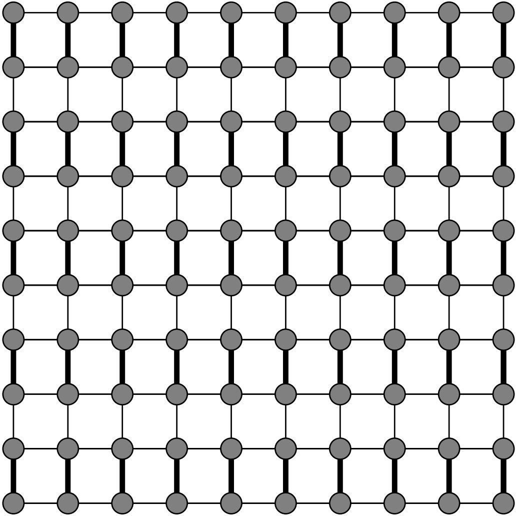

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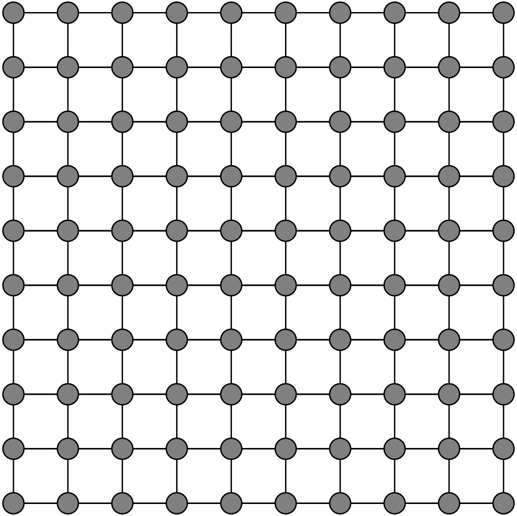
Feedback Vertex Set of a Grid



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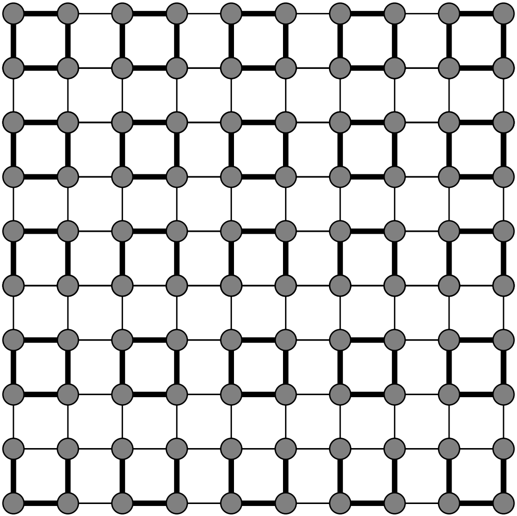

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  - Otherwise, the treewidth is bounded by $c\sqrt{k}$, and hence we can use a dynamic programming (DP) algorithm on graphs of bounded treewidth.
- If we have a DP algorithm for bounded treewidth running in time c^t or t^t, then it implies $2^{O(\sqrt{k})}$ or $2^{O(\sqrt{k}\log k)}$ algorithm.

Piecing everything together

Theorem

Let G be an H-minor-free graph, and let $\mathbf{p}$ be a minor bidimensional graph parameter computable in time $2^{O(\mathbf{tw}(G))} \cdot n^{O(1)}$.
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1. Compute (or approximate) $\mathbf{tw}(G)$.
2. If $\mathbf{tw}(G) = \Omega(\sqrt{k})$, then answer NO.
3. Otherwise $\mathbf{tw}(G) = O(\sqrt{k})$, and we solve the problem by DP.

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$\boxed{1}$ Compute (or approximate) $\mathbf{tw}(G)$.
We can use a fast FPT algorithm or a constant-factor approx.

$\boxed{2}$ If $\mathbf{tw}(G) = \Omega(\sqrt{k})$, then answer NO.

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Piecing everything together

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Minor Bidimensionality provides a meta-algorithm

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skip

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Definition

A parameter **p** is *contraction bidimensional* if

1. **p** is closed under taking of contractions (contraction-closed), and
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What is a $(k \times k)$-grid-like graph...?

Contraction bidimensionality: old setting

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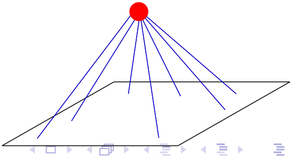
★ For apex-minor-free graphs, this is a $(k \times k)$-augmented grid, i.e., partially triangulated grid augmented with additional edges such that each vertex is incident to $O(1)$ edges to non-boundary vertices of the grid.

[Demaine, Fomin, Hajiaghayi, Thilikos. 2005]

H is an *apex graph* if $\exists v \in V(H)$: $H - v$ is planar



Contraction bidimensionality: new definition

Finally, the right "($k \times k$)-grid-like graph" was found:

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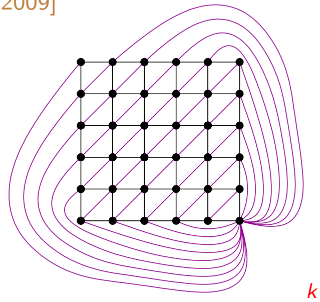

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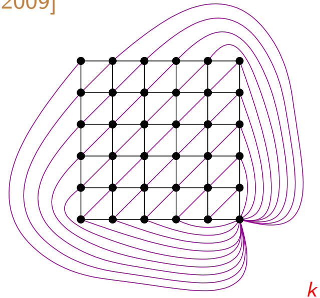

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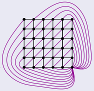
2. **p**( $_k$) $= \Omega(k^2)$.

Meta-algorithms for contraction bidimensional parameters

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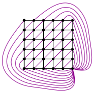
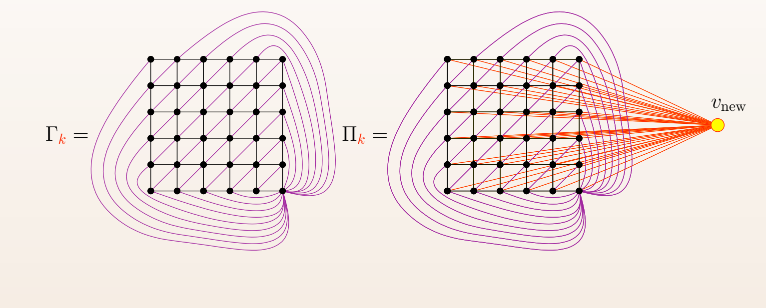
As for minor bidimensionality, we need to prove that

- If $\mathbf{tw}(G) = \Omega(k)$ then G contains  $_k$ as a contraction.

Two important grid-like graphs

Two pattern graphs Γ_k and Π_k:



$\Pi_k = \Gamma_k +$ a new universal vertex v_{new}.

The "contraction-certificates" for large treewidth

Theorem (Fomin, Golovach, Thilikos. 2009)

For any integer $\ell > 0$, there is c_ℓ such that every connected graph of treewidth at least c_ℓ contains K_ℓ, Γ_ℓ, or Π_ℓ as a contraction.

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Further applications of Bidimensionality

1. Bidimensionality + DP $\Rightarrow$ $\boxed{\text{Subexponential FPT algorithms}}$

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4. Bidimensionality + new Grid Theorems $\Rightarrow$ $\boxed{\text{Geometric graphs}}$

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   [Grigoriev, Koutsonas, Thilikos. 2013]

Next section is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

Basic principle of the irrelevant vertex technique

This technique was invented in [Robertson and Seymour. 1995]

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Input: a graph G and k pairs of vertices $T = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$.

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2. Otherwise, if $\mathrm{tw}(G) \leq f(k)$, solve the problem using dynamic programming (by Courcelle).

How to find an irrelevant vertex when the treewidth is large?

How to find an irrelevant vertex when the treewidth is large?

By using the Grid Exclusion Theorem!

How to find an irrelevant vertex when the treewidth is large?

By using the $\boxed{\text{Wall}}$ Exclusion Theorem!

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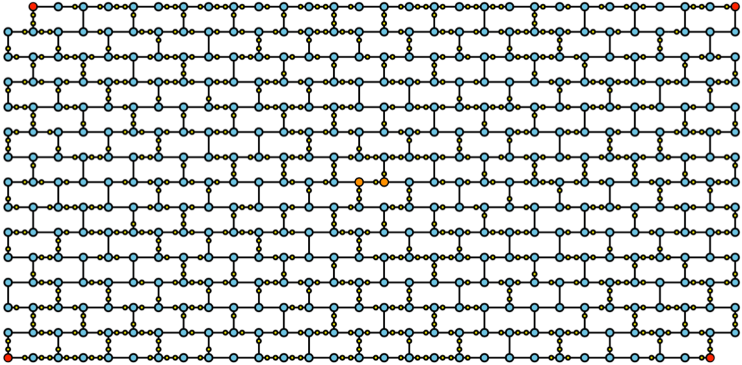
Theorem (Robertson and Seymour. 1986)

For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains an ℓ-wall as a minor.



[Figure by Dimitrios M. Thilikos]

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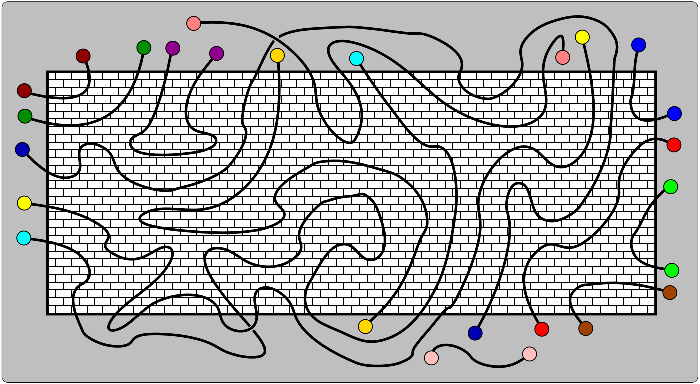

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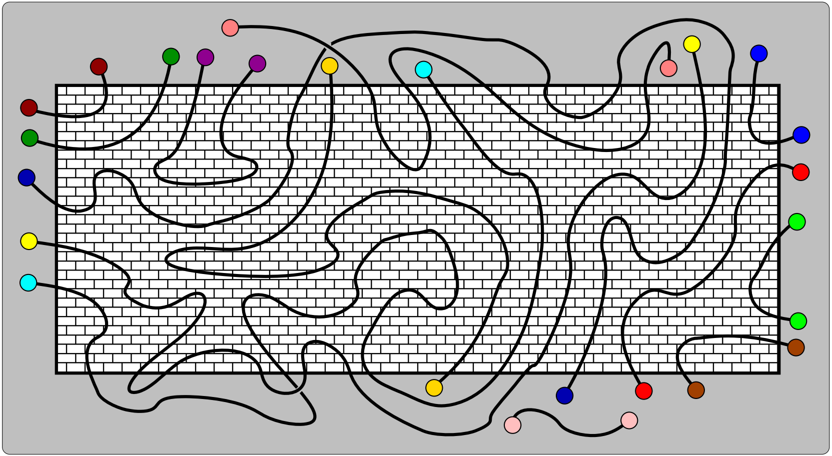
Goal: declare one of the central vertices of the wall irrelevant.



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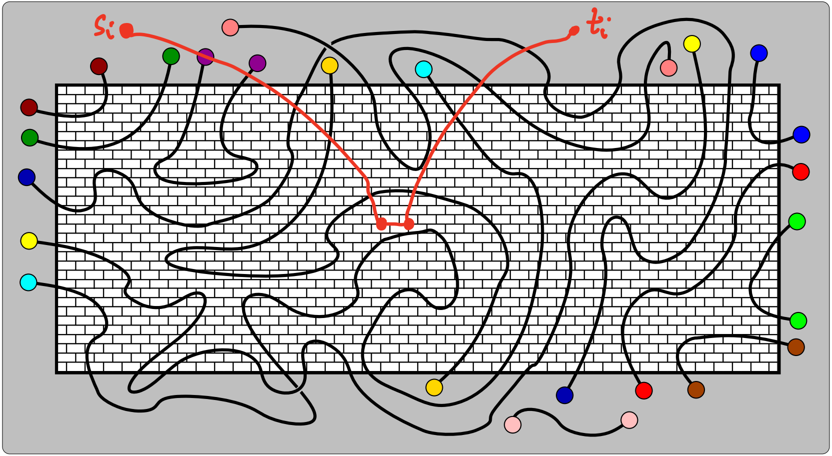
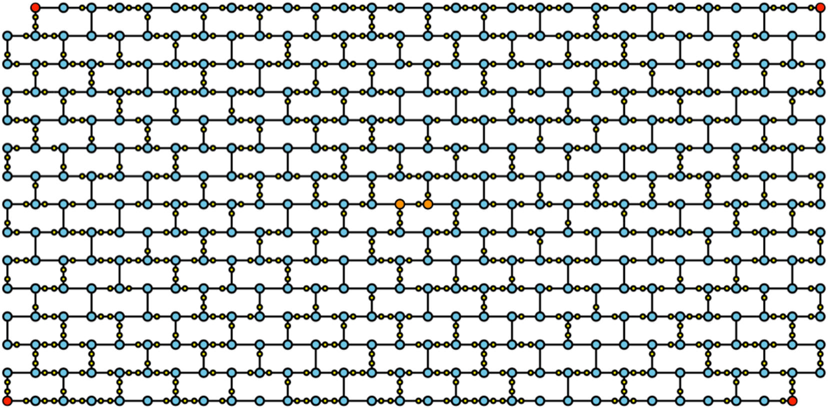


This is only possible if the wall is insulated from the exterior!

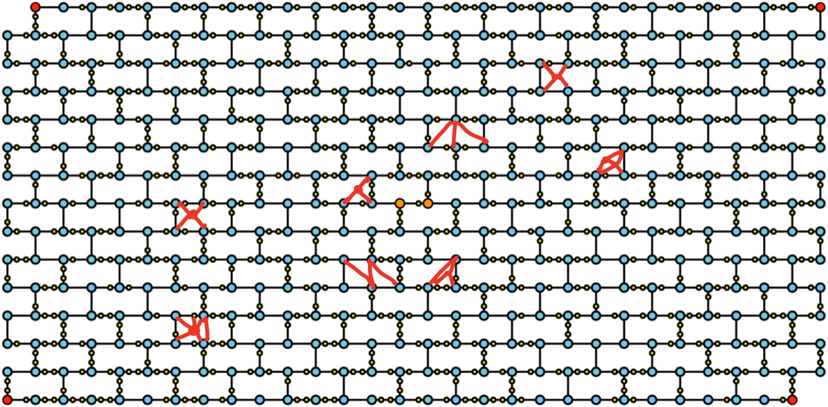
Flat walls

Goal: enrich the notion of wall so that we can insulate it from the exterior.



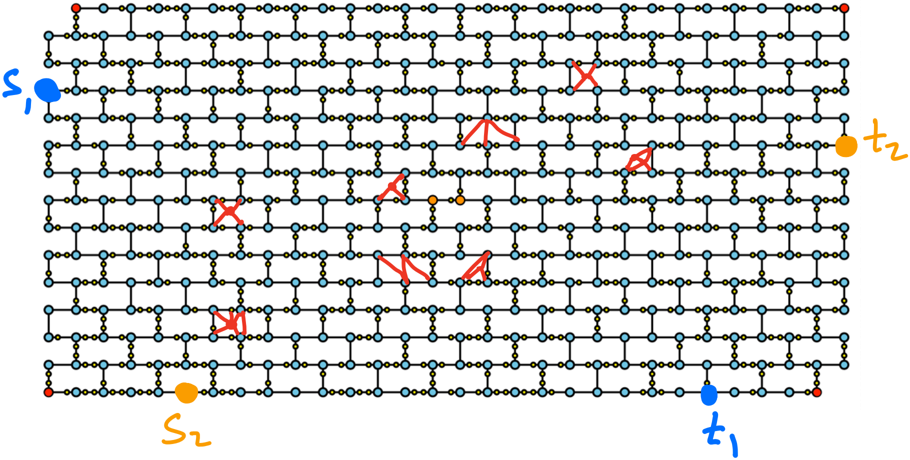
Flat walls

We need to allow some extra edges in the interior of the wall.



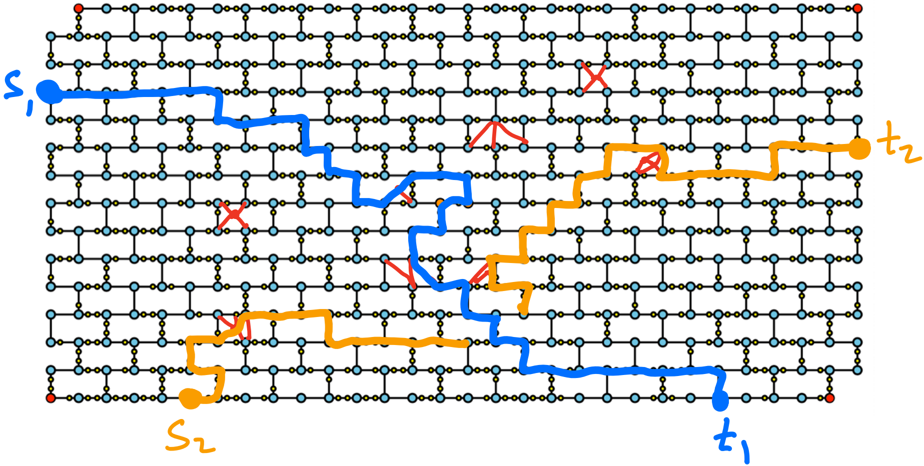
Flat walls

We impose a topological property that defines the "flatness" of the wall.

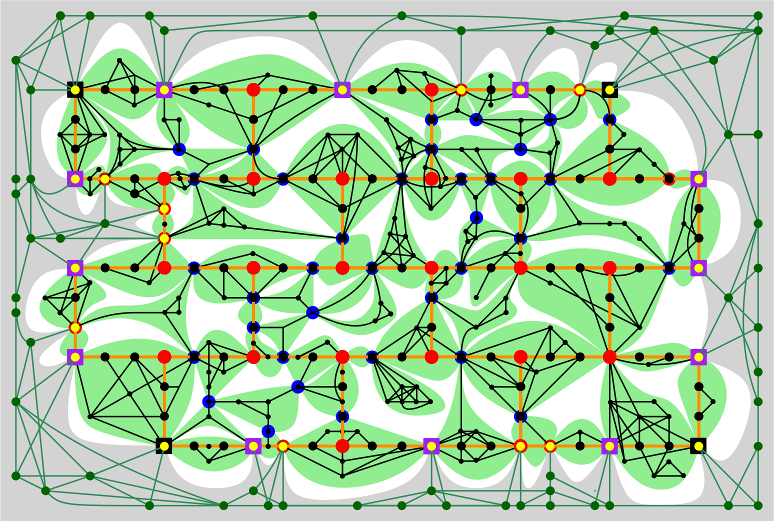


Flat walls

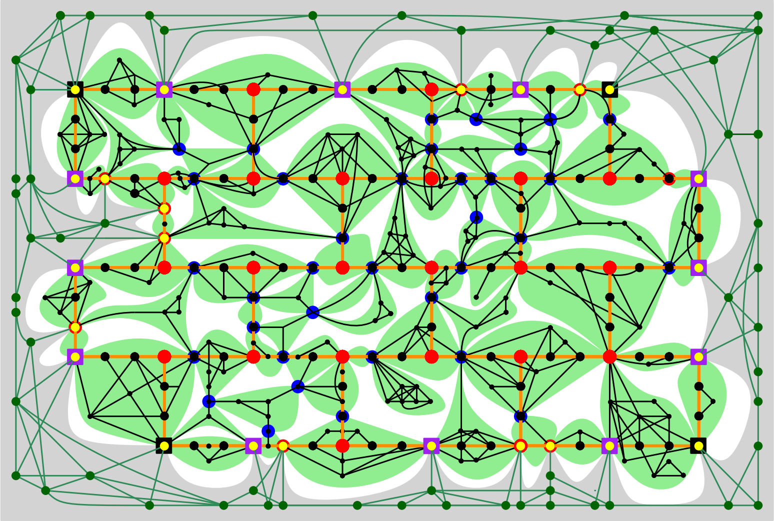
There are no crossing paths $s_1 - t_1$ and $s_2 - t_2$ from/to the perimeter.



A real flat wall can be quite wild... [Figure by Dimitrios M. Thilikos]



Flat walls: a bit more formal



[Figures by Dimitrios M. Thilikos]

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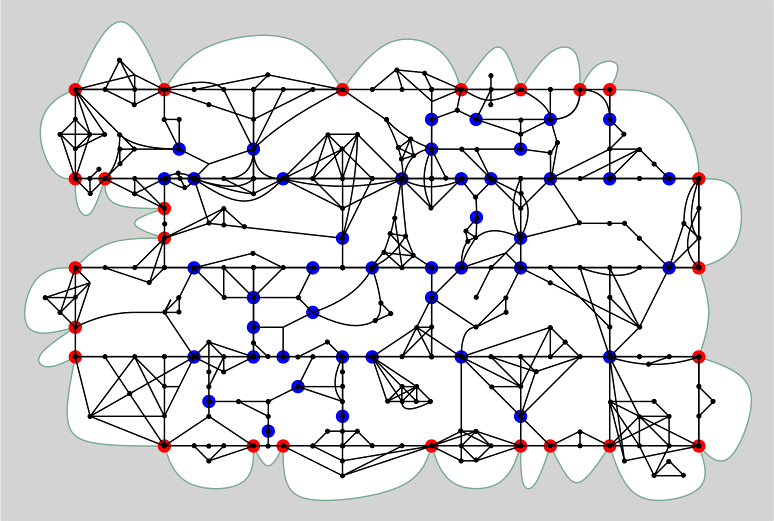

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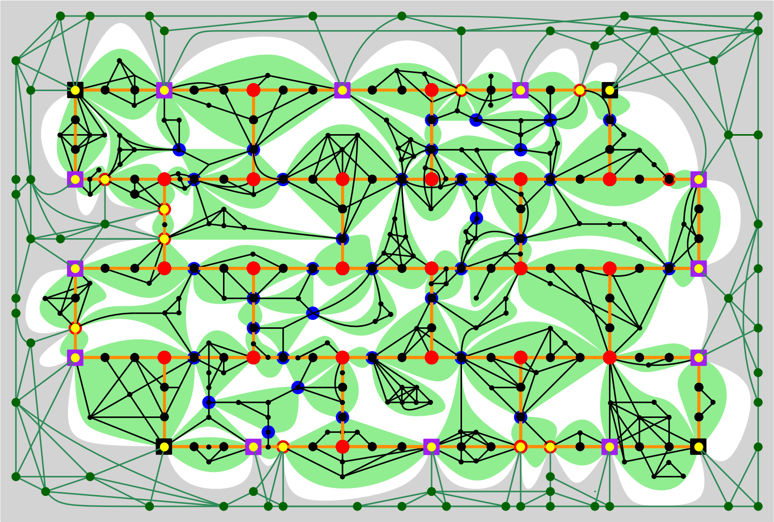

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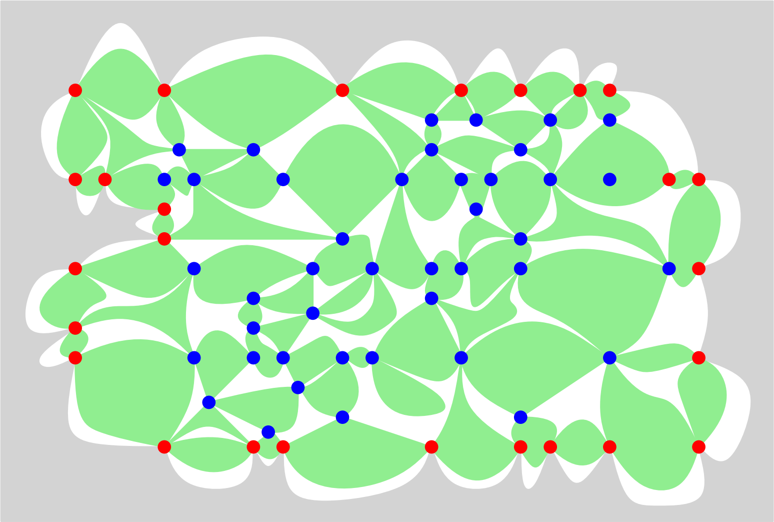

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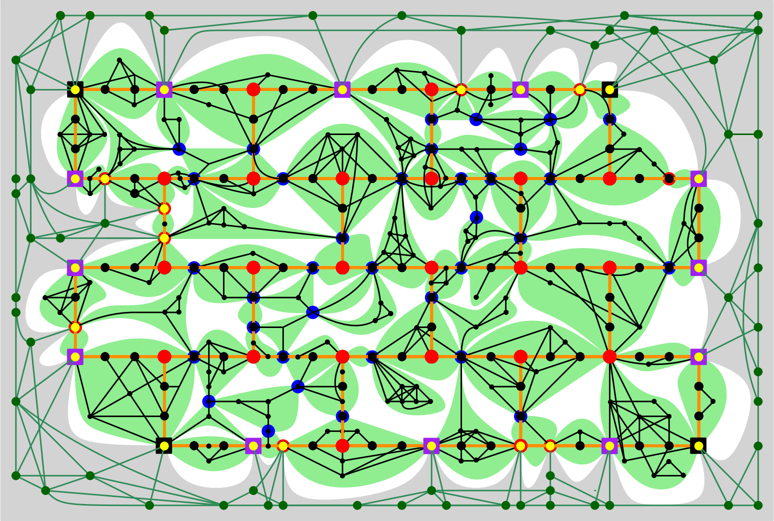

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The Weak Structure Graph Minors Theorem

Theorem (Robertson and Seymour. 1995)

There exist recursive functions $f_1 : \mathbb{N}^2 \to \mathbb{N}$ *and* $f_2 : \mathbb{N} \to \mathbb{N}$, *such that for every graph* G *and every* $q, r \in \mathbb{N}$, *one of the following holds:*

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[Chuzhoy. 2015]

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Important: possible to find one of the outputs in time $f(q, r) \cdot |V(G)|$.

Back to the DISJOINT PATHS problem

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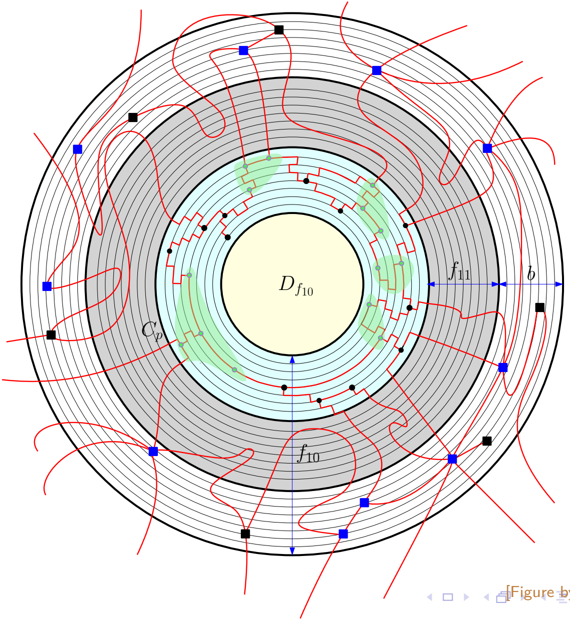
The irrelevant vertex technique has been applied to many problems... usually with a lot of technical pain.

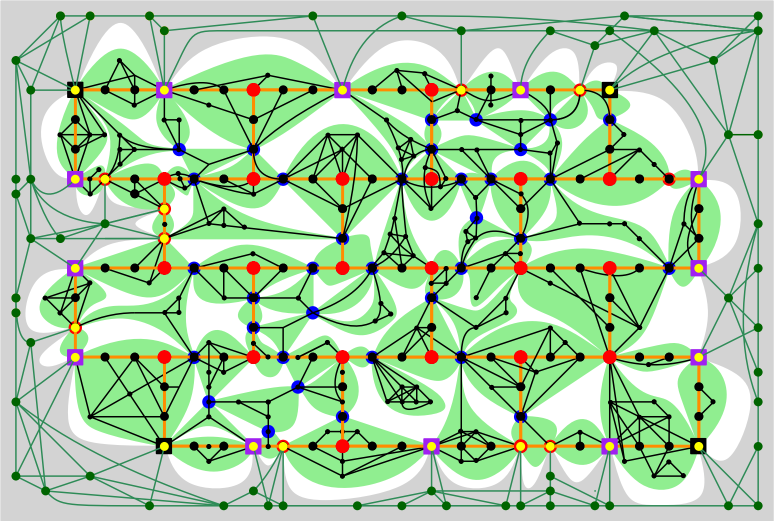
Rerouting inside a big flat wall...



[Figure by Dimitrios M. Thilikos]

Crucial notion: homogeneity

In order to declare a vertex irrelevant for some problem, usually we need to consider a homogenous flat wall, which we proceed to define.



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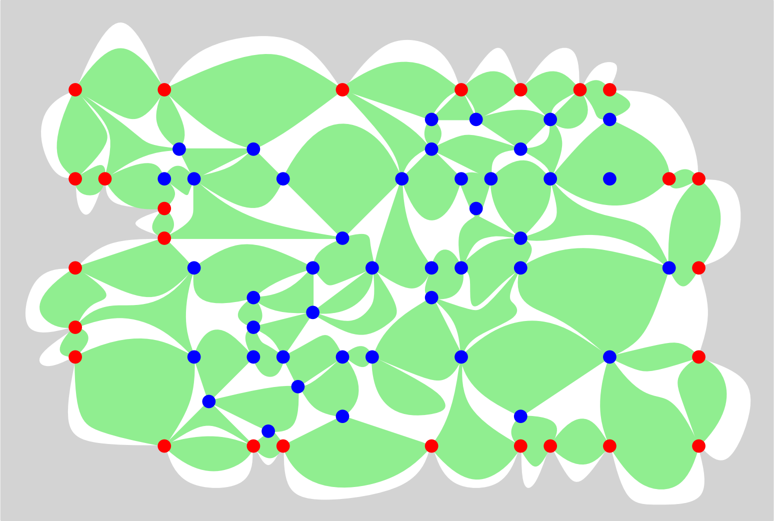
We consider a flap-coloring encoding the relevant information of our favorite problem inside each flap (similar to tables of DP).



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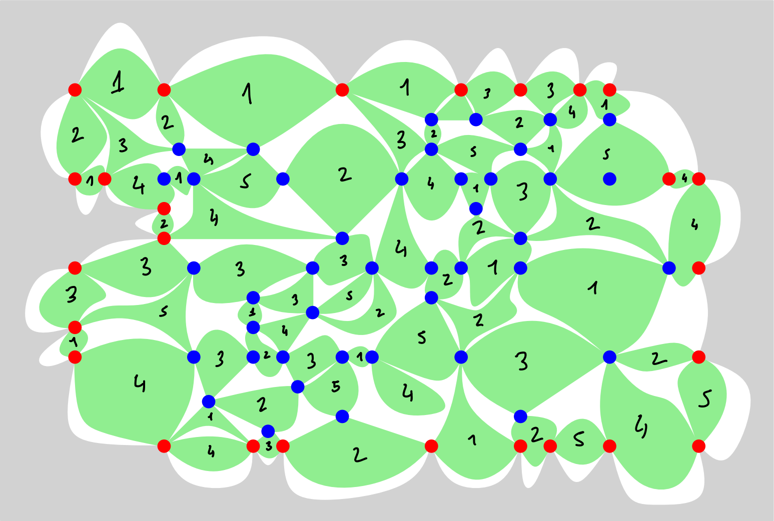

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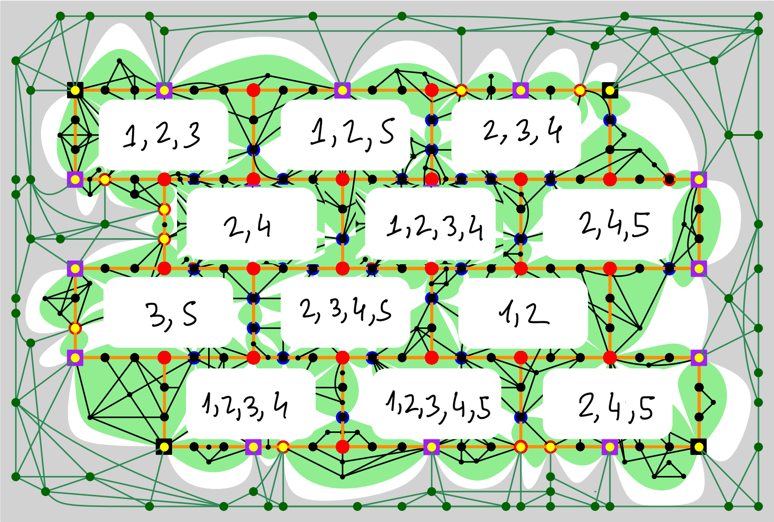
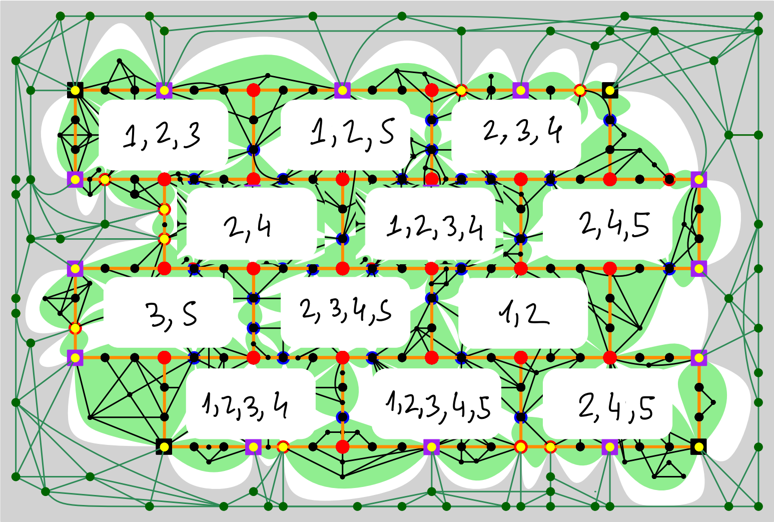
For every brick of the wall, we define its palette as the colors appearing in the flaps it contains.



Crucial notion: homogeneity

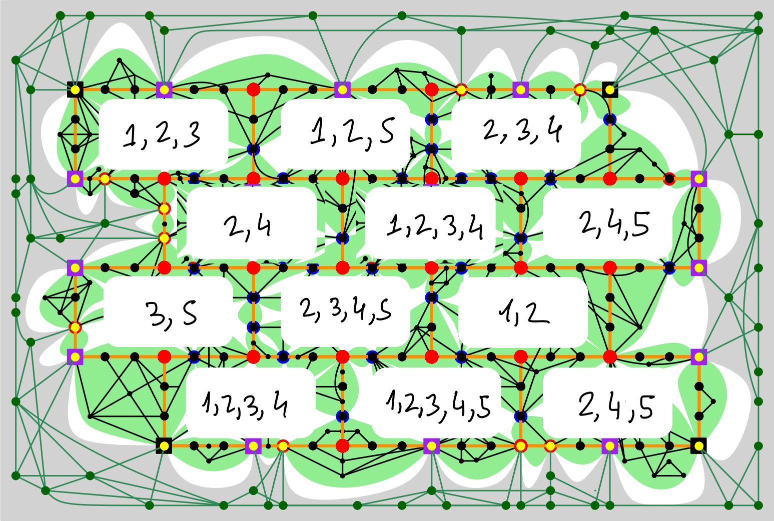
A flat wall is homogenous if every (internal) brick has the same palette.
Fact: every brick of a homogenous flat wall has the same "behavior".



Crucial notion: homogeneity

Price of homogeneity to obtain a homogenous flat r-wall (zooming):

If we have c colors, we need to start with a flat r^c-wall. (why?)



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2. Introduction to parameterized complexity
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Hitting forbidden minors

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We consider the following two parameterizations of $\mathcal{F}$-M-Deletion:

1. Structural parameter: $\mathrm{tw}(G)$.
2. Solution size: k.

Joint work with Dimitrios M. Thilikos, Julien Baste, Giannos Stamoulis, and Laure Morelle.

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ETH: The 3-SAT problem on n variables cannot be solved in time $2^{o(n)}$.

[Impagliazzo, Paturi. 1999]

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  [Jansen, Lokshtanov, Saurabh. 2014 + Pilipczuk. 2015]

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

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[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. I. General upper bounds**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. II. Single-exponential algorithms**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. III. Lower bounds**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. IV. An optimal algorithm**. 2021]

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A dichotomy for hitting a connected minor

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Let H be a connected graph.

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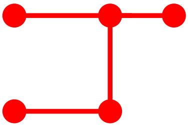

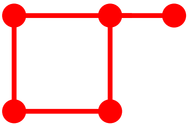

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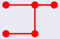
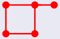
- $2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_{\mathsf{c}}$  *or* $H \leqslant_{\mathsf{c}}$  .

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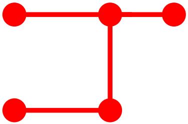

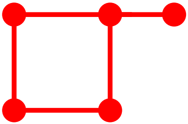

Theorem (Baste, S., Thilikos. 2016-2020)

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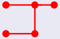
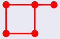
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_{\mathsf{c}}$  *or* $H \leqslant_{\mathsf{c}}$  .
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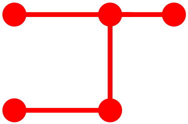

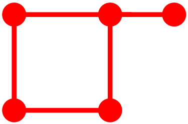

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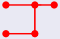
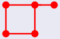
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_{\mathrm{c}}$  *or* $H \leqslant_{\mathrm{c}}$ .
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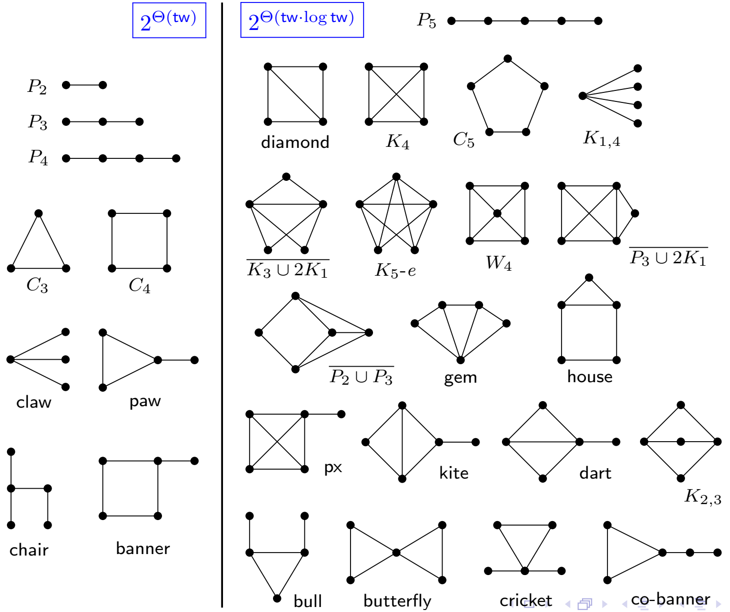
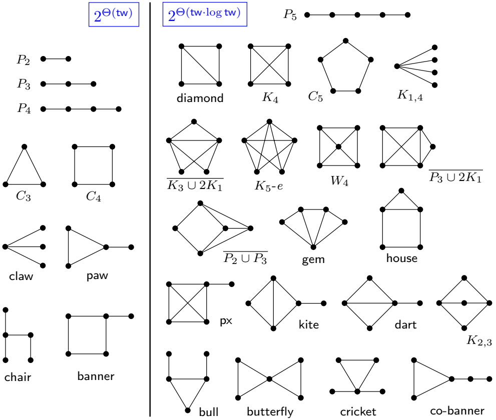
In both cases, the running time is asymptotically optimal under the ETH.

skip

Complexity of hitting a single connected minor H



A compact statement for a single connected graph



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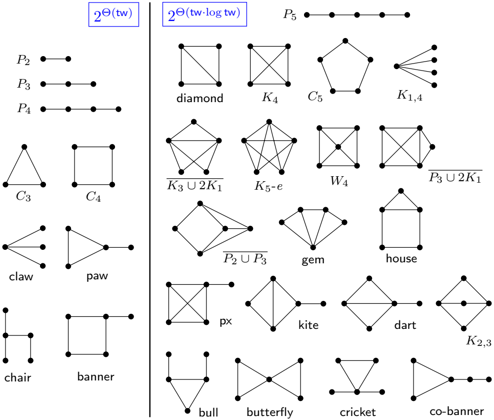

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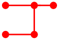
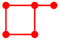
- All graphs on the left are contractions of  or 

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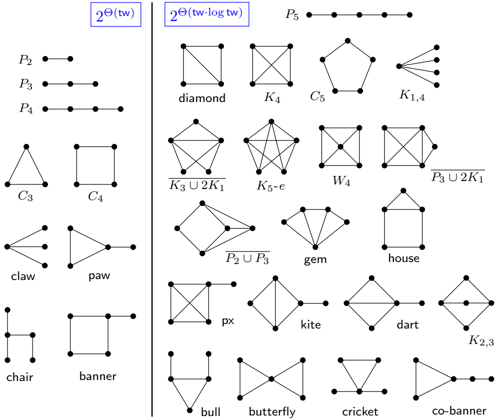

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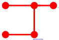
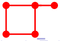
- All graphs on the right are not contractions of  or 

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   - For every $\mathcal{F}$: time $2^{2^{\mathcal{O}(\mathsf{tw}\cdot\log\mathsf{tw})}} \cdot n^{\mathcal{O}(1)}$.
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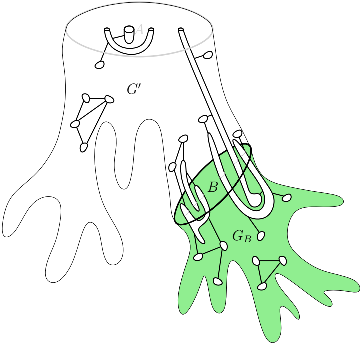
[Fig. by Valentin Garnero]



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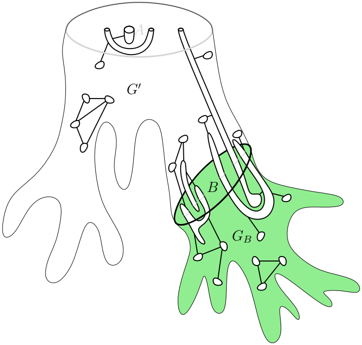

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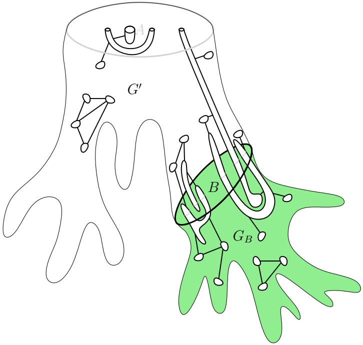

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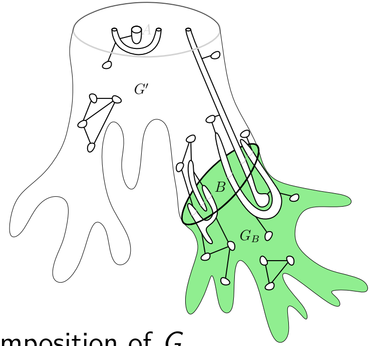

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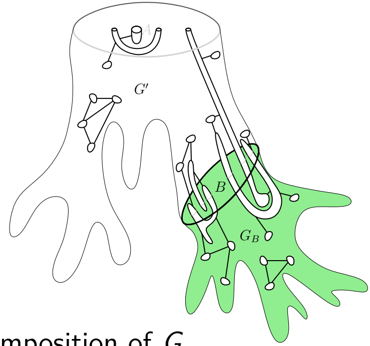

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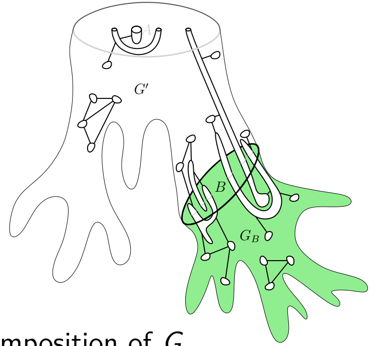

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- $\boxed{\text{Goal}}$ Bound the number of representatives: $|\mathcal{R}^{(\mathcal{F},t)}| = 2^{\mathcal{O}_{\mathcal{F}}(\text{tw}\cdot\log \text{tw})}$.

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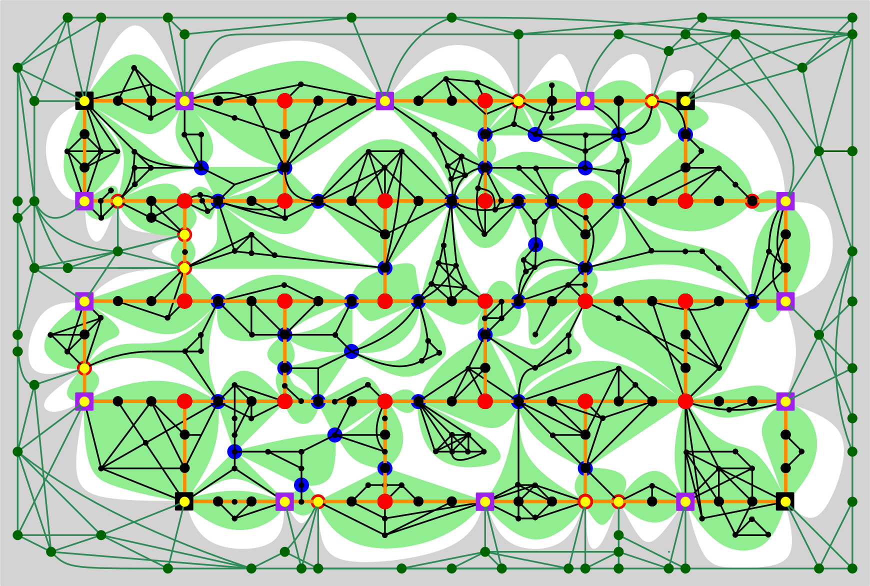
As we know, a flat wall can be quite wild...



[Figure by Dimitrios M. Thilikos]

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Hard part: finding an irrelevant vertex inside a flat wall

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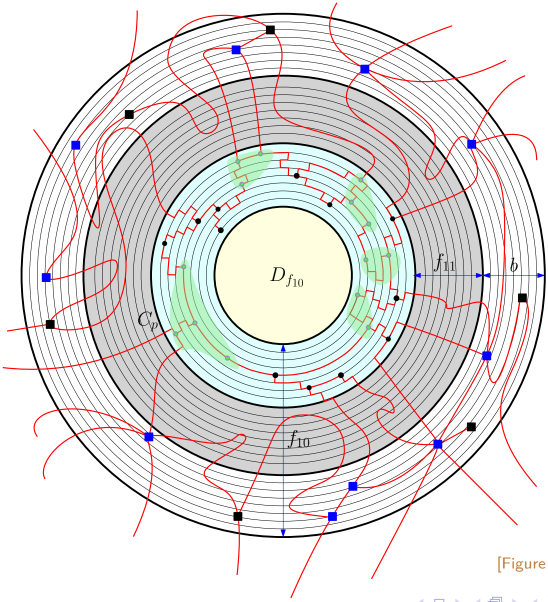

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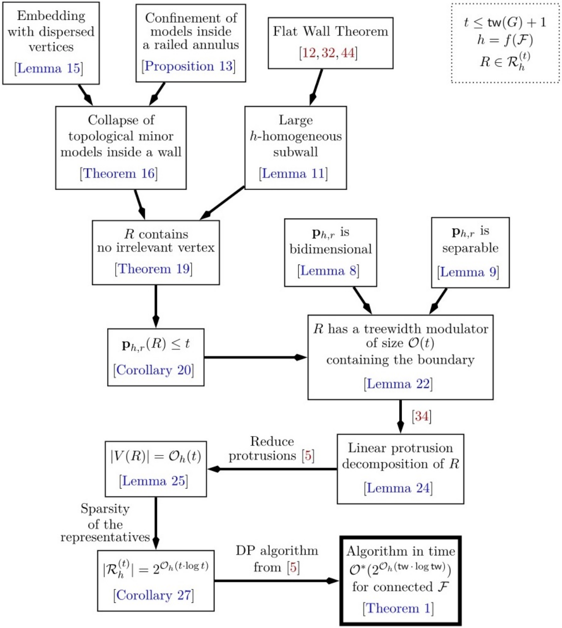
Diagram of the algorithm for a general collection $\mathcal{F}$



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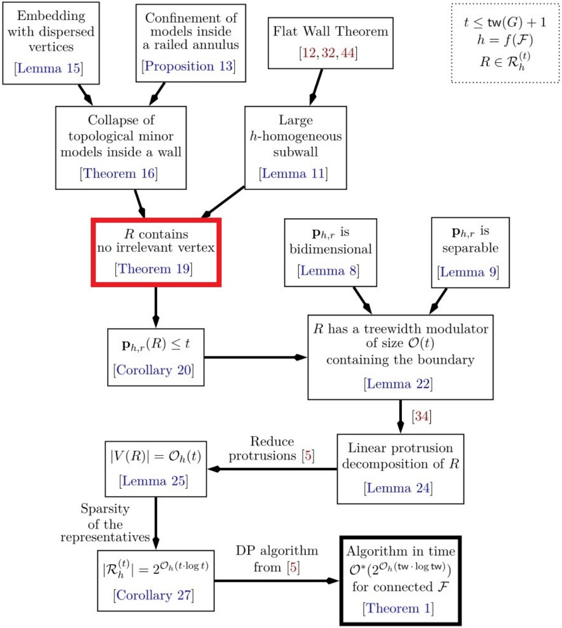

Next subsection is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

We parameterize by the size of the desired solution

$\mathcal{F}$-M-DELETION
Input: A graph G and an integer k.
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But... only existential, non-uniform, $f(\mathcal{C}_k)$ astronomical.

Can we do better?

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- For every $\mathcal{F}$, some enormous explicit function $f_{\mathcal{F}}(k)$ can be derived from an FPT algorithm for hitting topological minors:

$$f_{\mathcal{F}}(k) \cdot n^{\mathcal{O}(1)}.$$

[Fomin, Lokshtanov, Panolan, Saurabh, Zehavi. 2020]

Our results

Theorem (S., Stamoulis, Thilikos. 2020)

For all $\mathcal{F}$, the $\mathcal{F}$-M-Deletion problem can be solved in time $2^{\mathrm{poly}(k)} \cdot n^3$.

Here, $\mathrm{poly}(k)$ is a polynomial whose degree depends on $\mathcal{F}$.

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If $\mathcal{F}$ contains an apex graph, the $\mathcal{F}$-M-DELETION problem can be solved in time $2^{\mathsf{poly}(k)} \cdot n^2$.

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Theorem (Morelle, S., Stamoulis, Thilikos. 2022)

For all $\mathcal{F}$, the $\mathcal{F}$-M-DELETION problem can be solved in time $2^{\mathsf{poly}(k)} \cdot n^2$.

Sketch of the proofs

Let $\mathcal{F}$ be a fixed finite collection of graphs.

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Input: A graph G and an integer k.
Parameter: k.
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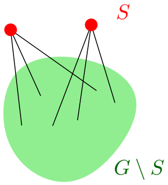

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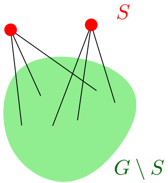

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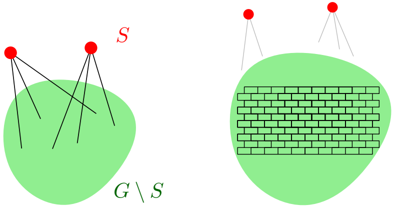

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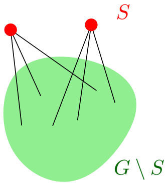

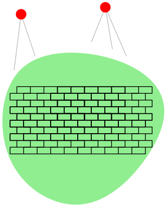

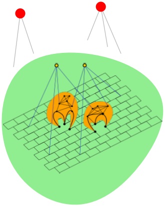

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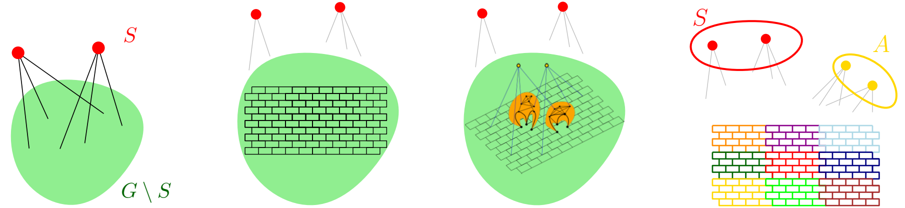

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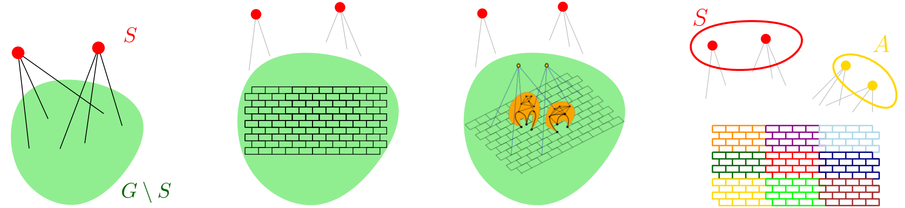

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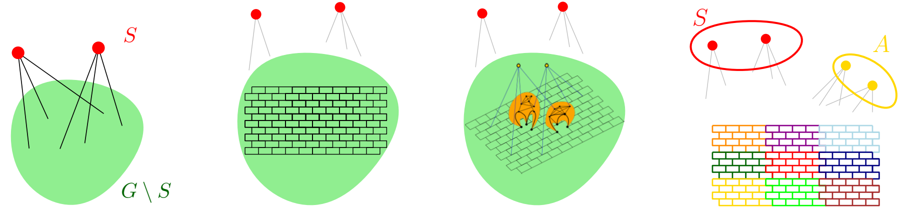

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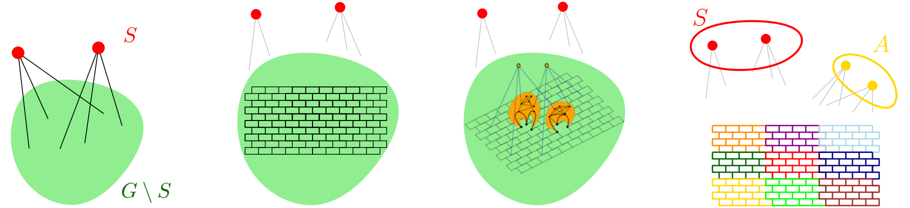

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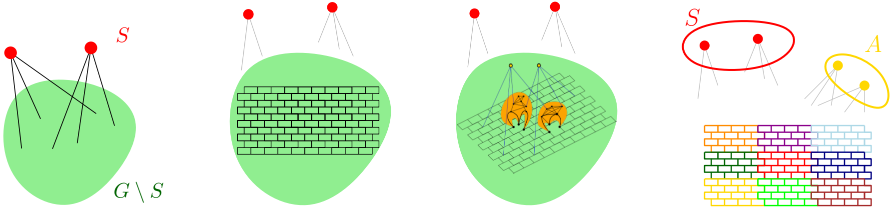

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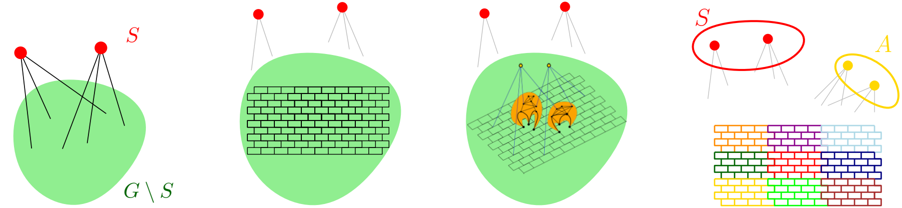

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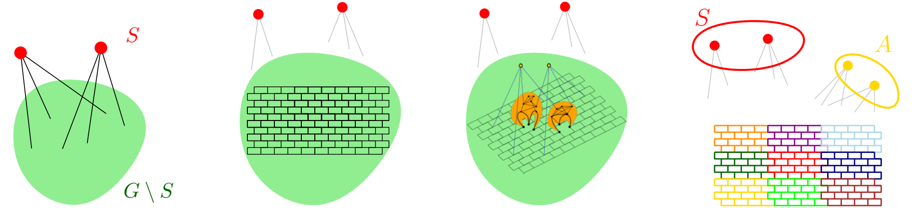

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Thus, $\mathrm{tw}(G \setminus S) = k^{\mathcal{O}_{\mathcal{F}}(1)}$: our previous FPT algo gives $2^{k^{\mathcal{O}_{\mathcal{F}}(1)}} \cdot n^2$.

Main idea of our improved algorithm

Theorem (Morelle, S., Stamoulis, Thilikos. 2022)

For all $\mathcal{F}$, the $\mathcal{F}$-M-Deletion problem can be solved in time $2^{\text{poly}(k)} \cdot n^2$.

Improvement from n^3 to n^2: avoiding iterative compression.

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How to achieve it?

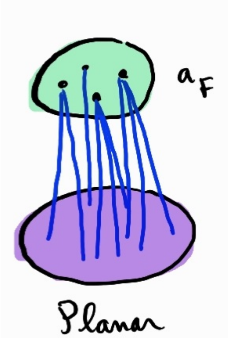
We are able to detect a vertex that must belong to every solution.

Approach inspired by [Marx, Schlotter. 2012] [S., Stamoulis, Thilikos. 2020]

skip

Let $\mathcal{F}$ be a finite collection of graphs.

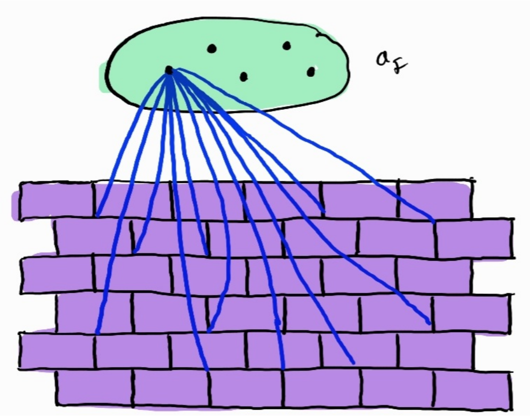
The **apex number** $a_{\mathcal{F}}$ is the smallest number of vertices that can be removed from a graph of $\mathcal{F}$ such that the remaining graph is planar.



[Figure by Laure Morelle]

$a_{\mathcal{F}} = 1 \rightarrow$ apex graph

Finding a vertex belonging to every solution of size k



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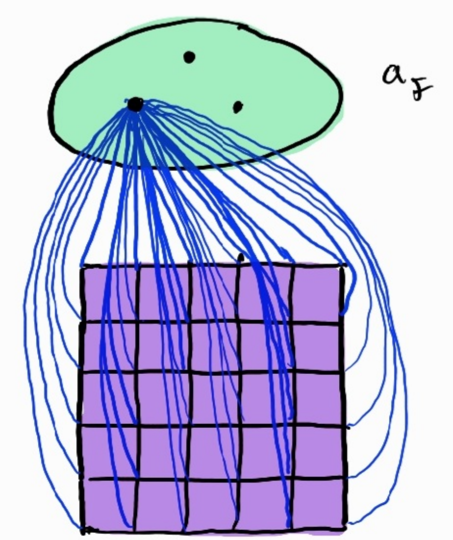

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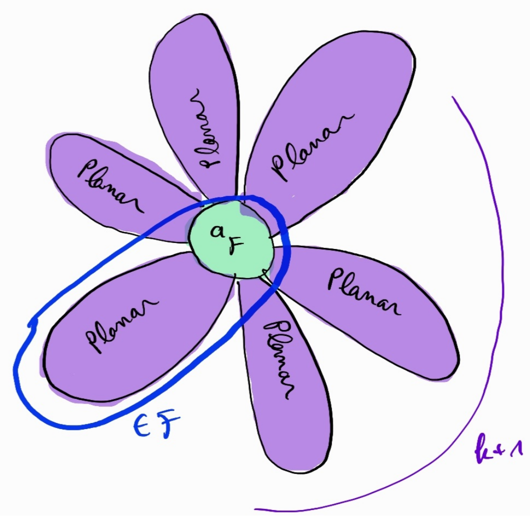

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    (Branching tree is of size $a_{\mathcal{F}}^k$, so we do *not* get an extra factor n).

skip

Next subsection is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

Motivation: distance from triviality

Distance from triviality: [Guo, Hüffner, Niedermeier. 2004]

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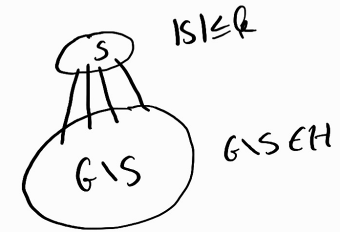

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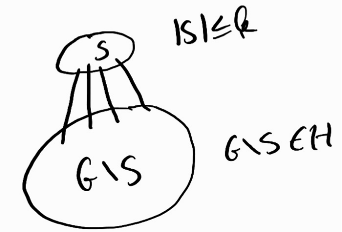

[Figure by Laure Morelle]

$\rightarrow$ ELIMINATION DISTANCE TO $\mathcal{H}$

Notion recently introduced by [Bulian, Dawar. 2016]

The **elimination distance** of a graph G to a graph class $\mathcal{H}$ is:

$$\mathrm{ed}_{\mathcal{H}}(G) = \begin{cases} 0 & \text{if } G \in \mathcal{H}, \\ 1 + \min\{\mathrm{ed}_{\mathcal{H}}(G \setminus \{v\}) \mid v \in V(G)\} & \text{if } G \text{ is connected}, \\ \max\{\mathrm{ed}_{\mathcal{H}}(H) \mid H \text{ is a connected component of } G\} & \text{otherwise.} \end{cases}$$

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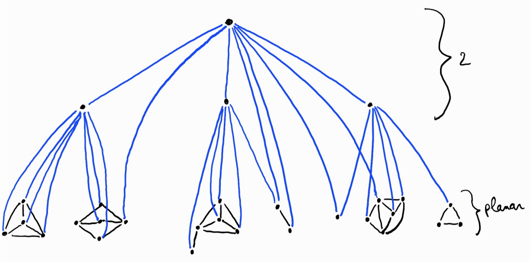

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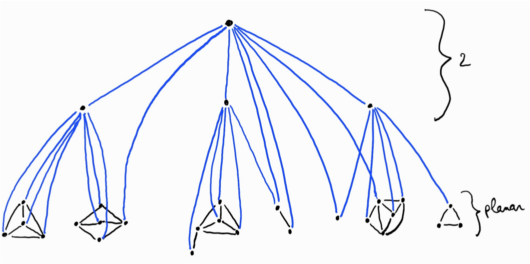

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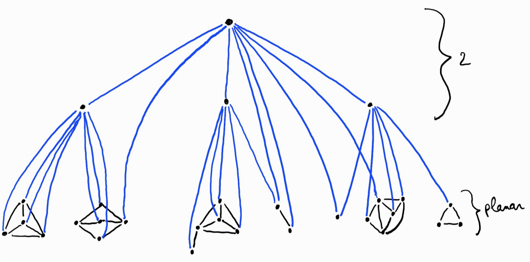

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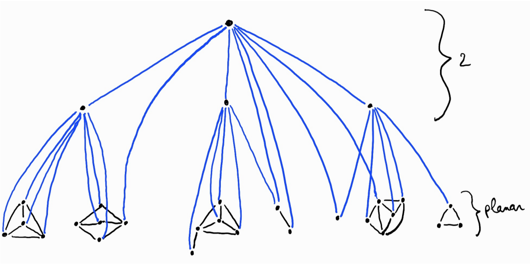

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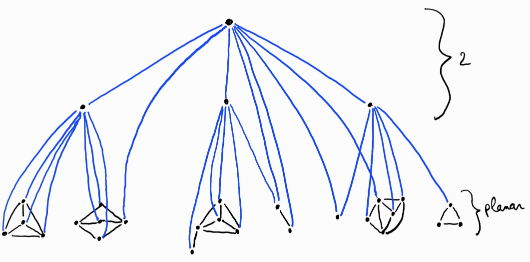

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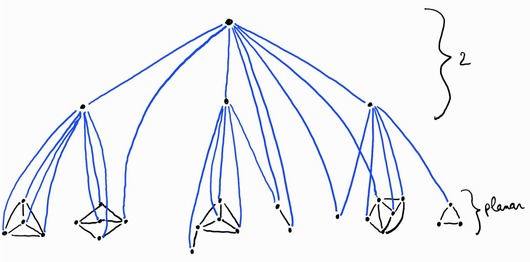

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ELIMINATION DISTANCE TO $\mathcal{H}$

Input: A graph G and a $k \in \mathbb{N}$.
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$\boxed{\text{Can we provide an explicit function } f(k)?}$

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Given a graph G on n vertices and with treewidth at most tw, and $k \in \mathbb{N}$, there is an algorithm that solves Elimination Distance to $\mathcal{H}$ *for the instance (G, k) in time $2^{\mathcal{O}_{\mathcal{H}}(k \cdot \mathrm{tw} + \mathrm{tw} \log \mathrm{tw})} \cdot n$.*

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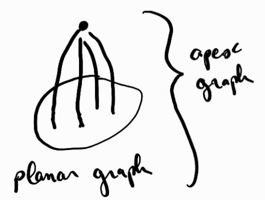

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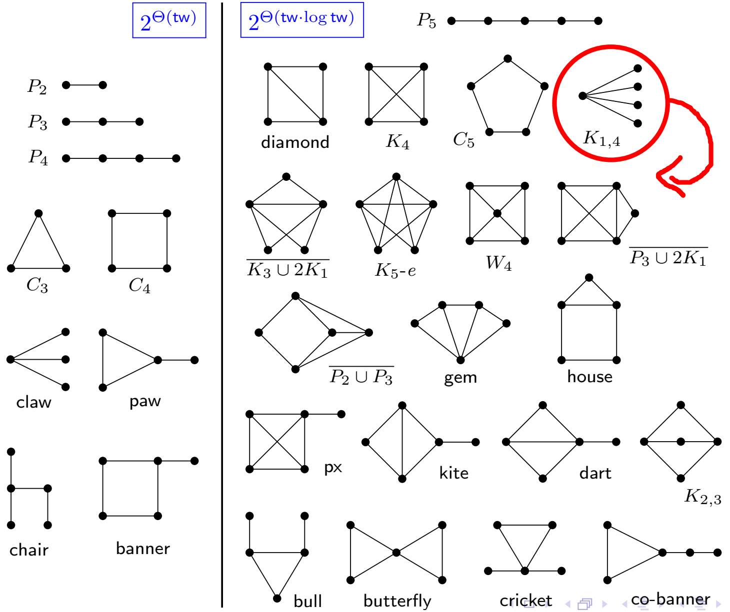
Is the price of homogeneity unavoidable?

skip

For topological minors, there is (at least) one change



Next section is...

1. Introduction to graph minors
2. Introduction to parameterized complexity
3. Treewidth
   - Definition and simple properties
   - Brambles and duality
   - Computing treewidth
   - Dynamic programming on tree decompositions
   - Exploiting topology in dynamic programming
4. Bidimensionality
   - Some ingredients and an illustrative example
   - Meta-algorithms
5. Irrelevant vertex technique
6. Application to hitting minors
   - Parameterized by treewidth
   - Parameterized by solution size
   - More general modification operations
7. Kernelization (?)

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Do all FPT problems admit polynomial kernels? $\boxed{\text{NO!}}$

Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)

Deciding whether a graph has a PATH *with* $\geq k$ *vertices is* FPT *but does not admit a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

Now, on the board!

- Definitions.
- Some simple kernels.
- Crown decompositions.
- Kernels based on linear programming.
- Sunflower lemma.

References

1. Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer 2015. DOI: 10.1007/978-3-319-21275-3.
2. Meirav Zehavi, Saket Saurabh, Daniel Lokshtanov, and Fedor V. Fomin. Kernelization: Theory of Parameterized Preprocessing. Cambridge University Press 2019. DOI: 10.1017/9781107415157.

Gràcies!