Algorithmic aspects of the theory of Graph Minors

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ForWorC

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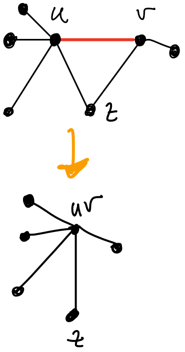
Outline of this mini-course

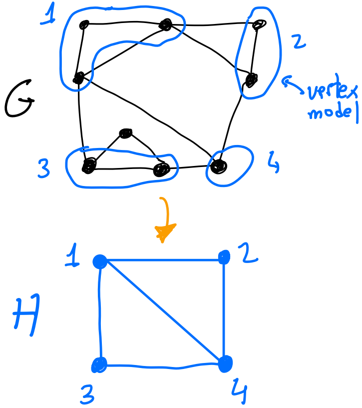
1. Introduction to graph minors
2. Bidimensionality
3. Irrelevant vertex technique

Next section is...

Graph minors

A graph H is a minor of a graph G, denoted by $H \leqslant_m G$, if H can be obtained by a subgraph of G by contracting edges.





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Note that, in general, this list $\mathcal{F}_{\mathcal{C}} = \{G_1, G_2, \ldots\}$ may be infinite.

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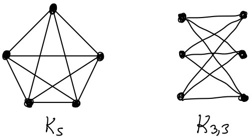

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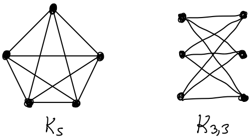

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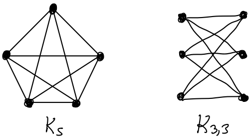

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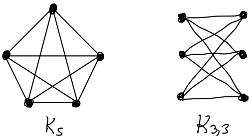

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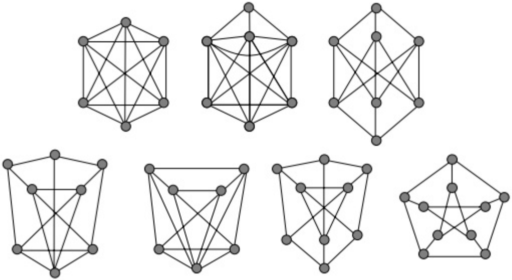

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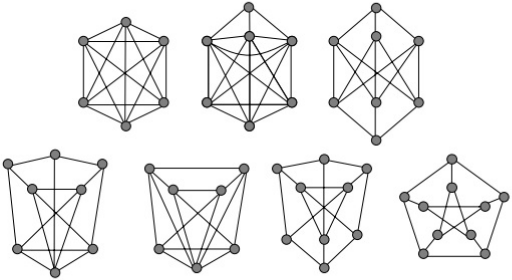

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$\mathcal{F}_{\mathcal{C}}$ seems to get complicated... but always finite!

Wagner's conjecture

Conjecture (Wagner. 1970)

For every minor-closed graph class $\mathcal{C}$, there exists a finite set of graphs $\mathcal{F}_{\mathcal{C}}$ such that $\mathcal{C} = \mathrm{exc}(\mathcal{F}_{\mathcal{C}})$.

Wagner's conjecture... now Robertson-Seymour's theorem

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Yet equivalent: Every infinite set $\{G_1, G_2, \ldots\}$ of finite graphs contains two graphs such that one is a minor of the other (there is no infinite antichain).

Well-quasi orders

A partially ordered set (poset) is a set P with a partial binary relation $\leq$:

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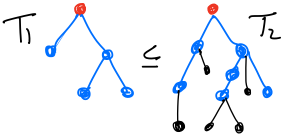
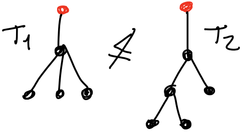
R&S theorem: Finite graphs are wqo with respect to the minor relation.

Illustrative example: rooted trees

Let T_1 and T_2 be two finite rooted trees.
Def: $T_1 \leq T_2$ if there is a subdivision of T_1 that occurs as a rooted subgraph of T_2 (the root of T_1 is not necessarily mapped to the root of T_2).





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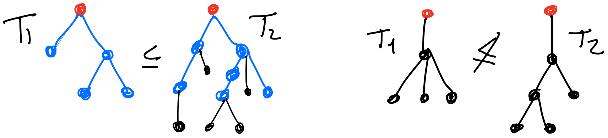

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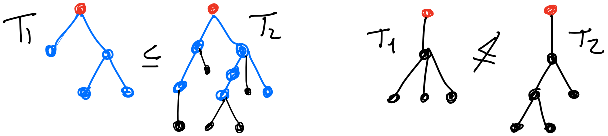

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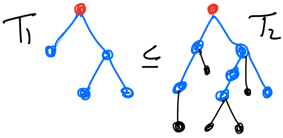

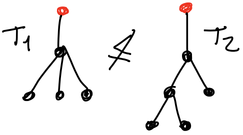

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We will now see a simple proof by [Nash-Williams. 1963]

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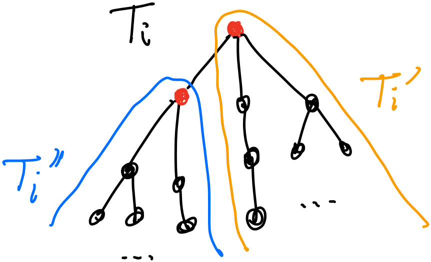
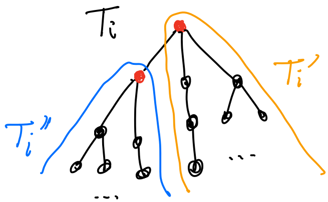
For $k \geq 1$:
Let T_i' be the tree obtained from T_i by deleting any branch from the root.
Let T_i'' be the deleted branch (rooted at a child of the root of T_i).





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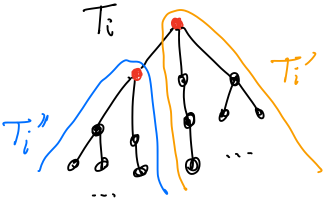

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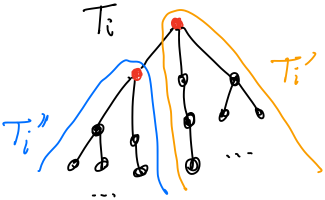

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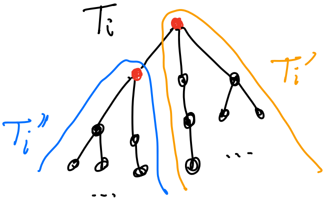

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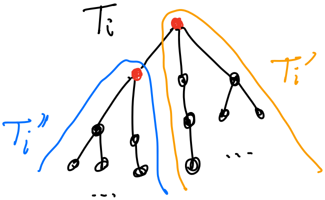

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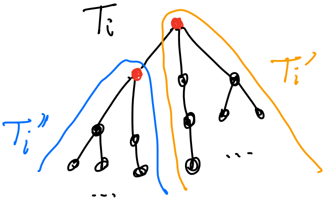

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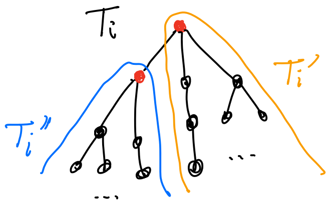

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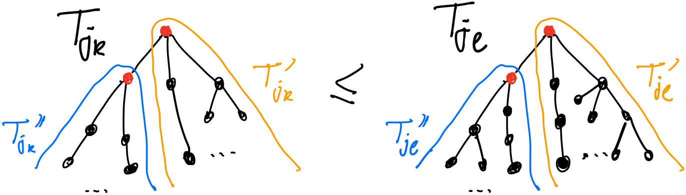

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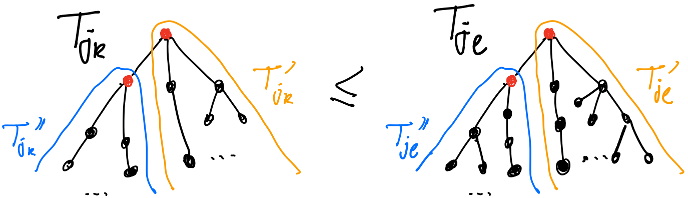

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A notion strongly linked to graph minors

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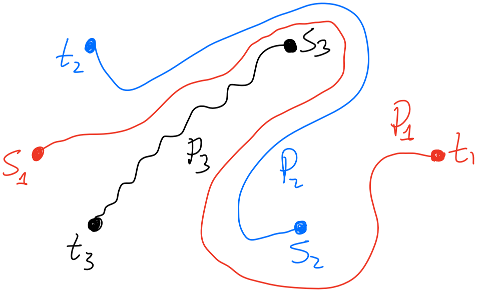

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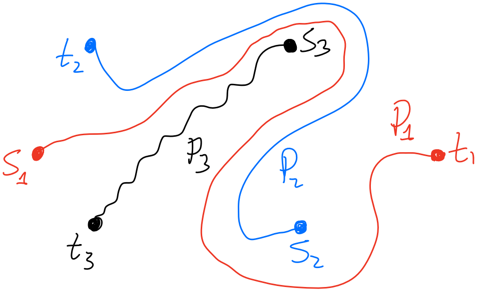

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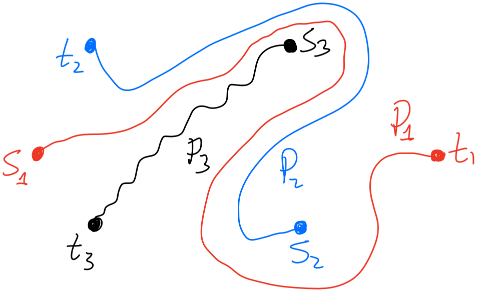

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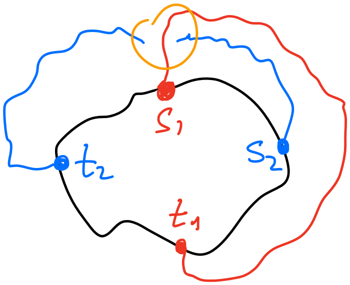
A graph G is k-linked if every instance of DISJOINT PATHS in G with k pairs is positive.

Topology appears naturally in linkages

Theorem (Thomassen and Seymour. 1980)

Let G be a 4-connected graph and $s_1, s_2, t_1, t_2 \in V(G)$. Then (s_1, s_2) and (t_1, t_2) are linked unless G is planar and s_1, s_2, t_1, t_2 are on the boundary of the same face, in this cyclic order.



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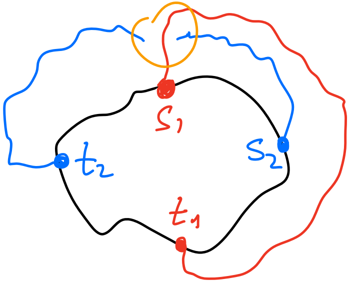

A combinatorial condition (linkage) is translated to a purely topological one (embedding).

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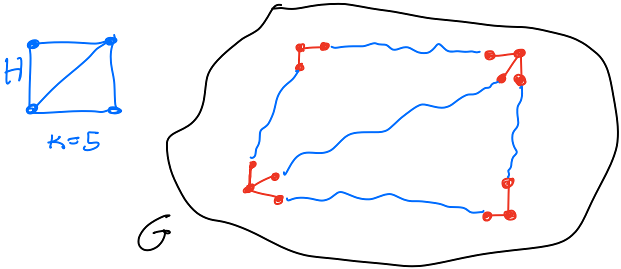

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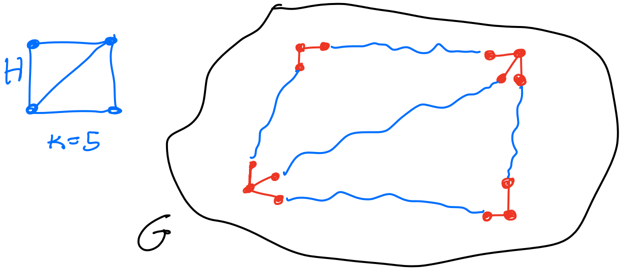

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Idea: if the goal is to decide whether $H \leq_m G$, if G is k-linked, then "yes".
Otherwise, we may exploit a topological obstruction to k-linkedness...

Another crucial notion: treewidth

Let G_1 and G_2 be two graphs, and let $S_i \subseteq V(G_i)$ be a k-clique.

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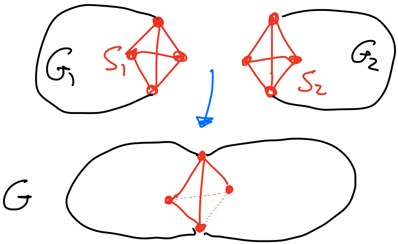

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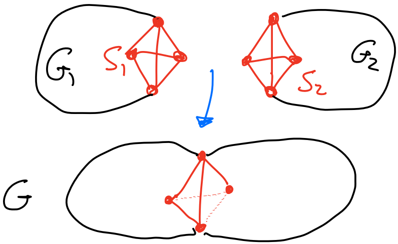

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We say that a graph G has treewidth at most k if it can be obtained by repeatedly taking a k-clique-sum with a graph on at most $k + 1$ vertices.

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Let H be a fixed graph. Recall that $\text{exc}(H)$ is the class of all graphs that do not contain H as a minor.

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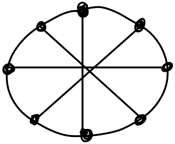
Theorem (Wagner. 1937)

A graph $G \in \mathrm{exc}(K_5)$ if and only if it can be obtained by 0-, 1-, 2- and 3-clique-sums from planar graphs and V_8.



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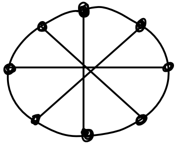

V_8

Paradigm: we find "pieces" that exclude K_5 for topological reasons (planarity), add some exceptions (V_8), and then define rules (clique-sums) that preserve being K_5-minor-free.

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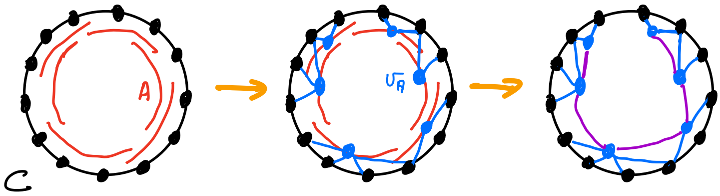
Note: this is an approximate characterization (i.e., not "iff").

Vortices



Adding a vortex of depth h to a cycle C:

- Select arcs on C so that each vertex is contained in at most h arcs.
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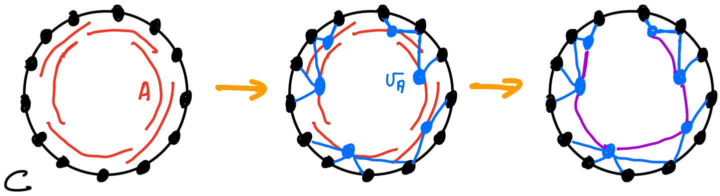

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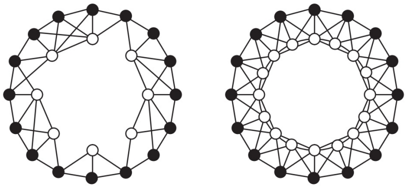

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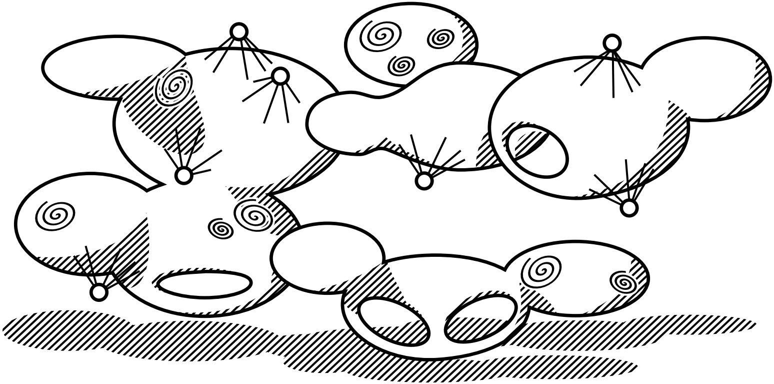
4. *Repeatedly construct the h-clique-sum of the current graph with another graph constructed using steps 1-2-3 above.*

A visualization of an H-minor-free graph



[Figure by Felix Riedl]

Sketch of sketch of sketch of proof of Wagner's conjecture

Let's try to mimic the proof for rooted trees by Nash-Williams:

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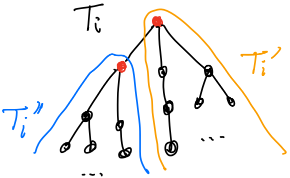

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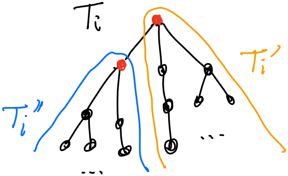

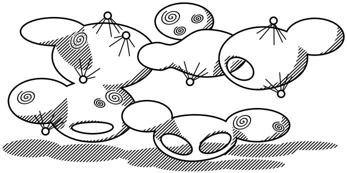

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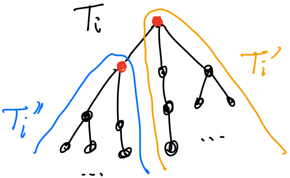

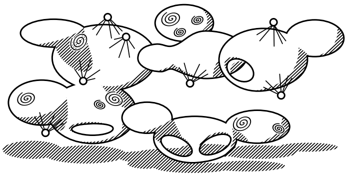

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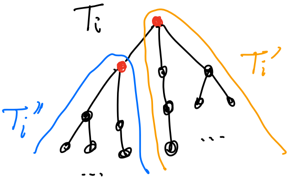

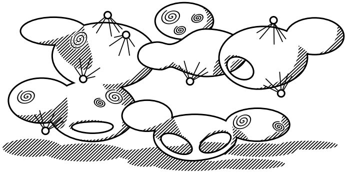

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- If G_1 is planar, every G_i has bounded treewidth: similar to trees.
- Otherwise, by the structure theorem: similar to "extended" surfaces (with apices and vortices), glued in a tree-like way.

Some algorithmic consequences

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This says that there $\boxed{\text{exists}}$ an algorithm... no idea how to $\boxed{\text{construct}}$ it!!

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Structure of sparse graphs

H-topological-minor-free

$\cup$

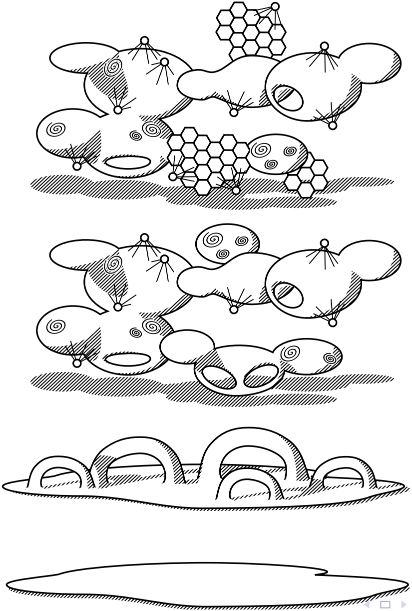
H-minor-free

$\cup$

bounded genus

$\cup$

planar



[Figure by Felix Riedl]

Next section is...

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1. Introduction to graph minors
2. Bidimensionality
   - Preliminaries
   - Some ingredients and an illustrative example
   - Meta-algorithms
3. Irrelevant vertex technique

Parameterized complexity in 2 slides

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed, finite alphabet.

For an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, k is called the parameter.

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- k-Vertex Cover: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, containing at least an endpoint of every edge?
- k-Clique: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \geq k$, of pairwise adjacent vertices?
- Vertex k-Coloring: Can the vertices of a graph be colored with $\leq k$ colors, so that any two adjacent vertices get different colors?

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These three problems are NP-hard, but are they $\boxed{\text{equally}}$ hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- k-Clique: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
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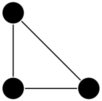
  $\boxed{\text{The problem is para-NP-hard}}$

skip

Treewidth via k-trees

Example of a 2-tree:



[Figure by Julien Baste]

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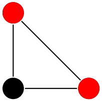

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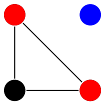

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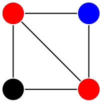

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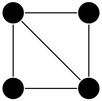

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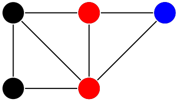

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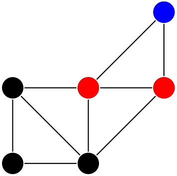

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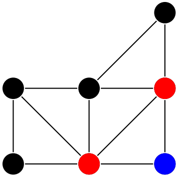

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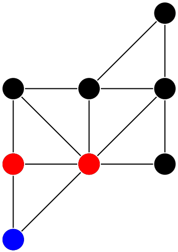

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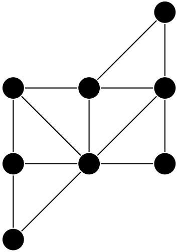

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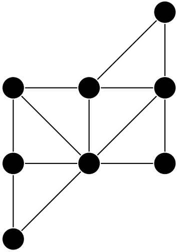

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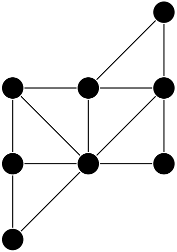

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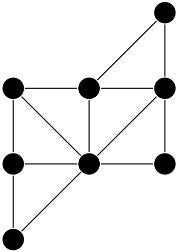

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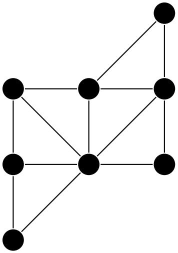

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Construction suggests the notion of tree decomposition: small separators.

Dynamic programming on tree decompositions

- Typically, FPT algorithms parameterized by treewidth are based on dynamic programming (DP) over a tree decomposition.

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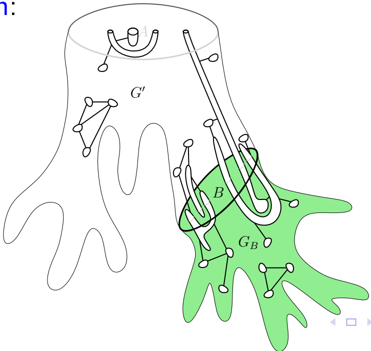

[Figure by Valentin Garnero]

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- Quantifiers $\exists$, $\forall$ on vertex/edge variables or vertex/edge **sets**.

($\text{MSO}_1/\text{MSO}_2$)

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- a graph being k-colorable (for fixed k), having a Hamiltonian cycle.

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   The vast majority, but not all of them:

   - List Coloring is W[1]-hard parameterized by treewidth.

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2. Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

Next subsection is...

A few representative problems

VERTEX COVER
Input: A graph $G = (V, E)$ and a positive integer k.
Parameter: k.
Question: Does there exist a subset $C \subseteq V$ of size at most k such that $G[V \setminus C]$ is an independent set?

A few representative problems

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Long Path
Input: A graph $G = (V, E)$ and a positive integer k.
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Question: Does there exist a path P in G of length at least k?

A few representative problems (II)

Feedback Vertex Set
Input: A graph $G = (V, E)$ and a positive integer k.
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Question: Does there exist a subset $F \subseteq V$ of size at most k such that for $G[V \setminus F]$ is a forest?

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DOMINATING SET
Input: A graph $G = (V, E)$ and a positive integers k.
Parameter: k.
Question: Does there exist a subset $D \subseteq V$ of size at most k such that for all $v \in V$, $N[v] \cap D \neq \emptyset$?

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- We say that a parameter P is *closed under taking of minors/contractions* (or, briefly, *minor/contraction-closed*) if for every graph H, $H \preceq_m G$ / $H \preceq_{cm} G$ implies that $P(H) \leq P(G)$.

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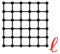
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: 

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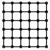
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.

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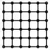
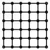
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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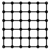
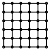
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. $\boxed{\text{Does the reverse implication hold?}}$

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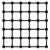
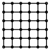
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_{\ell}$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_{\ell} \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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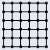
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_{\ell}$ *as a minor.*

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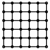
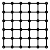
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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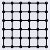
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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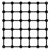
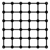
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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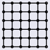
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_\ell$ *as a minor.*

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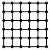
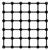
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\text{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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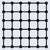
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_\ell$ *as a minor.*

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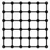
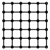
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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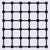
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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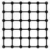
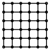
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. $\boxed{\text{Does the reverse implication hold?}}$

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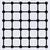
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_\ell$ *as a minor.*

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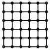
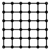
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. $\boxed{\text{Does the reverse implication hold?}}$

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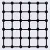
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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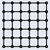
$\boxed{\text{Important message}}$ grid-minors are the certificate of large treewidth.

Grid Exclusion Theorems on sparse graphs

Theorem (Robertson, Seymour, Thomas. 1994)

Every planar graph of treewidth $\geq 6 \cdot \ell$ *contains*  $_{\ell}$ *as a minor.*

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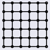
Every planar graph of treewidth $\geq 6 \cdot \ell$ *contains*  $_\ell$ *as a minor.*

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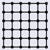
For every fixed g, *there is a constant* c_g *such that every graph of genus* g *and of treewidth* $\geq c_g \cdot \ell$ *contains*  $_\ell$ *as a minor.*

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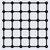
Every planar graph of treewidth $\geq 6 \cdot \ell$ contains $_\ell$ *as a minor.*

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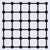
For every fixed g, there is a constant c_g such that every graph of genus g and of treewidth $\geq c_g \cdot \ell$ contains $_\ell$ *as a minor.*

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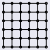
For every fixed graph H, there is a constant c_H such that every H-minor-free graph of treewidth $\geq c_H \cdot \ell$ contains $_\ell$ *as a minor.*

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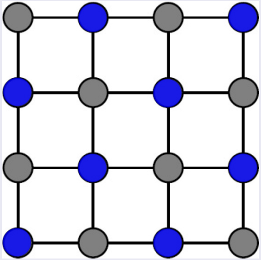
In sparse graphs: linear dependency between treewidth and grid-minors

How to use Grid Theorems algorithmically?

Example: FPT algorithm for Planar Vertex Cover

A vertex cover of a graph G is a set of vertices C such that every edge of G has at least one endpoint in C. Min size: $\mathbf{vc}(G)$.



Example: FPT algorithm for Planar Vertex Cover

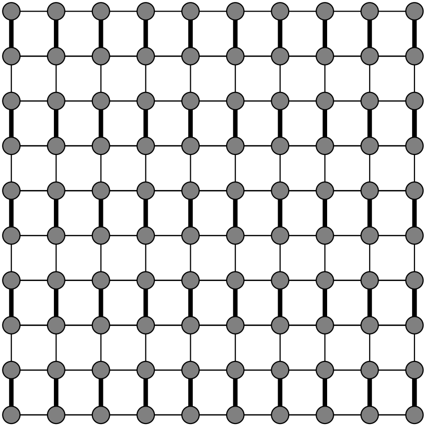
INPUT: Planar graph G on n vertices, and an integer k.

OUTPUT: Either a vertex cover of G of size $\leq k$, or a proof that G has no such a vertex cover.

RUNNING TIME: $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

$\boxed{\text{Objective}}$ subexponential FPT algorithm for Planar Vertex Cover.

Example: FPT algorithm for Planar Vertex Cover



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Let G be a planar graph of treewidth $\geq 6 \cdot \ell$ $\Longrightarrow$ G contains the $(\ell \times \ell)$-grid $H_{\ell,\ell}$ as a minor

- The size of any vertex cover of $H_{\ell,\ell}$ is at least $\ell^2/2$.
- Recall that Vertex Cover is a minor-closed parameter.
- Since $H_{\ell,\ell} \preceq_m G$, it holds that $\mathbf{vc}(G) \geq \mathbf{vc}(H_{\ell,\ell}) \geq \ell^2/2$.

We are already very close to an algorithm...

Recall:

- k is the parameter of the problem.
- We have that $\mathrm{tw}(G) = 6 \cdot \ell$ and ℓ is the size of a grid-minor of G.
- Therefore, $\mathbf{vc}(G) \geq \ell^2/2$.

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This gives a subexponential FPT algorithm!

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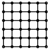
minor-closed $+$ **vc**($_k$) $= \Omega(k^2)$.

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★ Only the linear Grid Exclusion Theorem!

Arguments go through up to H-minor-free graphs.

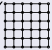
Next subsection is...

Minor Bidimensionality:

[Demaine, Fomin, Hajiaghayi, Thilikos. 2005]

Definition

A parameter **p** is *minor bidimensional* if

1. **p** is closed under taking of minors (minor-closed), and
2. **p** $\Bigg($  $_{k}\Bigg) = \Omega(k^2)$.

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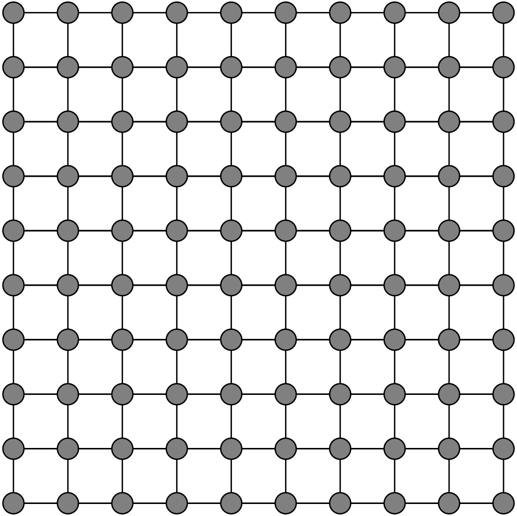

$H_{\ell,\ell}$ for $\ell = 10$

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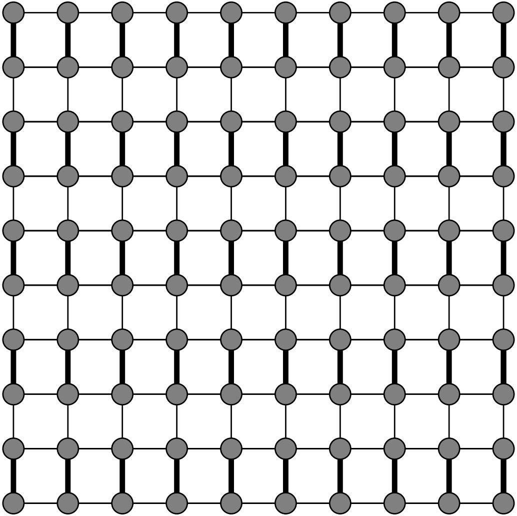

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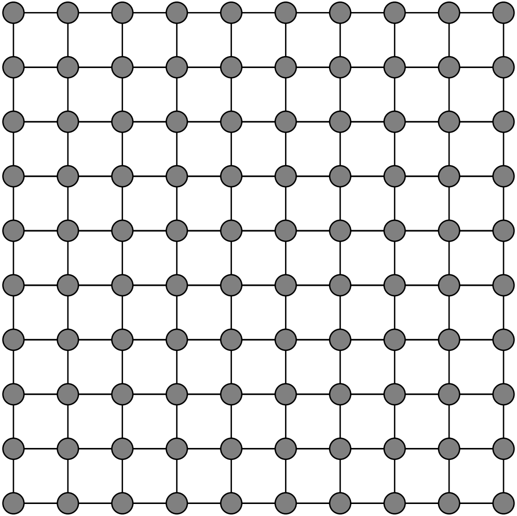

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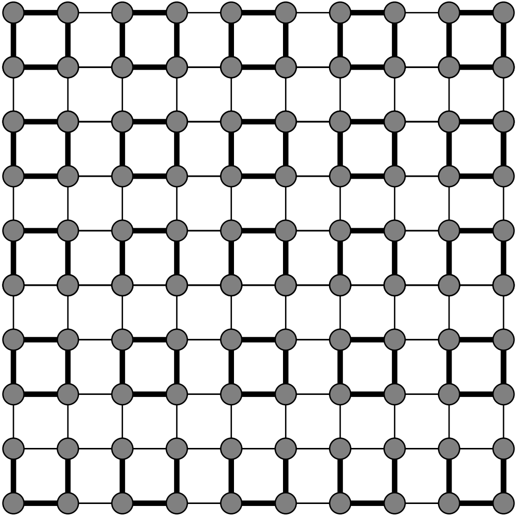

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- If we have a DP algorithm for bounded treewidth running in time c^t or t^t, then it implies $2^{O(\sqrt{k})}$ or $2^{O(\sqrt{k}\log k)}$ algorithm.

Piecing everything together

Theorem

Let G be an H-minor-free graph, and let $\mathbf{p}$ be a minor bidimensional graph parameter computable in time $2^{O(\mathbf{tw}(G))} \cdot n^{O(1)}$.
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- Randomized algorithms using Cut&Count. [Cygan et al. 2011]
- Deterministic algorithms based on matrix rank. [Boadlaender et al. 2012]
- Deterministic algorithms based on matroids. [Fomin et al. 2013]

Further applications of Bidimensionality

1. Bidimensionality + DP $\Rightarrow$ $\boxed{\text{Subexponential FPT algorithms}}$

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4. Bidimensionality + new Grid Theorems $\Rightarrow$ $\boxed{\text{Geometric graphs}}$

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   [Grigoriev, Koutsonas, Thilikos. 2013]

Next section is...

Basic principle of the irrelevant vertex technique

This technique was invented in [Robertson and Seymour. 1995]

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1. If $\mathrm{tw}(G) > f(k)$, find an irrelevant vertex:

   A vertex $v \in V(G)$ such that (G, T, k) and $(G \setminus v, T, k)$ are equivalent instances.

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2. Otherwise, if $\text{tw}(G) \leq f(k)$, solve the problem using dynamic programming (by Courcelle).

How to find an irrelevant vertex when the treewidth is large?

How to find an irrelevant vertex when the treewidth is large?

By using the Grid Exclusion Theorem!

How to find an irrelevant vertex when the treewidth is large?

By using the $\boxed{\text{Wall}}$ Exclusion Theorem!

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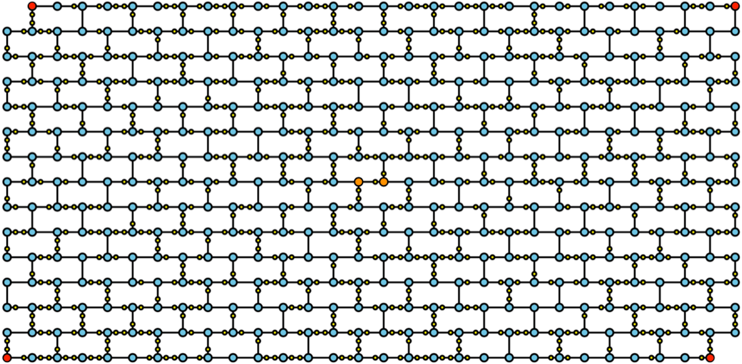
Theorem (Robertson and Seymour. 1986)

For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains an ℓ-wall as a minor.



[Figure by Dimitrios M. Thilikos]

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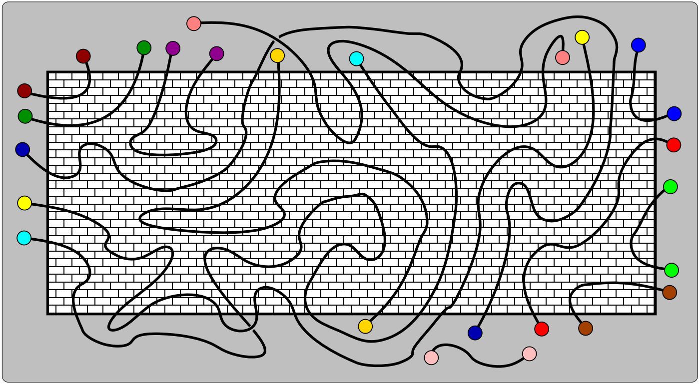
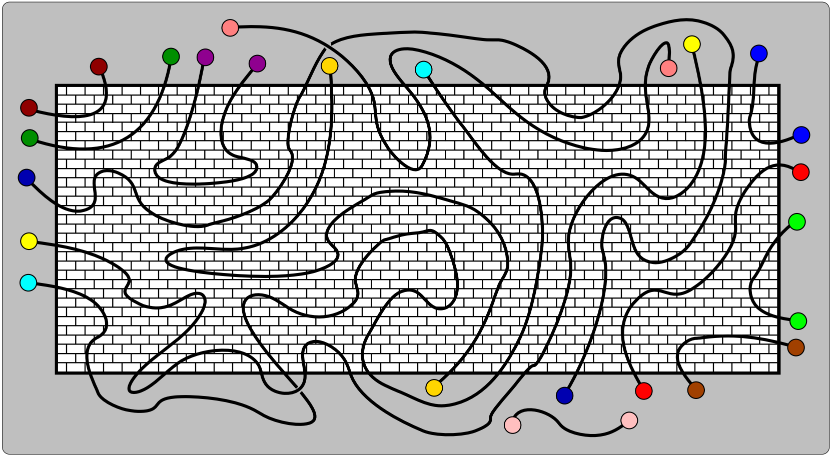


[Figure by Dimitrios M. Thilikos]

Goal: declare one of the central vertices of the wall irrelevant.



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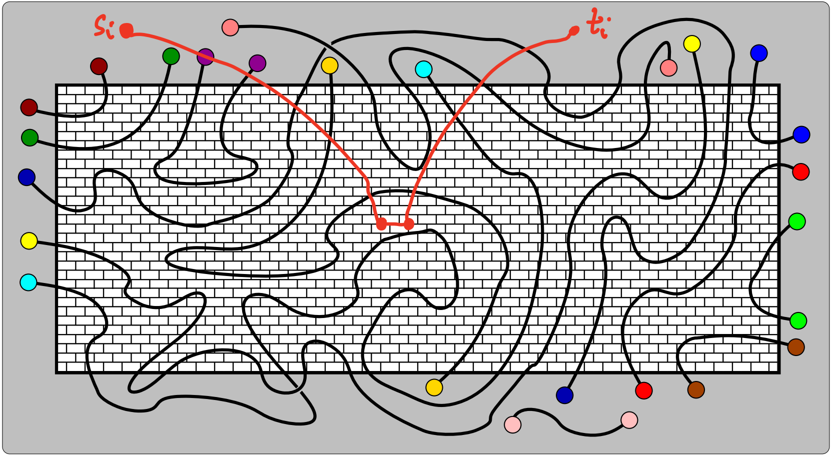
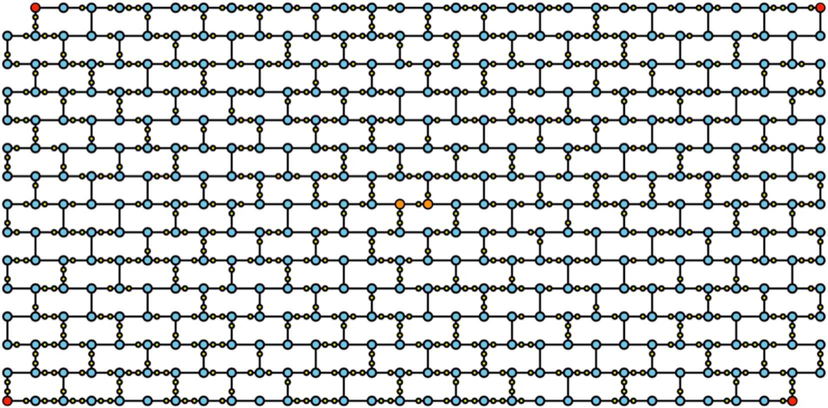


This is only possible if the wall is insulated from the exterior!

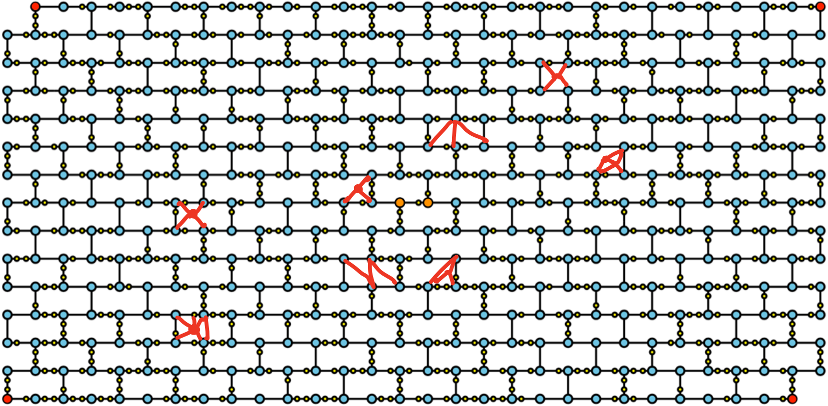
Flat walls

Goal: enrich the notion of wall so that we can insulate it from the exterior.



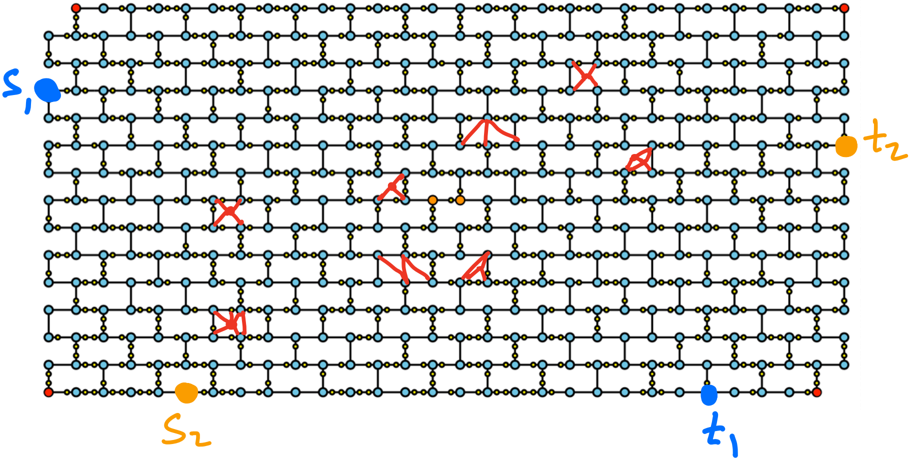
Flat walls

We need to allow some extra edges in the interior of the wall.



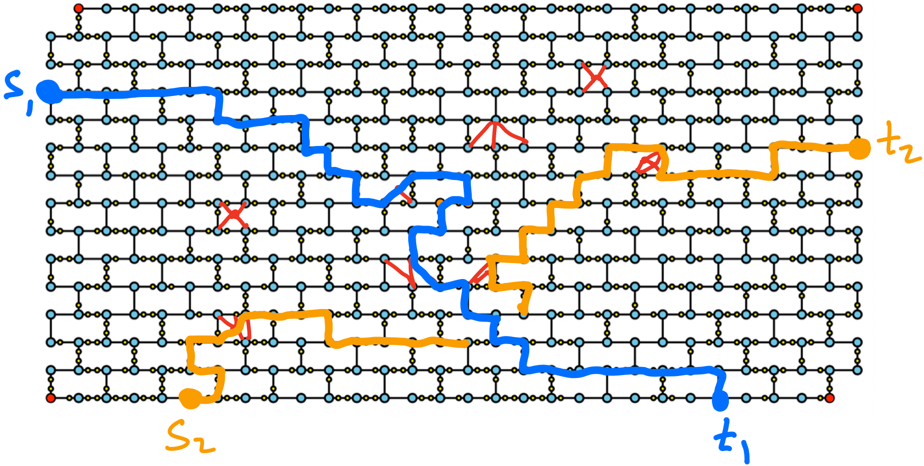
Flat walls

We impose a topological property that defines the "flatness" of the wall.



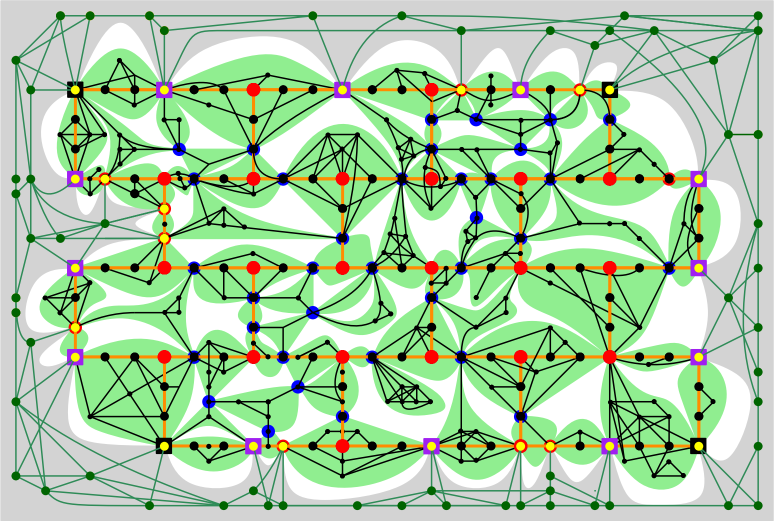
Flat walls

There are no crossing paths $s_1 - t_1$ and $s_2 - t_2$ from/to the perimeter.

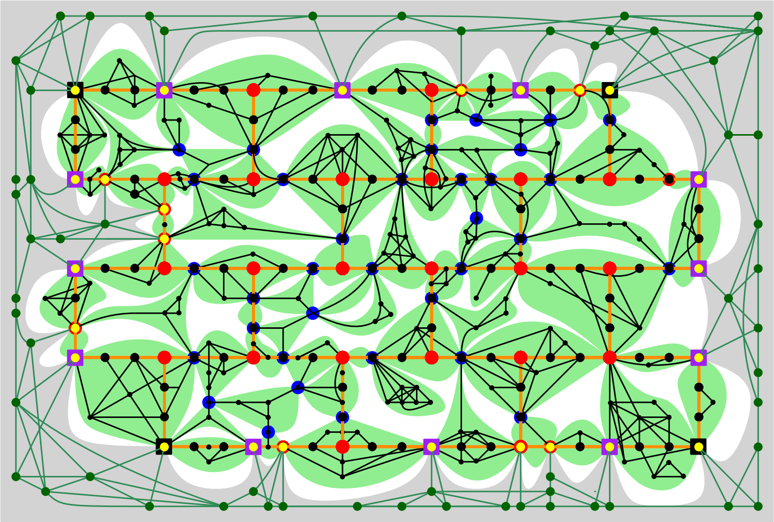


Flat walls

A real flat wall can be quite wild... [Figure by Dimitrios M. Thilikos]



Flat walls: a bit more formal



[Figures by Dimitrios M. Thilikos]

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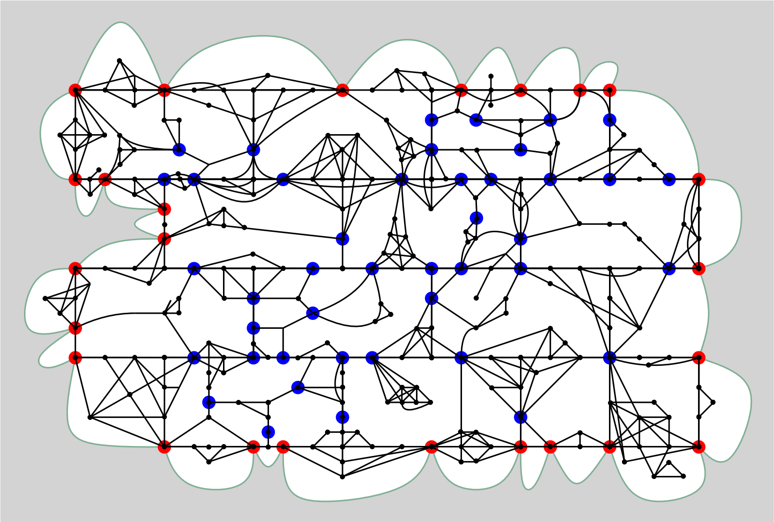

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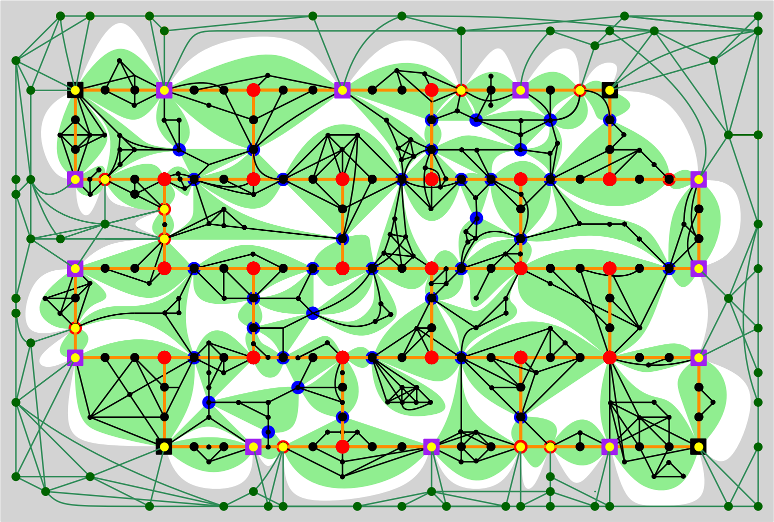

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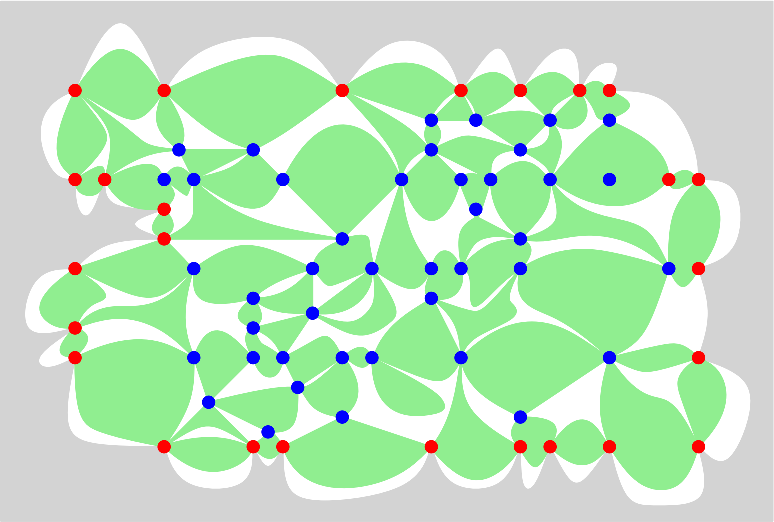

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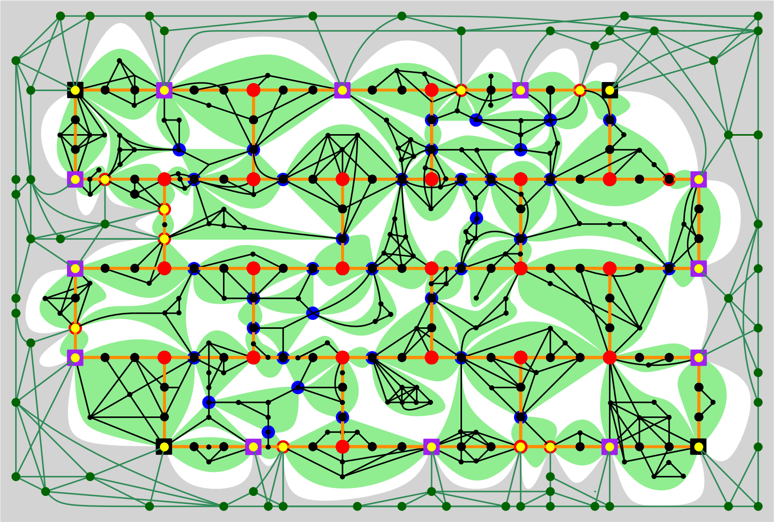

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The Weak Structure Graph Minors Theorem

Theorem (Robertson and Seymour. 1995)

There exist recursive functions $f_1 : \mathbb{N}^2 \to \mathbb{N}$ *and* $f_2 : \mathbb{N} \to \mathbb{N}$, *such that for every graph* G *and every* $q, r \in \mathbb{N}$, *one of the following holds:*

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Important: possible to find one of the outputs in time $f(q, r) \cdot |V(G)|$.

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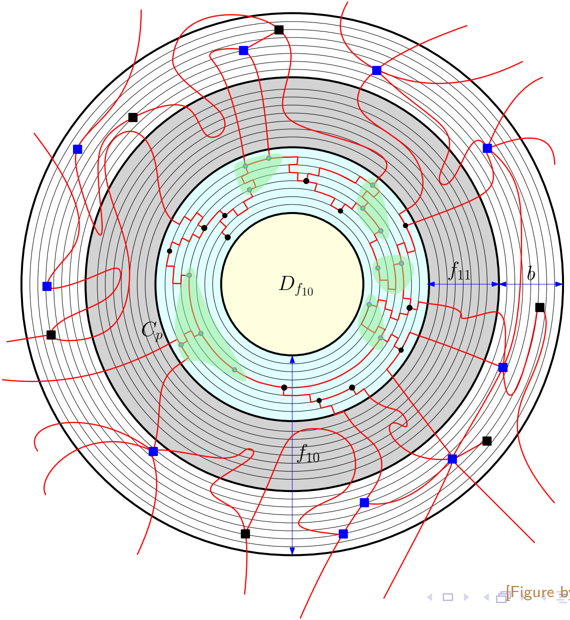
The irrelevant vertex technique has been applied to many problems... usually with a lot of technical pain.

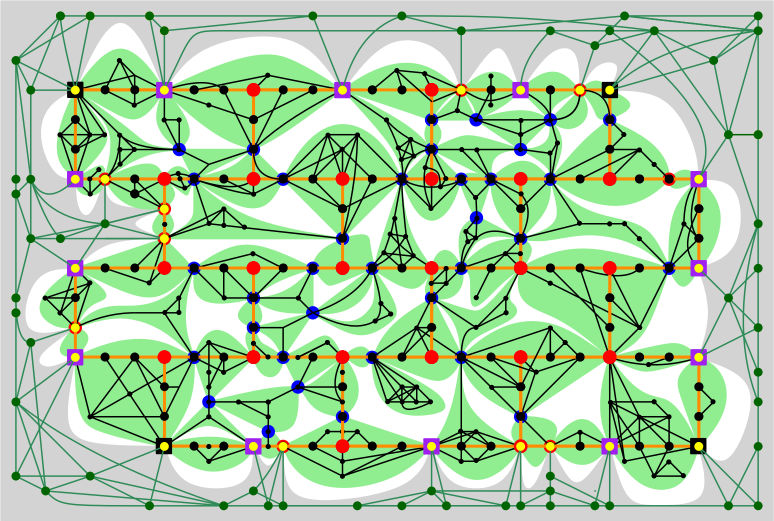
Rerouting inside a big flat wall...



[Figure by Dimitrios M. Thilikos]

Crucial notion: homogeneity

In order to declare a vertex irrelevant for some problem, usually we need to consider a homogenous flat wall, which we proceed to define.



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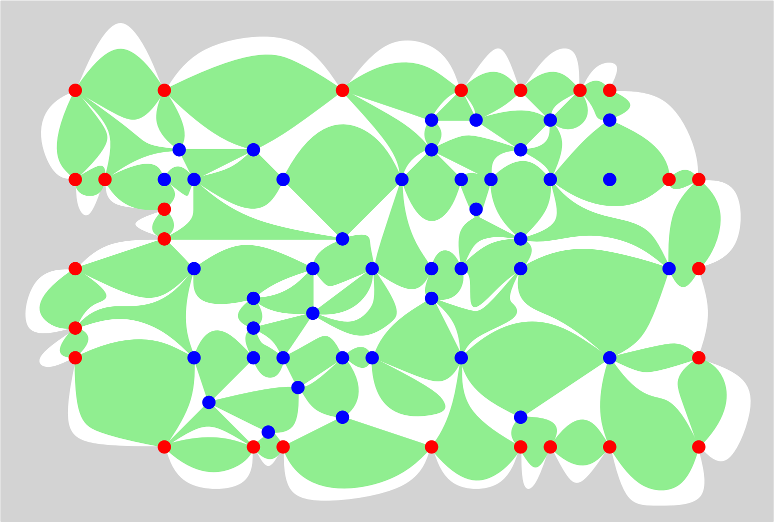
We consider a flap-coloring encoding the relevant information of our favorite problem inside each flap (similar to tables of DP).



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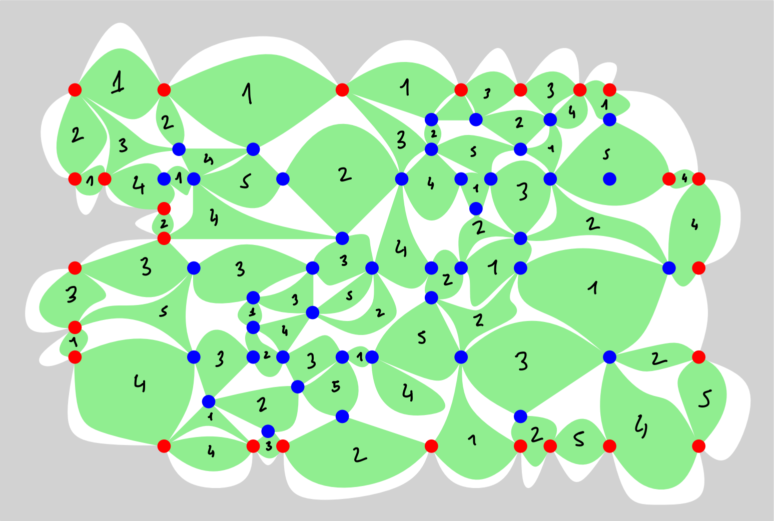
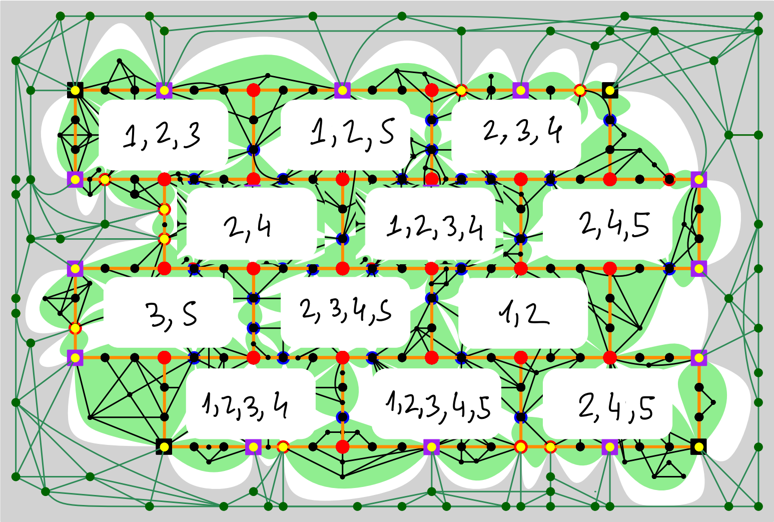


Crucial notion: homogeneity

For every brick of the wall, we define its palette as the colors appearing in the flaps it contains.



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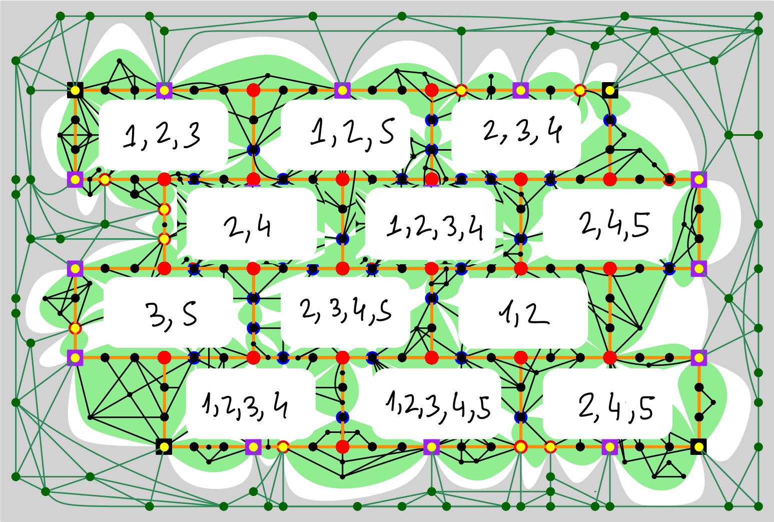
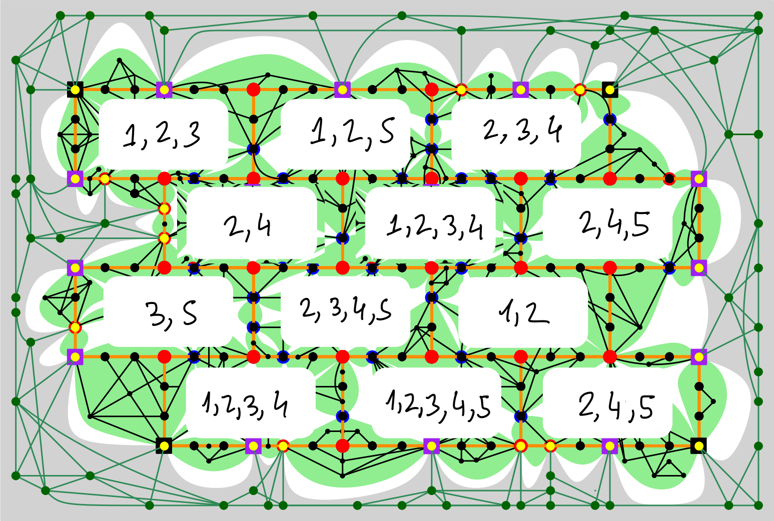
A flat wall is homogenous if every (internal) brick has the same palette.
Fact: every brick of a homogenous flat wall has the same "behavior".



Crucial notion: homogeneity

Price of homogeneity to obtain a homogenous flat r-wall (zooming):

If we have c colors, we need to start with a flat r^c-wall. (why?)



Gràcies!