Hitting minors on bounded treewidth graphs

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Joint work with Julien Baste and Dimitrios M. Thilikos

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Theorem (Courcelle. 1990)

Every problem expressible in MSOL can be solved in time $f(\mathrm{tw}) \cdot n$ on graphs on n vertices and treewidth at most tw.

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In parameterized complexity: FPT parameterized by treewidth.

Examples: Vertex Cover, Dominating Set, Hamiltonian Cycle, Clique, Independent Set, k-Coloring for fixed k, ...

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Dynamic programming on tree decompositions

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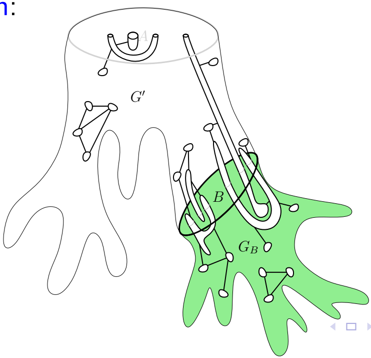
- Typically, FPT algorithms parameterized by treewidth are based on dynamic programming (DP) over a tree decomposition.
- Starting from the leaves of the tree decomposition, a set of appropriately defined partial solutions is computed recursively until the root, where a global solution is obtained.
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

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It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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1. Relax the connectivity requirement by considering a set of cuts that contain the relevant (connected) solutions.
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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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There are other examples of such problems...

The $\mathcal{F}$-M-Deletion problem

Let $\mathcal{F}$ be a fixed finite collection of graphs.

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Both problems are NP-hard if $\mathcal{F}$ contains some edge. [Lewis, Yannakakis. 1980]

FPT by Courcelle's Theorem.

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-Deletion/$\mathcal{F}$-TM-Deletion can be solved in time

$$f_{\mathcal{F}}(\mathsf{tw}) \cdot n^{\mathcal{O}(1)}$$

on n-vertex graphs.

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- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

Summary of our results: arXiv 1704.07284+1907.04442

[1] Connected collection $\mathcal{F}$: all the graphs are connected.

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- $\mathcal{F} = \{H\}$, H connected: complete tight dichotomy...

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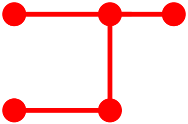

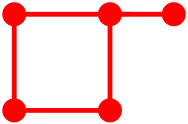

Theorem

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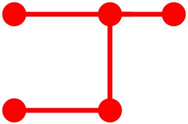

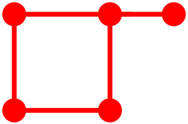

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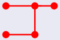
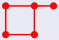
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathsf{c}}$  *or* $H \preceq_{\mathsf{c}}$  .

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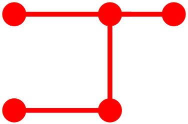

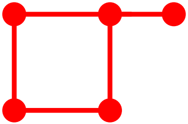

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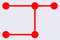
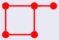
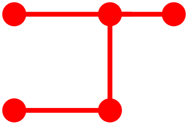
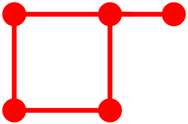
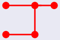
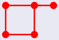
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_c$  *or* $H \preceq_c$ .
- $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

A dichotomy for hitting a connected minor

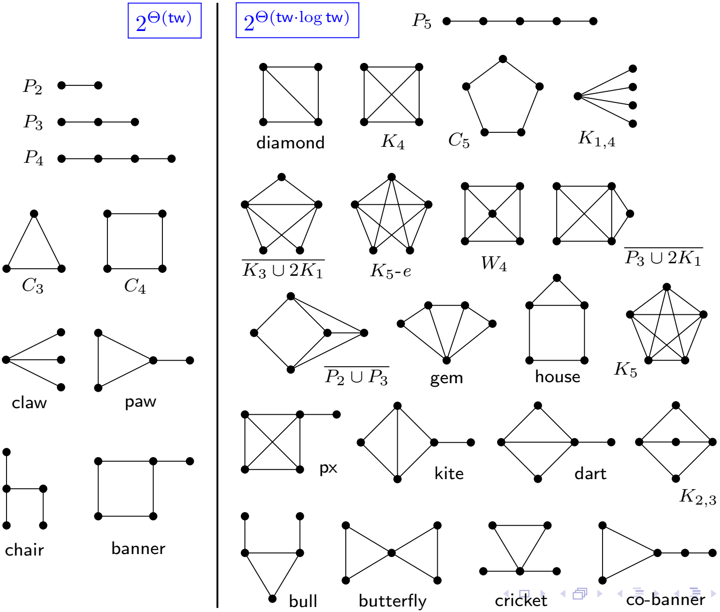
Theorem

Let H *be a connected graph.*
The $\{H\}$-M-DELETION *problem is solvable in time*

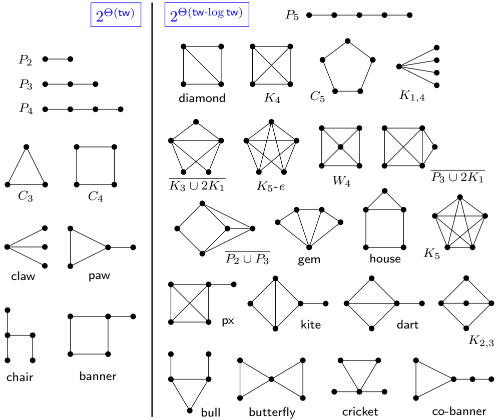
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{c}}$  *or* $H \preceq_{\mathrm{c}}$ .
- $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

In both cases, the running time is asymptotically optimal under the ETH.

Complexity of hitting a single connected minor H



A compact statement for a single connected graph



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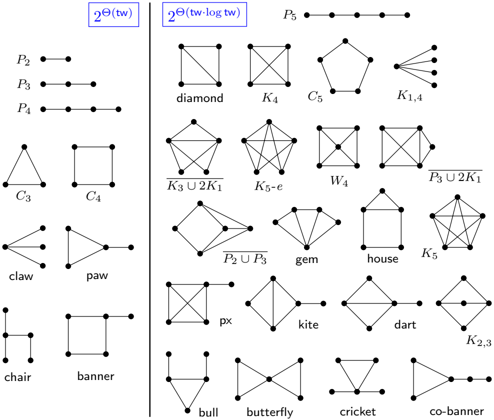

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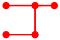
- All graphs on the left are contractions of  or 

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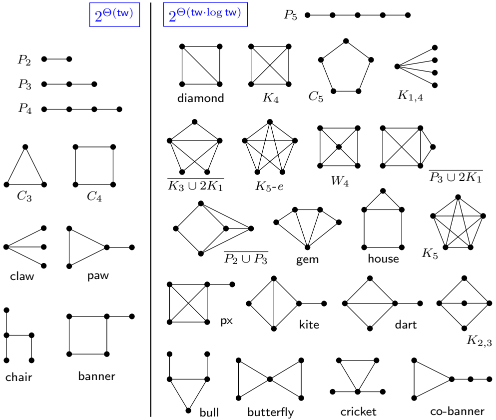

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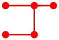
- All graphs on the left are contractions of  or 
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   - For every $\mathcal{F}$: time $2^{2^{\mathcal{O}(\text{tw}\cdot\log\text{tw})}} \cdot n^{\mathcal{O}(1)}$.
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3. Lower bounds under the ETH
   - $2^{o(\mathrm{tw})}$ is "easy".
   - $2^{o(\mathrm{tw}\cdot\log \mathrm{tw})}$ is much more involved and we get ideas from:

     [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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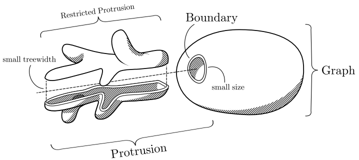

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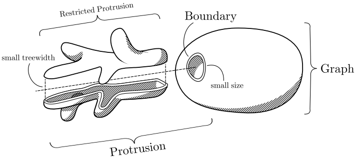

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  Extra: Bidimensionality, irrelevant vertices, protrusion decompositions...

skip

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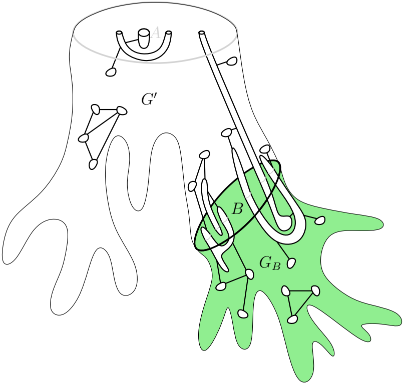

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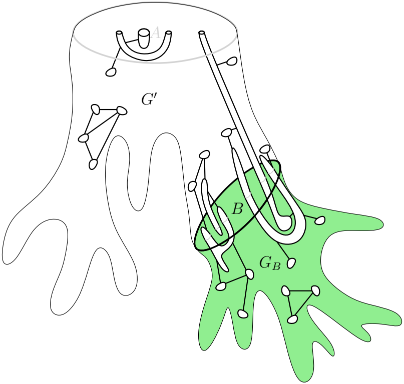

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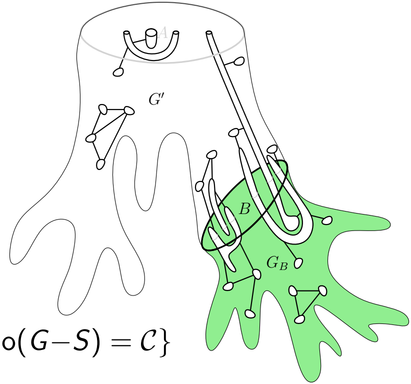

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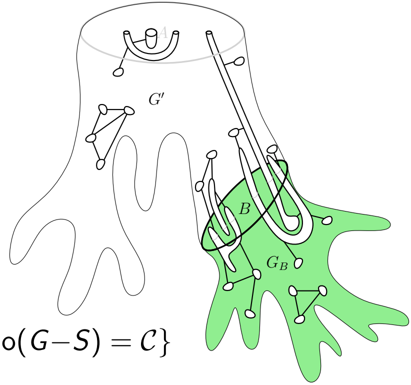

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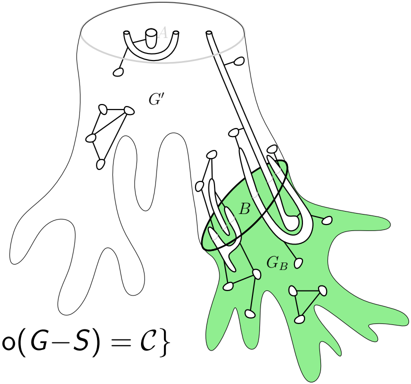

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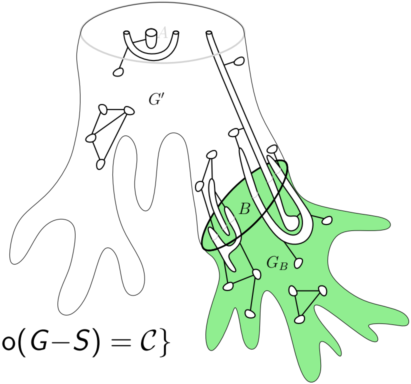

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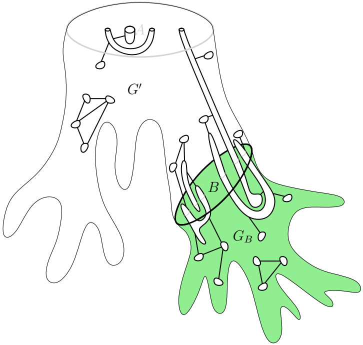

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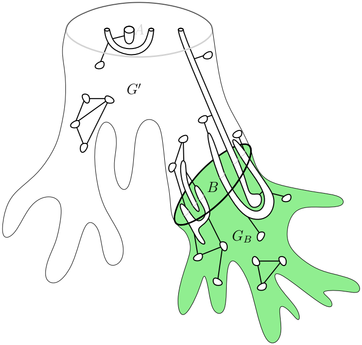

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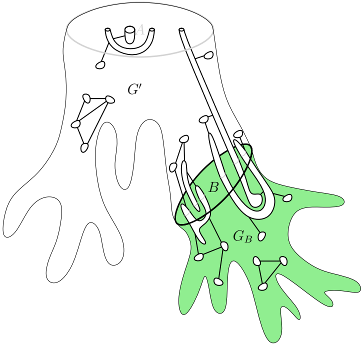

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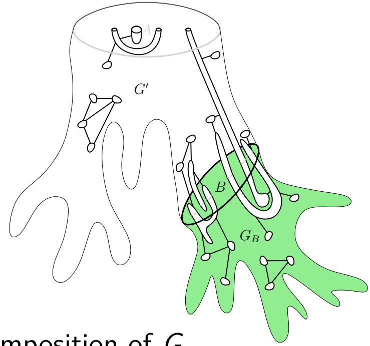

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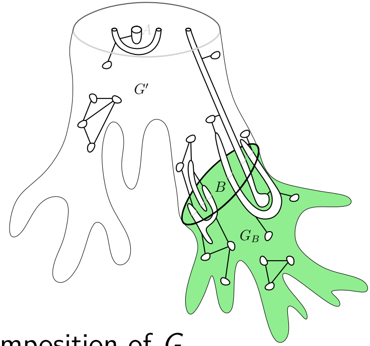

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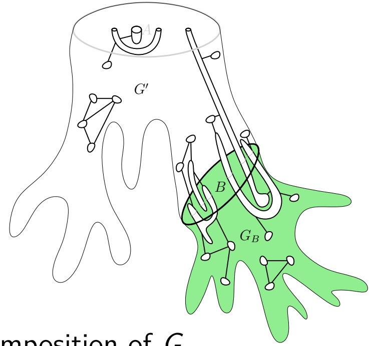

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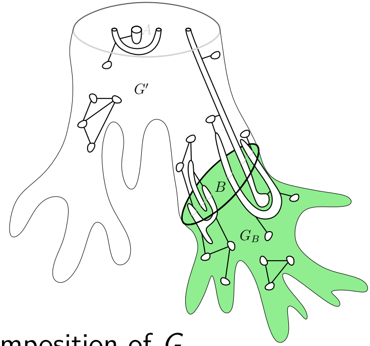

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- We can then find a linear protrusion decomposition of R.

Algorithm for any connected collection $\mathcal{F}$

- $\mathcal{R}^{(\mathcal{F},t)}$: set of minimum-size representatives of $\equiv^{(\mathcal{F},t)}$.
- Suppose that we can prove that, for every $R \in \mathcal{R}^{(\mathcal{F},t)}$,
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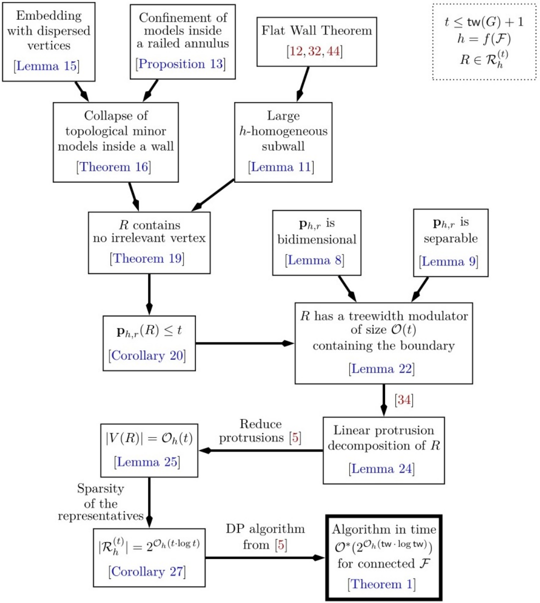

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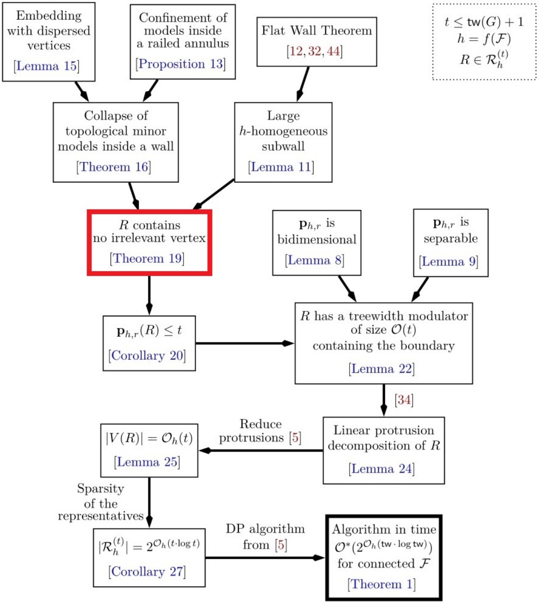

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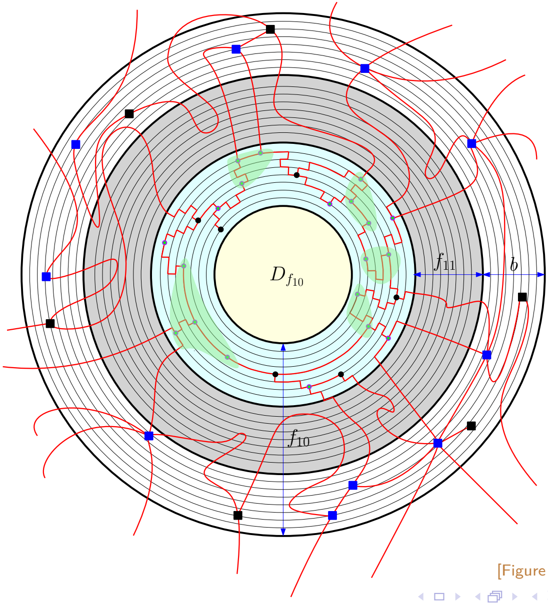

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[Figure by Dimitrios M. Thilikos]

Algorithm when the input graph G is planar

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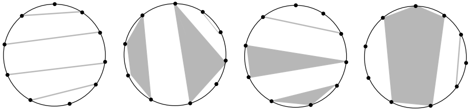

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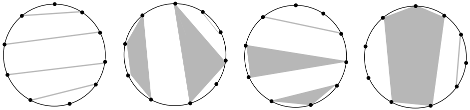

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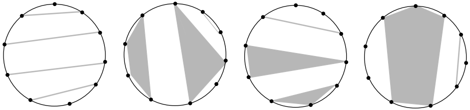

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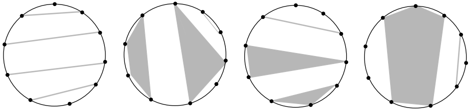

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. skip [Rué, S., Thilikos. 2014]

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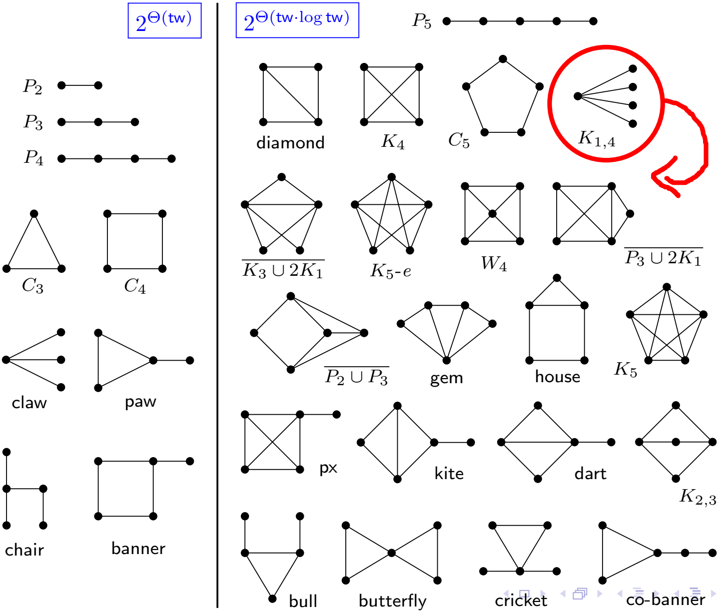
  - $\boxed{\text{Conjecture}}$ For every (connected) family $\mathcal{F}$, the $\mathcal{F}$-TM-Deletion problem is solvable in time $2^{\mathcal{O}(\mathsf{tw} \cdot \log \mathsf{tw})} \cdot n^{\mathcal{O}(1)}$.

For topological minors, there is (at least) one change



Gràcies!