Hitting minors on bounded treewidth graphs

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Joint work with Julien Baste and Dimitrios M. Thilikos

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Outline of the talk

1. Introduction
   - Parameterized complexity
   - Treewidth
   - FPT algorithms parameterized by treewidth

2. The $\mathcal{F}$-DELETION problem

3. Further research

Next section is...

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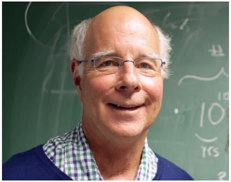
The area of parameterized complexity

Idea Measure the complexity of an algorithm in terms of the input size and an additional integer parameter, expected to be small.

This theory started in the late 80's, by Downey and Fellows:

Today, it is a well-established area with hundreds of articles published every year in the most prestigious TCS journals and conferences.

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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

1. k-Vertex Cover: solvable in time $2^k \cdot n^2$

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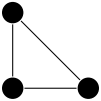
   The problem is para-NP-hard

Next subsection is...

Treewidth via k-trees

Example of a 2-tree:



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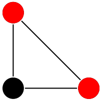

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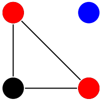

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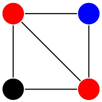

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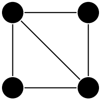

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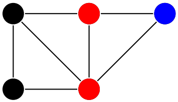

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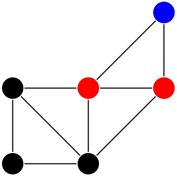

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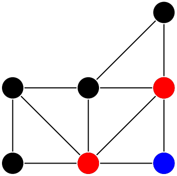

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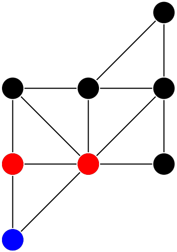

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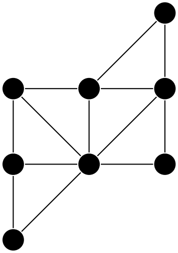

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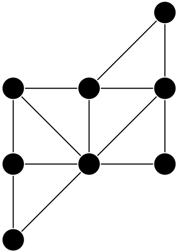

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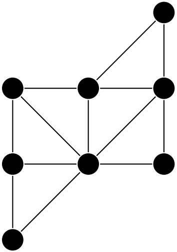

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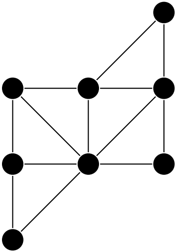

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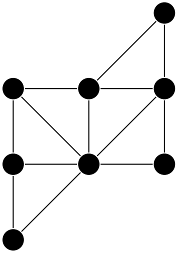

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{B_t \mid t \in V(T)\})$, where
  T is a tree, and
  $B_t \subseteq V(G) \;\; \forall t \in V(T)$ (bags),

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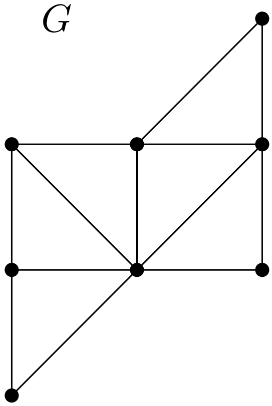

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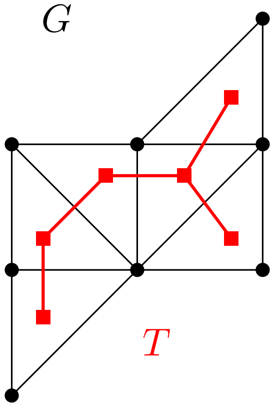

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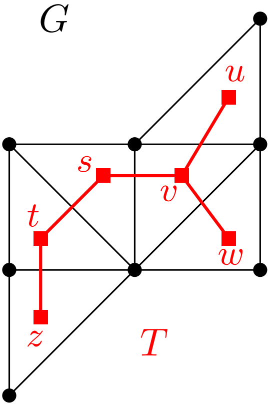

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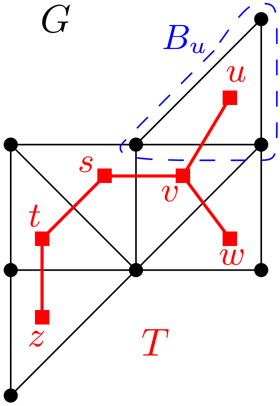

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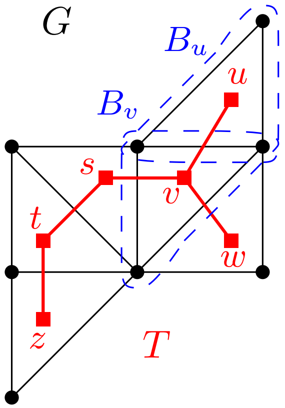

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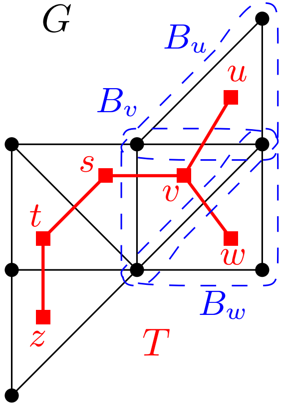

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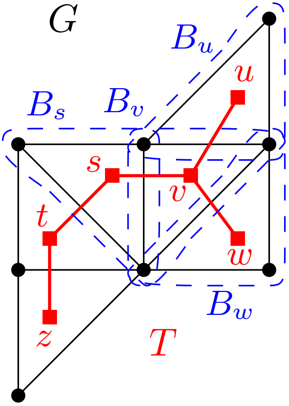

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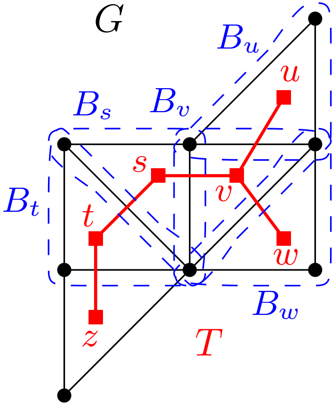

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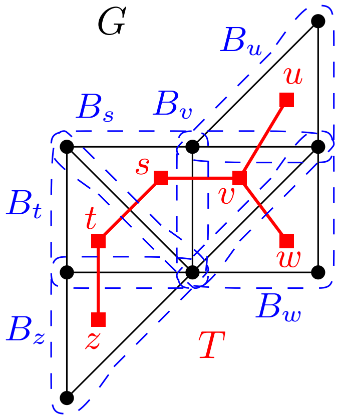

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3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

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Examples: Vertex Cover, Dominating Set, Hamiltonian Cycle, Clique, Independent Set, k-Coloring for fixed k, ...

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- List Coloring is W[1]-hard parameterized by treewidth.

  [Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Szeider, Thomassen. 2007]

- Some problems involving weights or colors are even NP-hard on graphs of constant treewidth (even on trees!).

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   Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

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$\boxed{\text{ETH: The 3-SAT problem on } n \text{ variables cannot be solved in time } 2^{o(n)}}$

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Typical statements:

ETH $\Rightarrow$ k-Vertex Cover cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$.

ETH $\Rightarrow$ Planar k-Vertex Cover cannot in time $2^{o(\sqrt{k})} \cdot n^{O(1)}$.

Dynamic programming on tree decompositions

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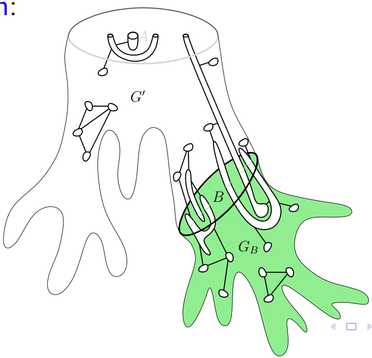
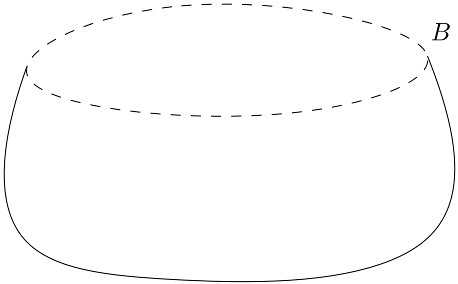
- Typically, FPT algorithms parameterized by treewidth are based on dynamic programming (DP) over a tree decomposition.
- Starting from the leaves of the tree decomposition, a set of appropriately defined partial solutions is computed recursively until the root, where a global solution is obtained.
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Two behaviors for problems parameterized by treewidth

Local problems VERTEX COVER, DOMINATING SET, CLIQUE, INDEPENDENT SET, q-COLORING for fixed q.



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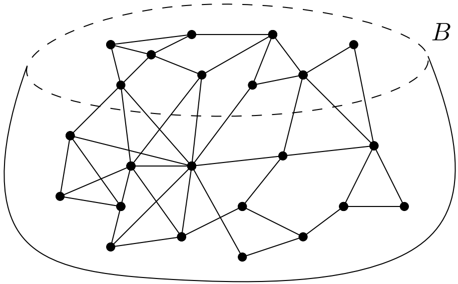

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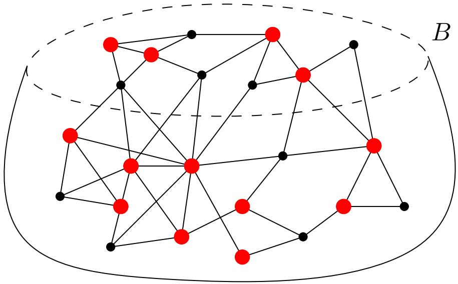

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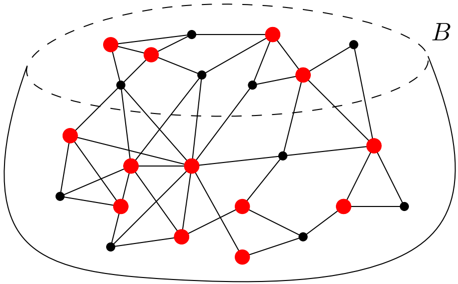

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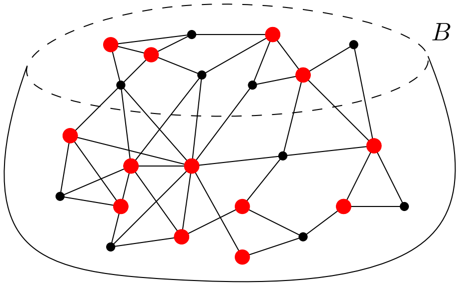
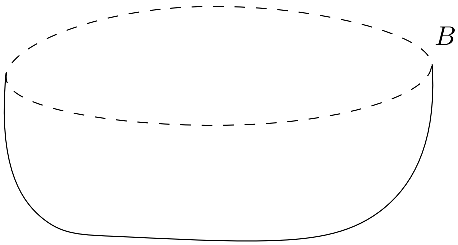


- It is sufficient to store, for each bag B, the subset of vertices of B that belong to a partial solution: $\boxed{2^{\text{tw}} \text{ choices}}$
- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

$$2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}.$$

Connectivity problems seem to be more complicated...

$\boxed{\text{Connectivity problems}}$ Hamiltonian Cycle, Longest Path, Steiner Tree, Connected Vertex Cover.



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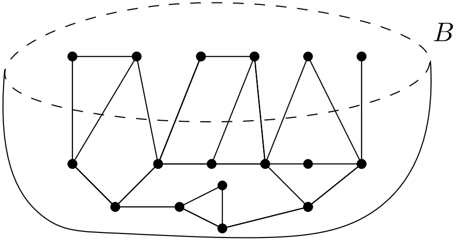

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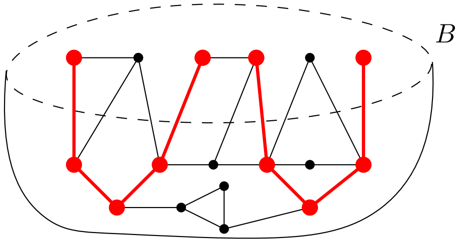

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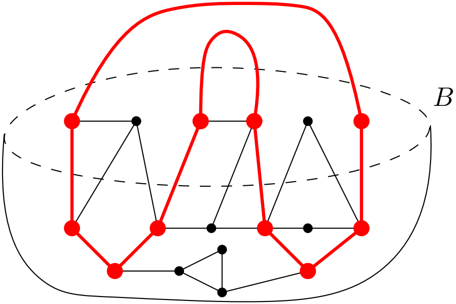

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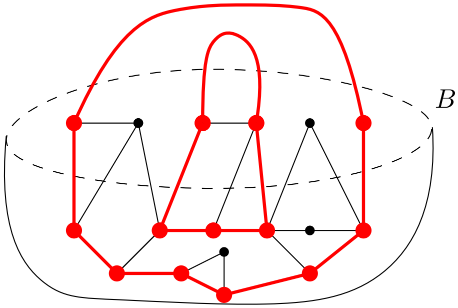

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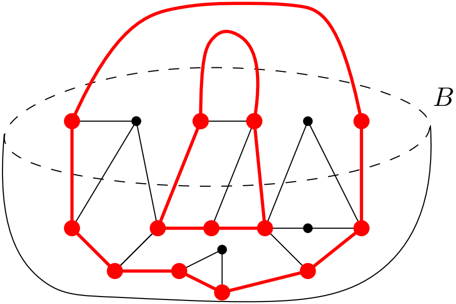

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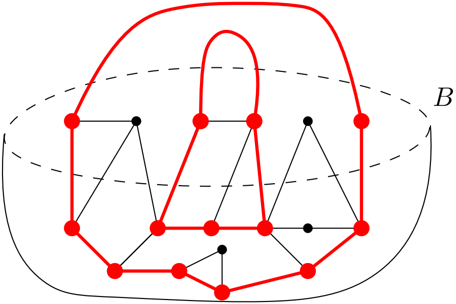

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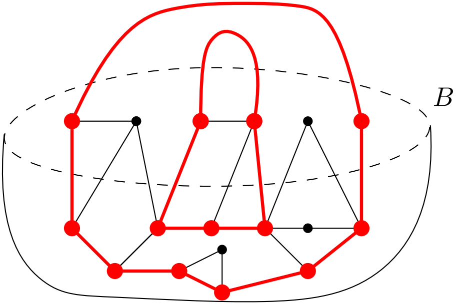

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- The "natural" DP algorithms provide only time $2^{\mathcal{O}(\mathsf{tw} \cdot \log \mathsf{tw})} \cdot n^{\mathcal{O}(1)}$.

Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

$$\boxed{2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

Vertex Cover, Dominating Set, ...

- Connectivity problems:

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The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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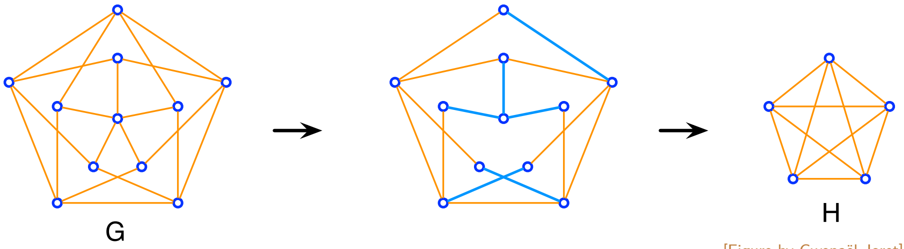
There are other examples of such problems...

Next section is...

1. Introduction
   - Parameterized complexity
   - Treewidth
   - FPT algorithms parameterized by treewidth
2. The $\mathcal{F}$-Deletion problem
3. Further research

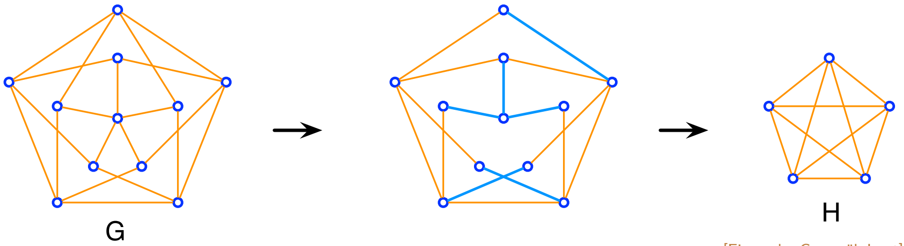
Minors and topological minors



[Figure by Gwenaël Joret]

- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

Minors and topological minors



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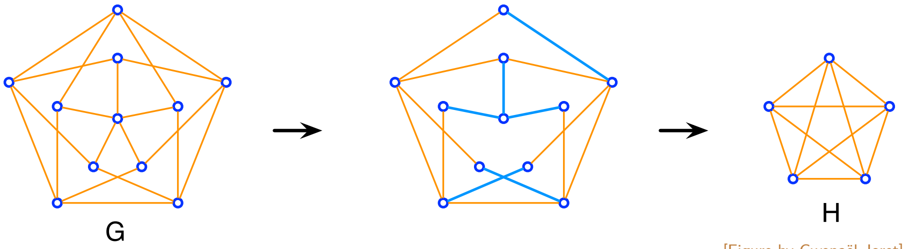

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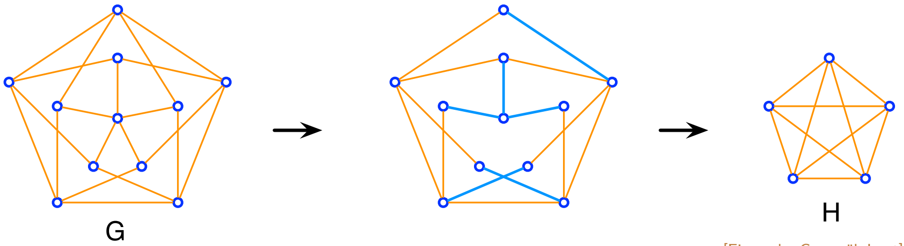

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- Therefore: $\boxed{H \text{ topological minor of } G \not\Leftarrow H \text{ minor of } G}$

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Let $\mathcal{F}$ be a fixed finite collection of graphs.

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  Solvable in time $2^{\Theta(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$. [Jansen, Lokshtanov, Saurabh. 2014 + Pilipczuk. 2015]

Covering topological minors

Let $\mathcal{F}$ be a fixed finite collection of graphs.

$\mathcal{F}$-M-DELETION
Input: A graph G and an integer k.
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FPT by Courcelle's Theorem.

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

$\boxed{\text{Objective}}$

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-Deletion/$\mathcal{F}$-TM-Deletion can be solved in time

$$f_{\mathcal{F}}(\mathsf{tw}) \cdot n^{\mathcal{O}(1)}$$

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- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

Summary of our results: arXiv 1704.07284+1907.04442

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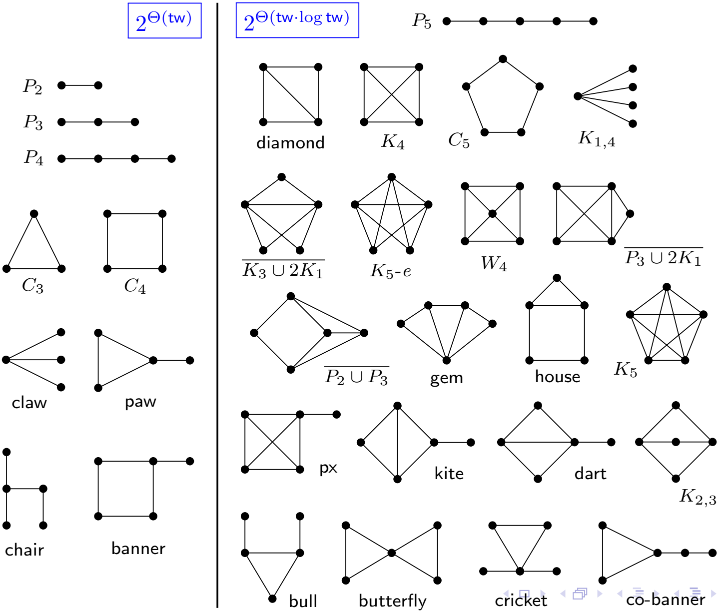
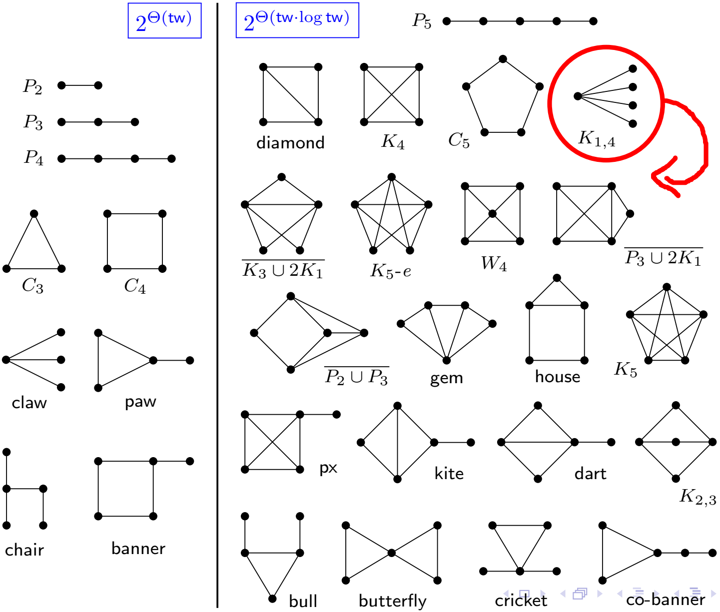
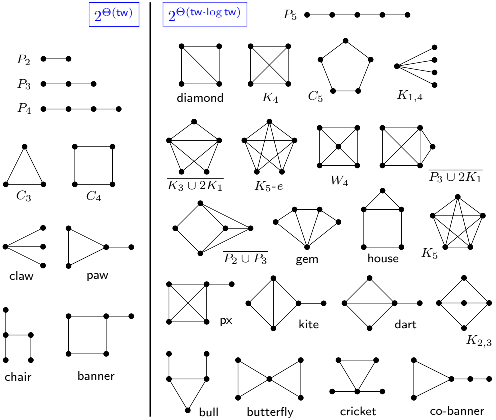
Complexity of hitting a single connected minor H



For topological minors, there is (at least) one change



A compact statement for a single connected graph



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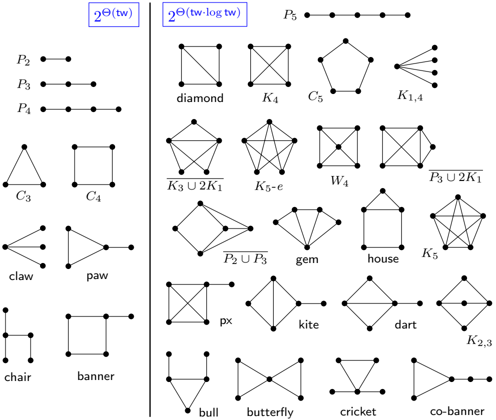

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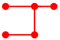
- All graphs on the left are contractions of  or 

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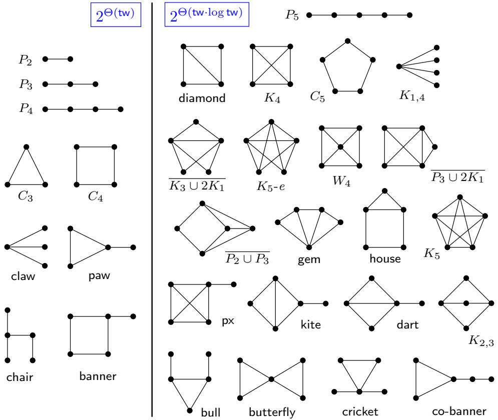

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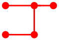
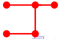
- All graphs on the left are contractions of  or 
- All graphs on the right are not contractions of  or 

A dichotomy for hitting connected minors

We can prove that any connected H with $|V(H)| \geq 6$ is $\boxed{\text{hard}}$:
$\{H\}$-M-Deletion cannot be solved in time $2^{o(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ under the ETH.

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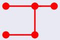
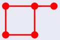
- $2^{\mathcal{O}(\mathsf{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathsf{c}}$  *or* $H \preceq_{\mathsf{c}}$ .

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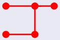
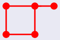
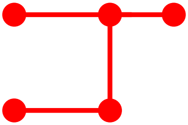
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_c$  *or* $H \preceq_c$ .
- $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

In both cases, the running time is asymptotically optimal under the ETH.

skip

Why the chair and the banner??



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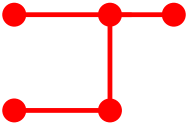

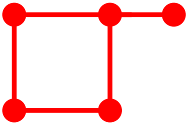

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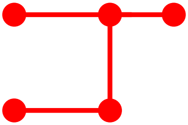

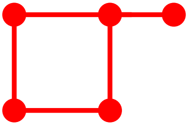

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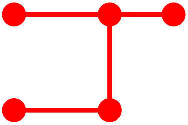

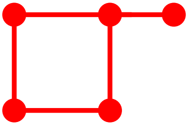

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- Both such types of components can be maintained by a dynamic programming algorithm in single-exponential time.
- If the characterization of the allowed connected components is enriched in some way, such as restricting the length of the allowed cycles or forbidding certain degrees, the problem becomes harder.

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3. Lower bounds under the ETH
   - $2^{o(\mathrm{tw})}$ is "easy".
   - $2^{o(\mathrm{tw}\cdot\log\mathrm{tw})}$ is much more involved and we get ideas from:

     [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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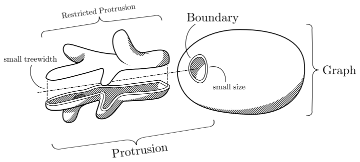

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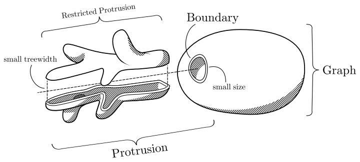

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  Extra: Bidimensionality, irrelevant vertices, protrusion decompositions...

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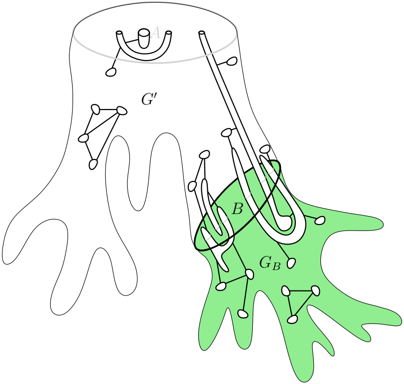

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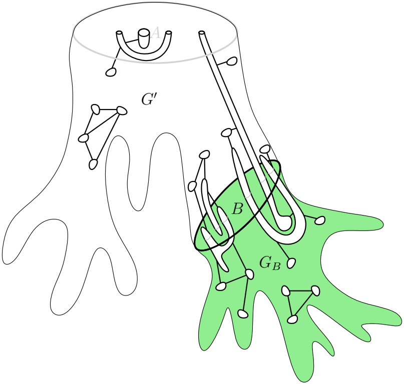

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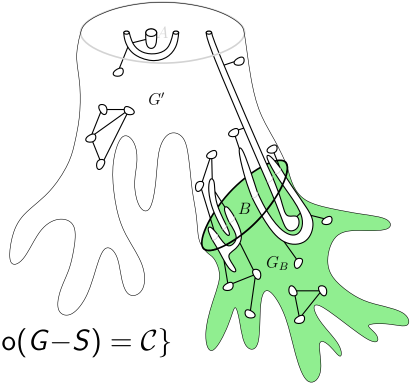
- We compute, using DP over a tree decomposition of G, the following parameter for every folio $\mathcal{C}$:

$$\mathbf{p}(G,\mathcal{C}) = \min\{|S| \ : \ S \subseteq V(G) \wedge \text{folio}(G - S) = \mathcal{C}\}$$



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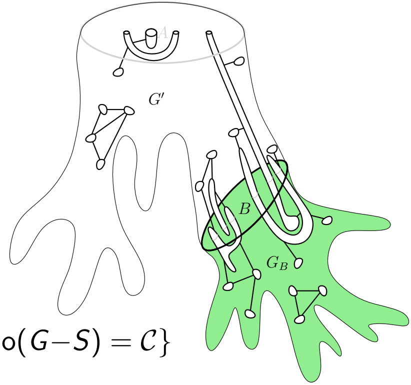

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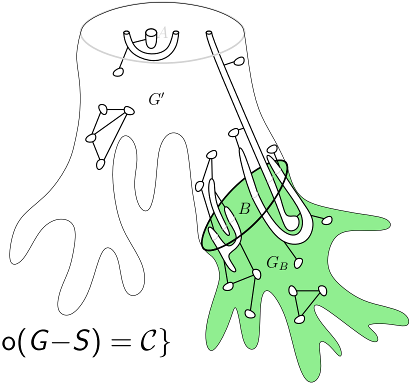

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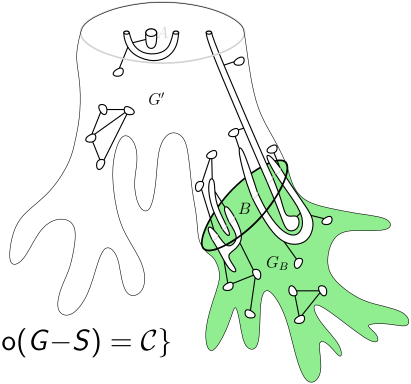

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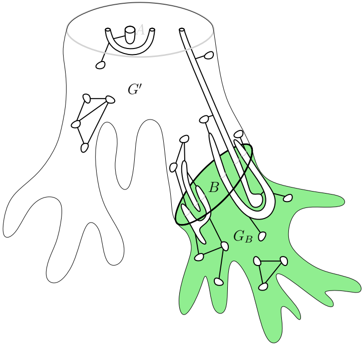

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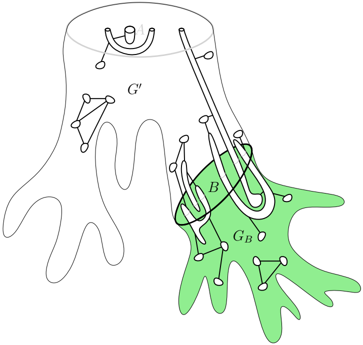

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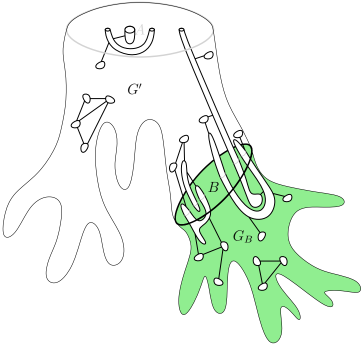

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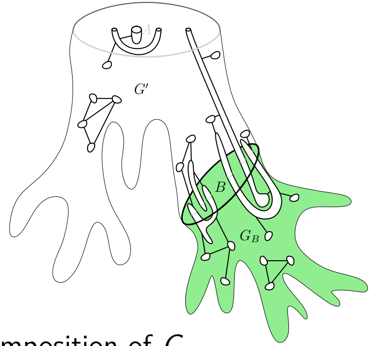

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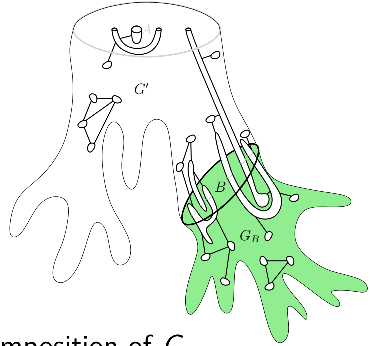

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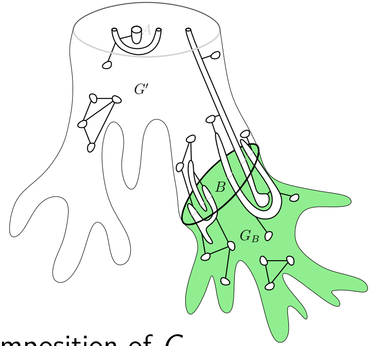

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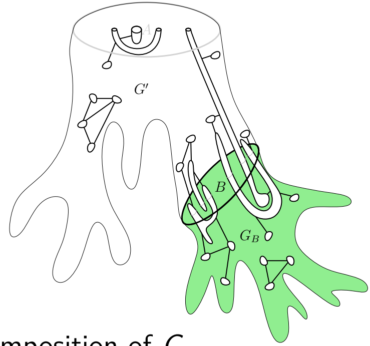

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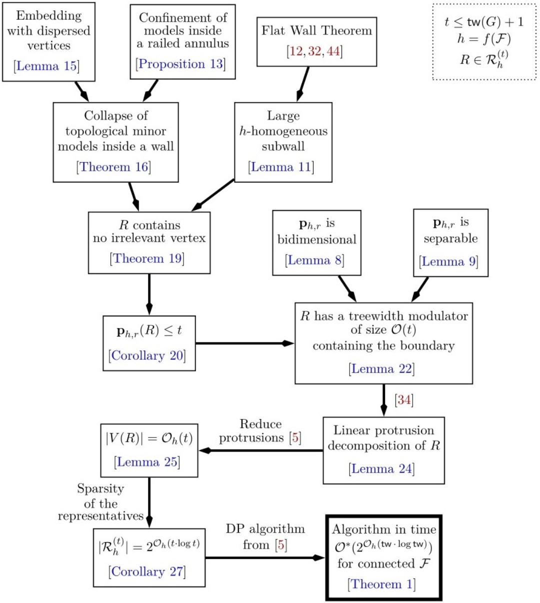

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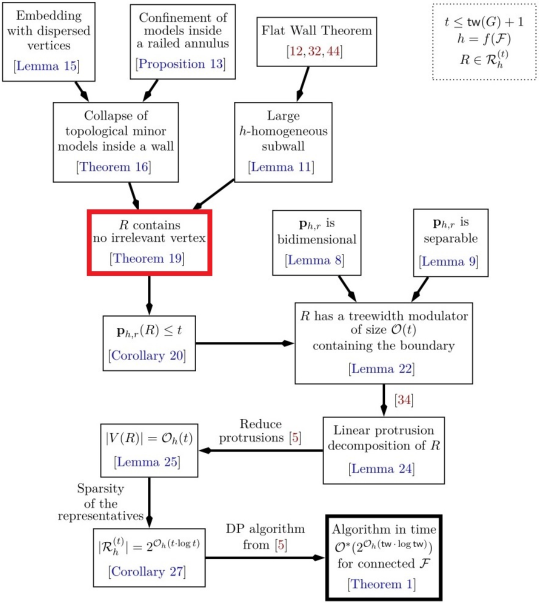

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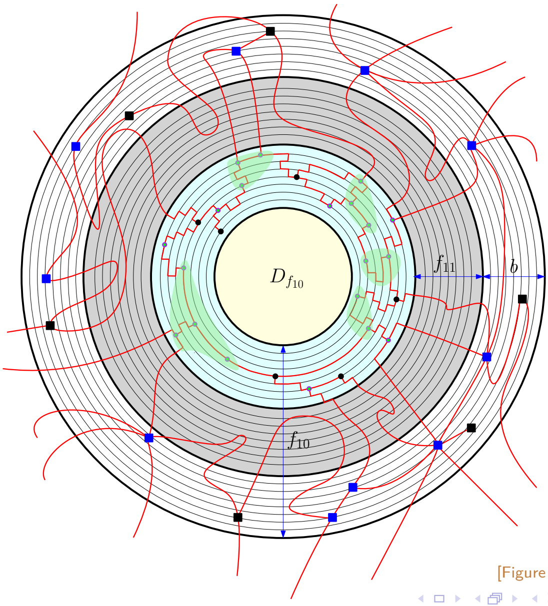

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[Figure by Dimitrios M. Thilikos]

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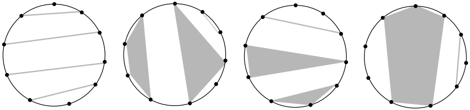

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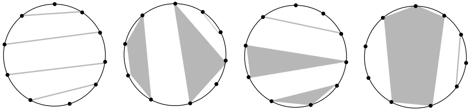

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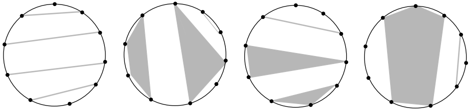

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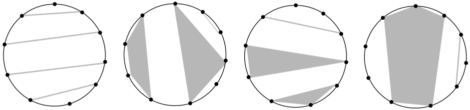

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. skip [Rué, S., Thilikos. 2014]

Next section is...

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Gràcies!