Enumeration kernels for Vertex Cover and Feedback Vertex Set

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Enumeration

Enumeration problem

List the set $\text{Sol}(x)$ of all solutions associated with the instance x that satisfy your problem's constraints.

An example - Vertex Cover

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Input: A graph G and an integer k.
Question: Does G have a vertex cover of size at most k?

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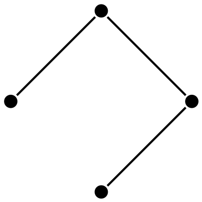

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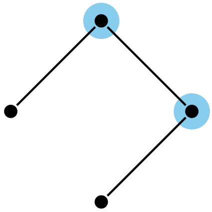

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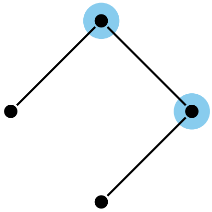

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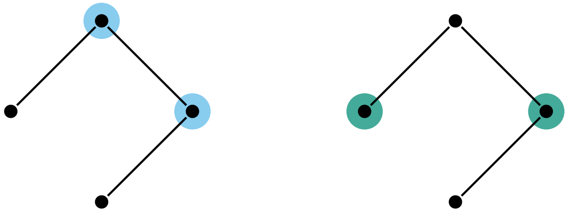

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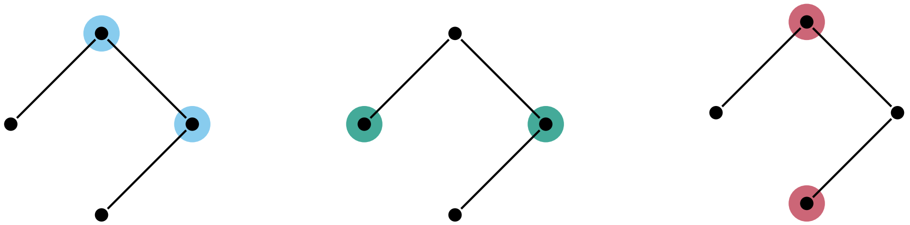

More on enumeration

Applications

Bioinformatics, robotics, optimization, ...

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- Polynomial-delay: time between consecutive outputs in $\text{poly}(|x|)$.

Parameterized complexity for decision

Decision problems & FPT

Each instance x of problem is given with a parameter k, and Π is said to be fixed-parameter tractable if it can be solved in $f(k) \cdot |x|^{\mathcal{O}(1)}$-time.

Preprocessing as kernelization

A kernelization algorithm takes (x, k) as input, runs in polynomial time, and outputs an equivalent instance (y, ℓ) with $|y|, \ell \leq g(k)$.

Theorem

A parameterized problem admits an FPT algorithm $\Leftrightarrow$ it admits a kernel. It is in P $\Leftrightarrow$ $g(k) \in \mathcal{O}(1)$.

Goal of kernelization: minimize $g(k)$.

Parameterized enumeration

FPT-delay

If Π is a parameterized enumeration problem, then FPT-delay is commonly accepted as the "right" notion of tractability:

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Kernelization

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A k^2 enum-kernel for Enum Vertex Cover

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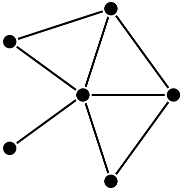

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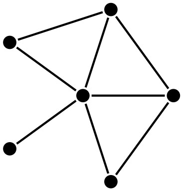

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Rule 1

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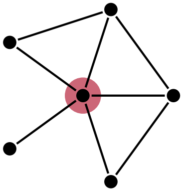

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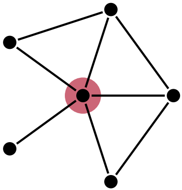

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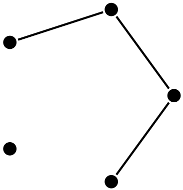

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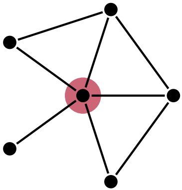

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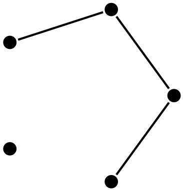

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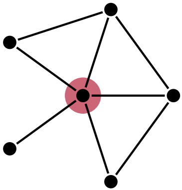

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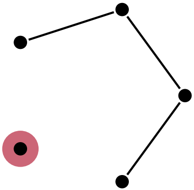

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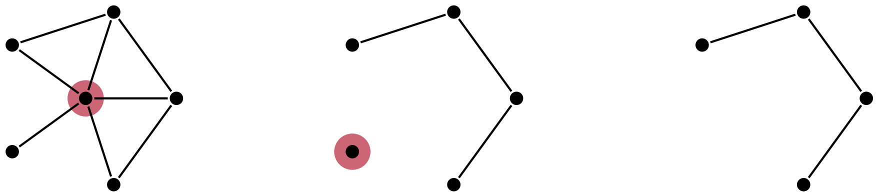

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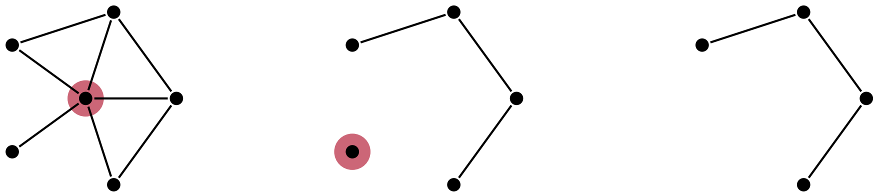

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A bounding criterion

No applicable rule $\rightarrow$ max degree k. $|E(G)| > k^2 \rightarrow$ NO-instance.

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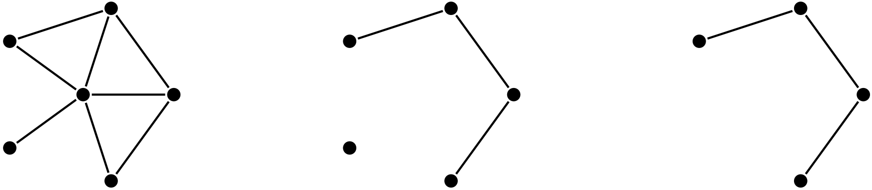

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Lifting

Take $Y \in \mathrm{Sol}(G', k')$; we never remove vertices from it.

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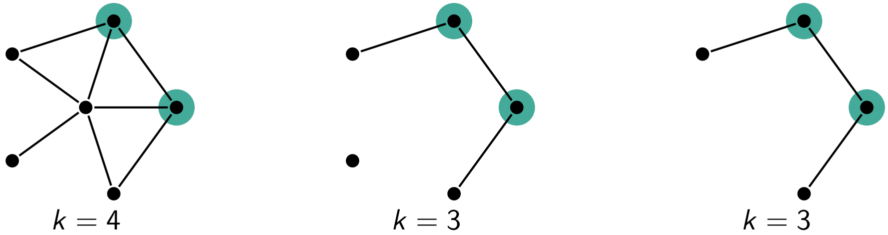

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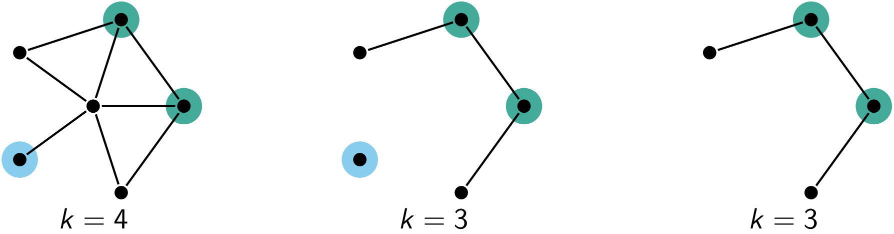

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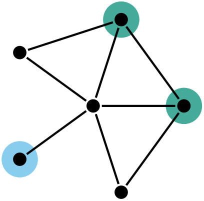

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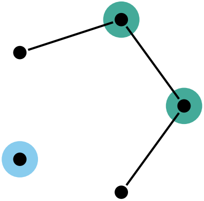

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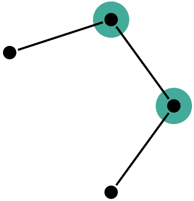

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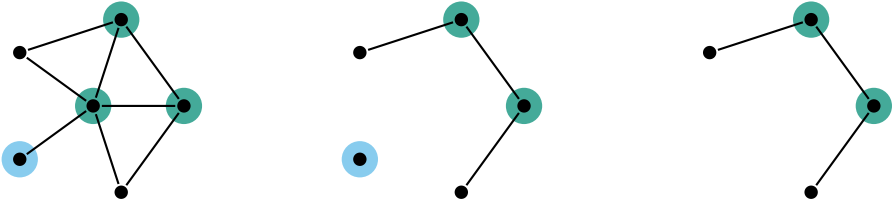

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A new model for enumeration kernels needed to be introduced.

Polynomial-delay (PD) kernels

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Theorem (Golovach, Komusiewicz, Kratsch, Le. 2022)

Problem Π admits a PD kernel $\Leftrightarrow$ it admits an FPT-delay algorithm.
Moreover, $g(k) \in \mathcal{O}(1) \Leftrightarrow \Pi$ is solvable with polynomial-delay.

Known polynomial PD kernels

Problem	Parameter	Kernel size

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Kernels found by [Gomes, Juliano, Martins, dos Santos. IPEC 2024]

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These were all the known PD kernels.

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Theorem

ENUM FEEDBACK VERTEX SET *admits a PD kernel with* $\mathcal{O}(k^3)$ *vertices when parameterized by the solution size.*

Sketch of the linear kernel for Enum Vertex Cover

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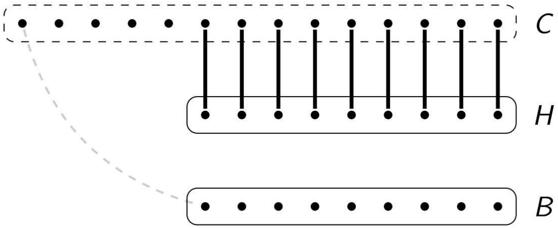
Crown decomposition of a graph G

A partition (C, H, B) of $V(G)$ such that:

- C is an independent set.
- H separates C and B.
- there is an H-saturating matching between H and C;



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Theorem (Nemhauser and Trotter, 1975 + Chlebík and Chlebíková, 2008)

Let G be a graph without isolated vertices and at least $2k + 1$ vertices. Then, there is a polynomial-time algorithm that either:

- *Decides that no vertex cover of size at most k exists.*
- *Or finds a crown decomposition of G.*

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Let (G, k) be the input instance. If v is an isolated vertex of G:

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Rule 2

If G has a crown decomposition (C, H, B):

- Remove C, H from G.
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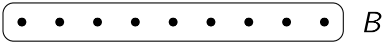
The resulting graph has at most $2k$ vertices.

The lifting algorithm is the hard part!



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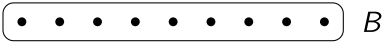

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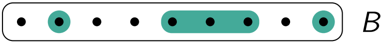

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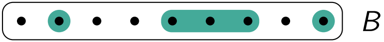

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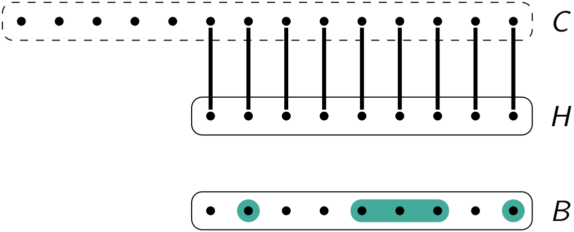

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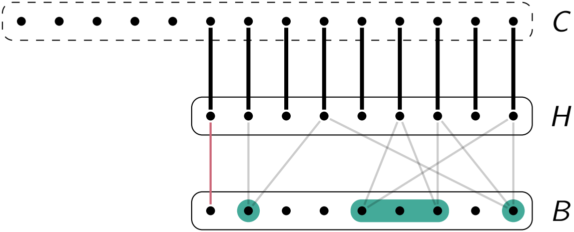

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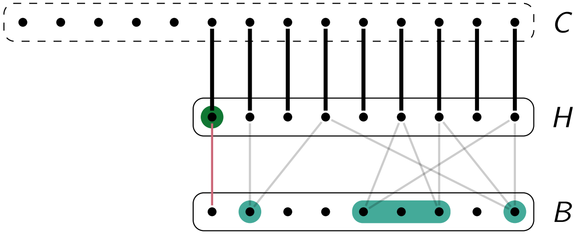

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A linear kernel for Enum Vertex Cover

Task

Enumerate all vertex covers of $G[H \cup C \setminus Y]$ of size $\leq k - |Y| = |H| + s$.

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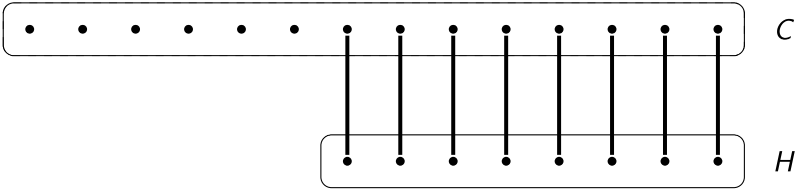

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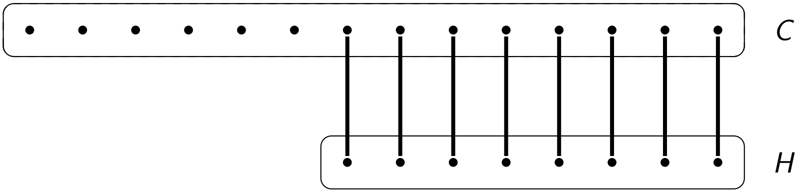

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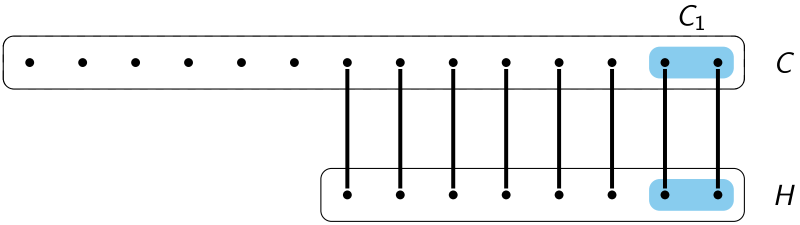

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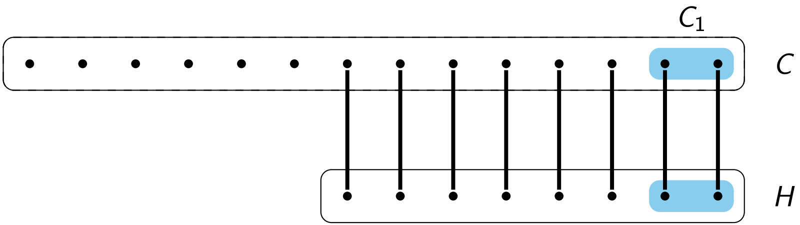

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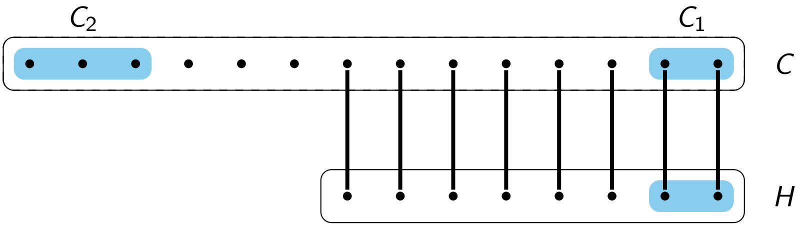

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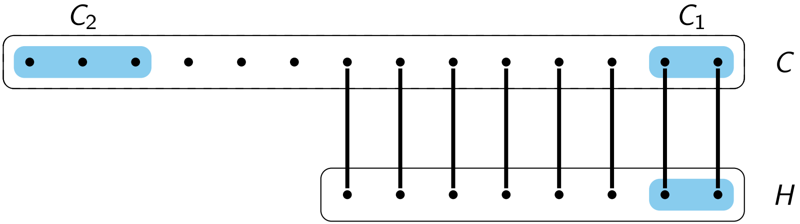

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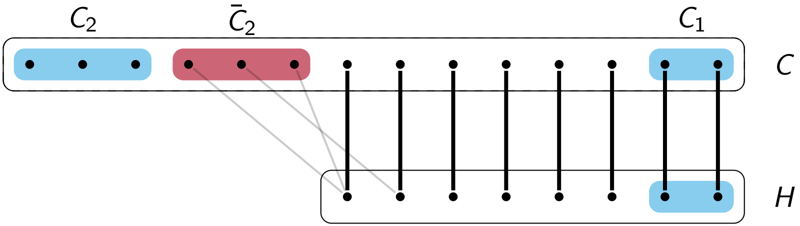

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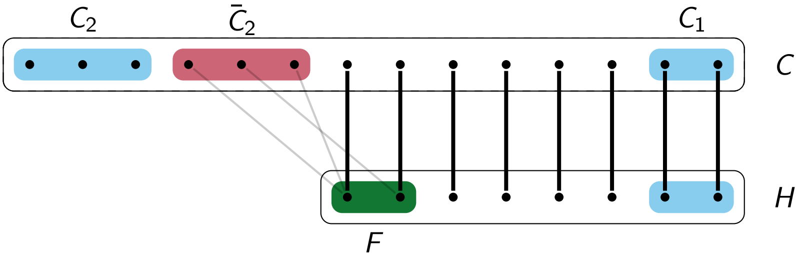

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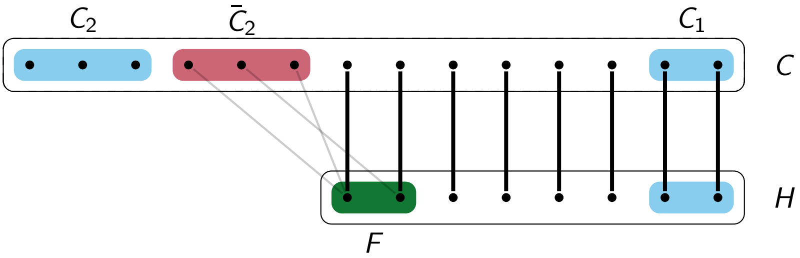

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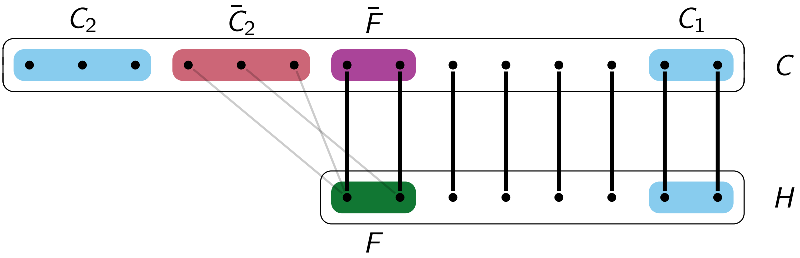

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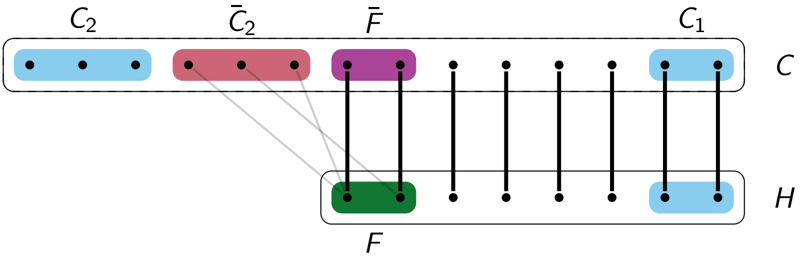

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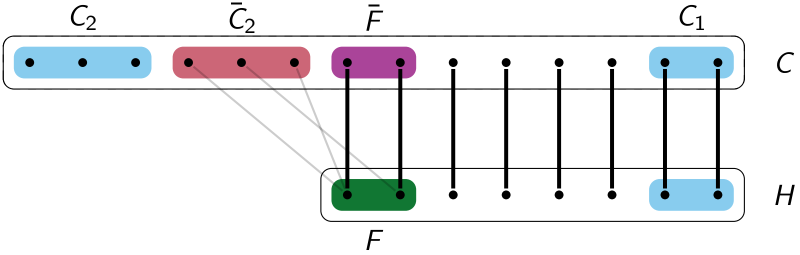

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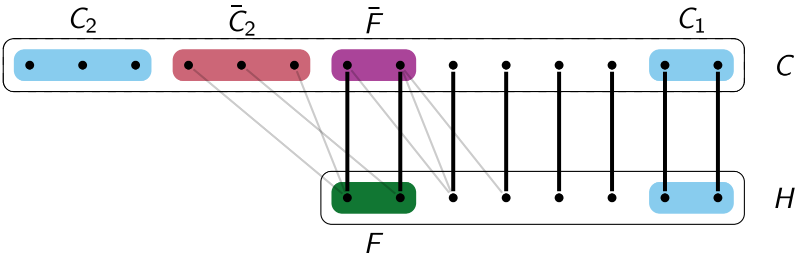

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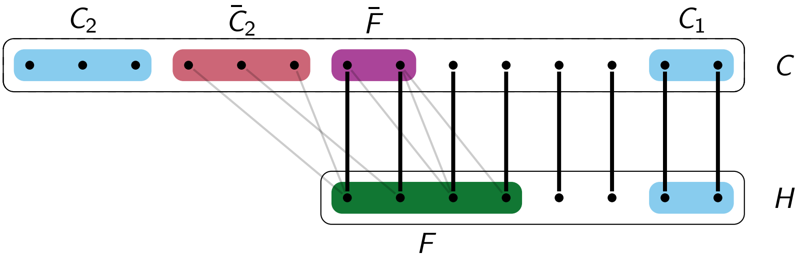

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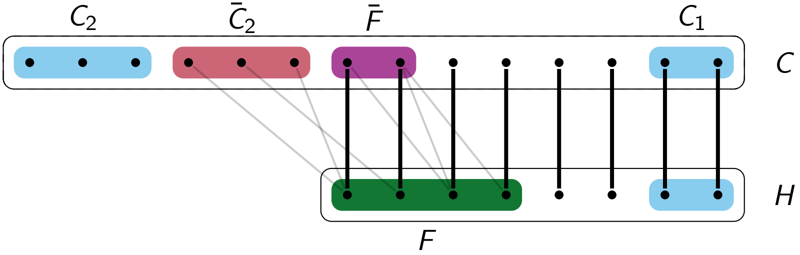

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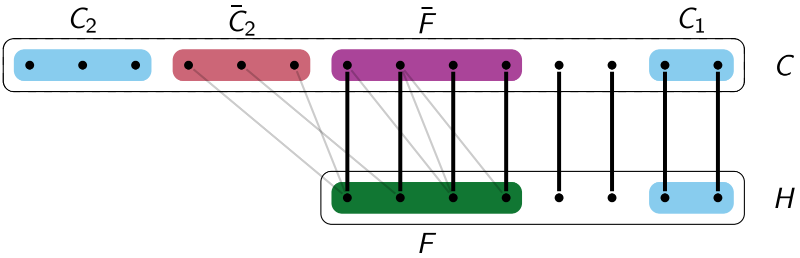

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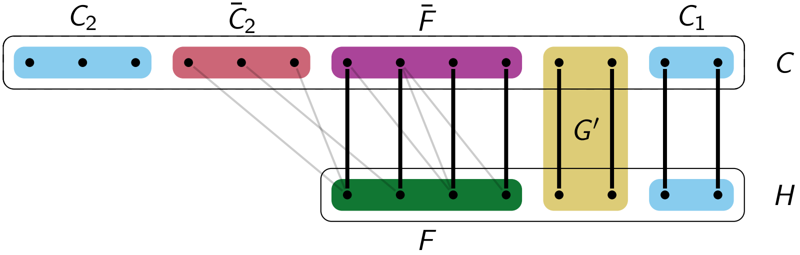

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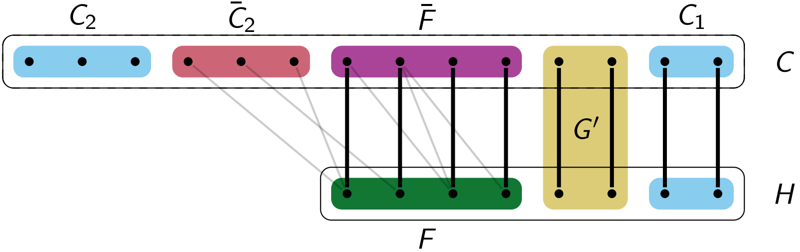

What remains

Enumerate vertex covers of G' of size $|V(G')|/2$.

A linear kernel for Enum Vertex Cover

After a long branching algorithm on G' and even longer correctness argument...

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Enum Vertex Cover *admits a PD kernel with at most* $2k$ *vertices when parameterized by the solution size.*

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Observation

Requiring that every solution of the compressed instance outputs some solution of (G, k) significantly complicates the lifting procedure.

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- As a result, we obtain a maximum degree of $\mathcal{O}(k^2)$ (instead of $\mathcal{O}(k)$).

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