Hitting (topological) minors on bounded treewidth graphs - part I

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[arXiv 1704.07284]

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Monadic Second Order Logic (MSOL):
Graph logic that allows quantification over sets of vertices and edges.

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Theorem (Courcelle, 1990)

Every problem expressible in MSOL can be solved in time $f(\mathrm{tw}) \cdot n$ on graphs on n vertices and treewidth at most tw.

In parameterized complexity: FPT parameterized by treewidth.

Examples: Vertex Cover, Dominating Set, Hamiltonian Cycle, Clique, Independent Set, k-Coloring for fixed k, ...

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Remark: Algorithms parameterized by treewidth appear very often as a "black box" in all kinds of parameterized algorithms.

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But for the so-called connectivity problems, like Longest Path or Steiner Tree, the "natural" DP algorithms provide only time

$$2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}.$$

Single-exponential algorithms on sparse graphs

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- Planar graphs: [Dorn, Penninkx, Bodlaender, Fomin. 2005]
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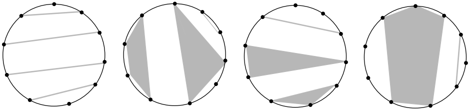
partial solutions $\Longleftrightarrow$ non-crossing partitions



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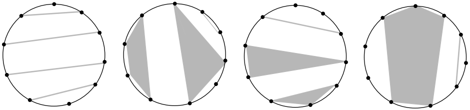

$$\text{CN}(k) = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}} \leq 4^k.$$

The revolution of single-exponential algorithms

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Randomized single-exponential algorithms for connectivity problems.

Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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ETH: The 3-SAT problem on n variables cannot be solved in time $2^{o(n)}$

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There are other examples of such problems...

The $\mathcal{F}$-M-Deletion problem

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FPT by Courcelle, or by Graph Minors theory.

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- $\mathcal{F} = \{H\}$, H planar + connected: complete tight dichotomy.

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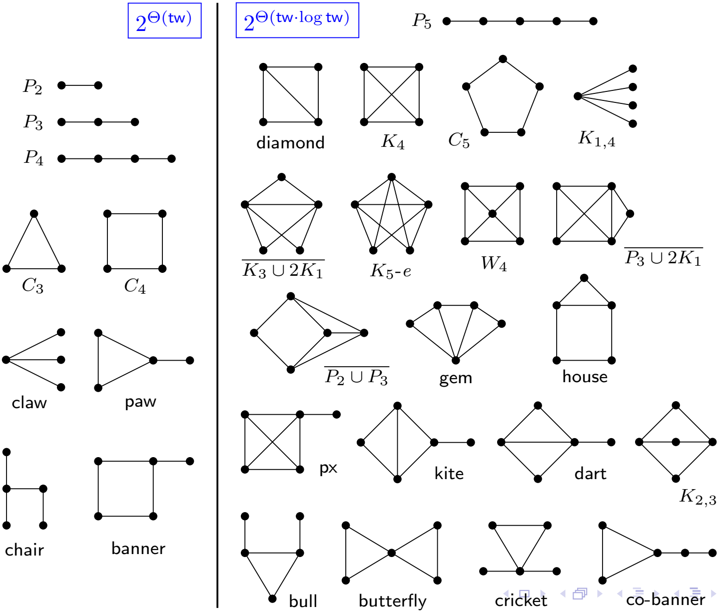
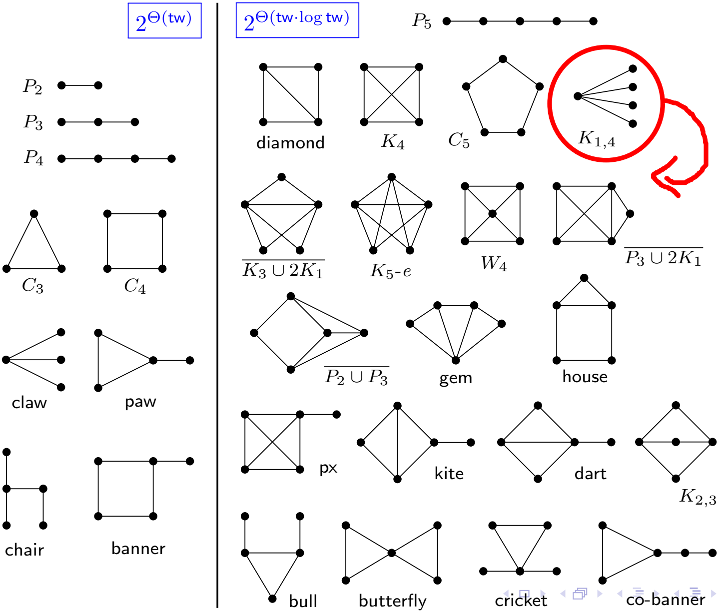
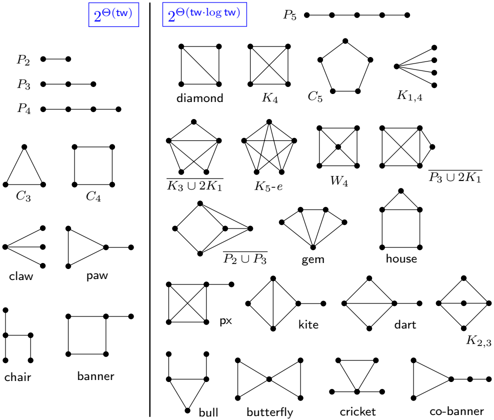
Complexity of hitting small planar minors H



For topological minors, there is only one change



A compact statement for small planar minors



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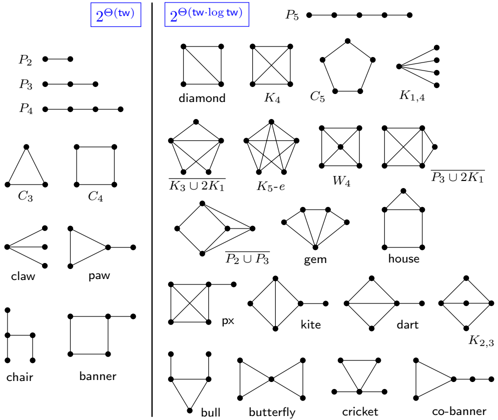

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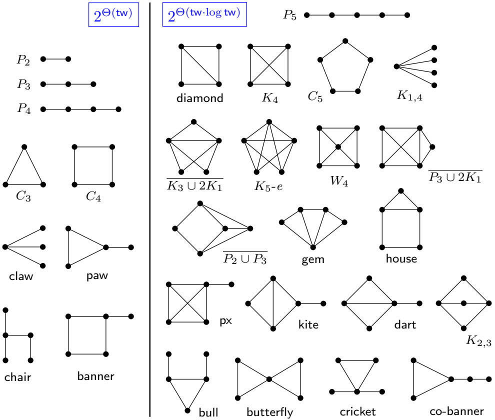

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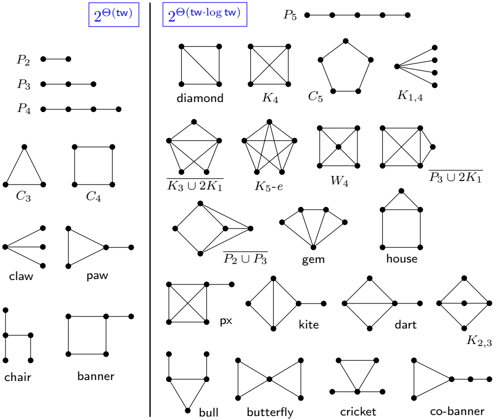

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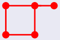
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{m}}$  *and* $H \neq P_5$.

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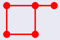
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{m}}$  *and* $H \neq P_5$.
- $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

In both cases, the running time is asymptotically optimal under the ETH.

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We have three types of results

1. General algorithms

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3. Lower bounds under the ETH

   - $2^{o(\text{tw})}$ is "easy".
   - $2^{o(\text{tw} \cdot \log \text{tw})}$ is much more involved and we get ideas from:

     [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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We build on the machinery of boundaried graphs and representatives:

[Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh, Thilikos. 2009]

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[Kim, Langer, Paul, Reidl, Rossmanith, S., Sikdar. 2013]

[Garnero, Paul, S., Thilikos. 2014]

Algorithm for a general collection $\mathcal{F}$

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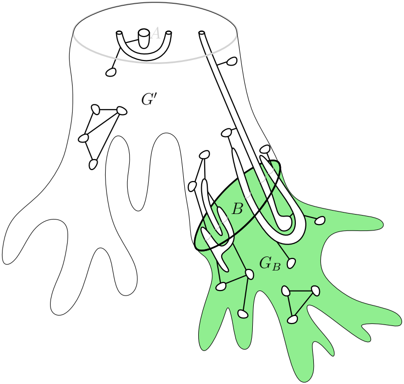

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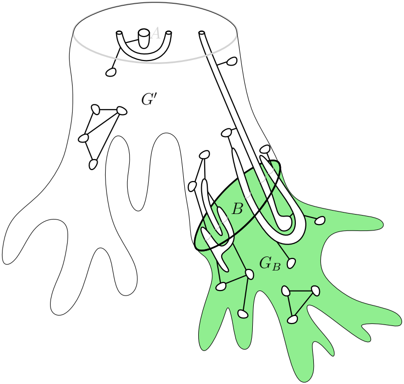

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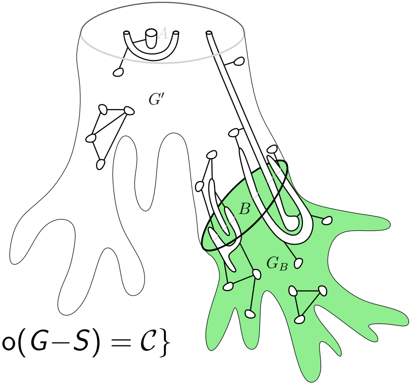

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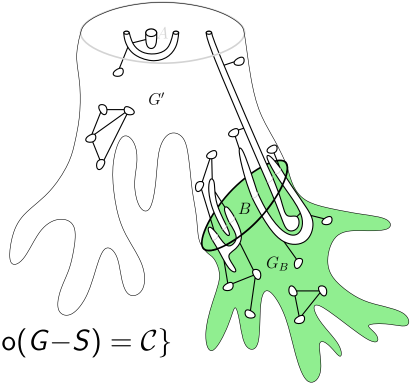

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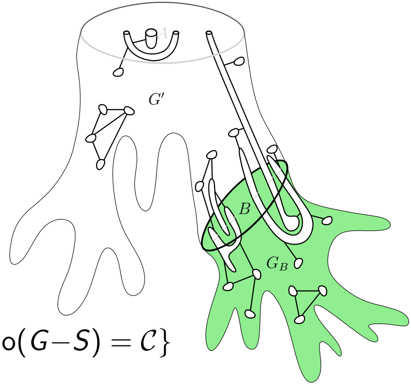

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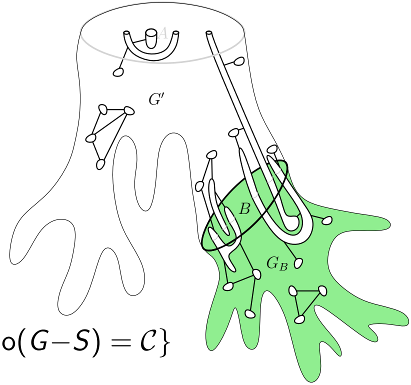

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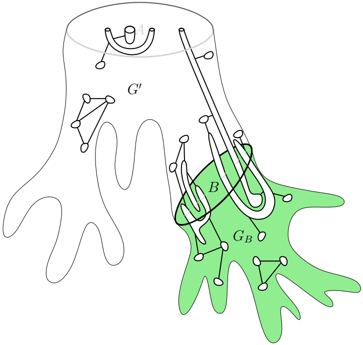

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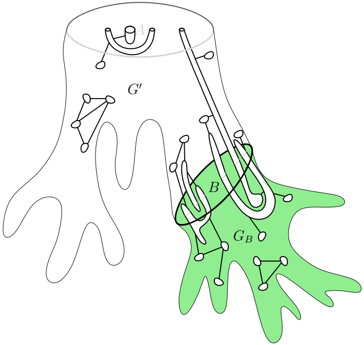

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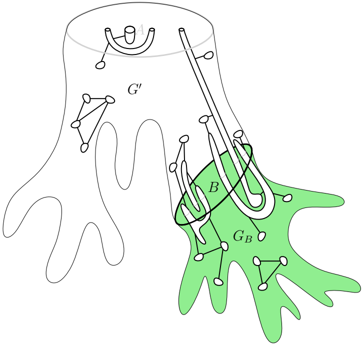

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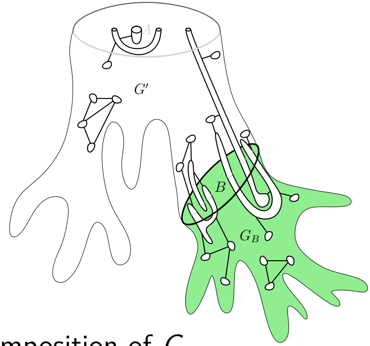

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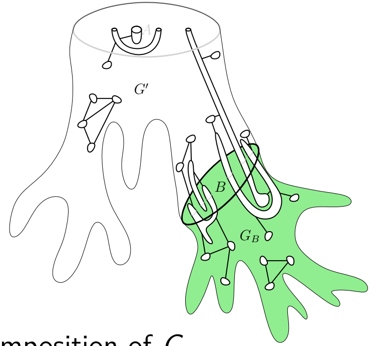

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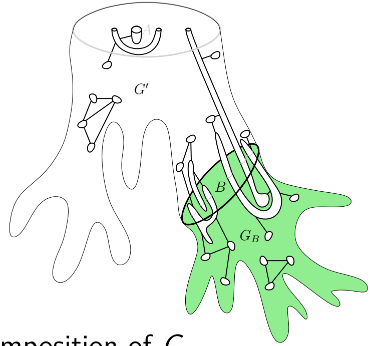

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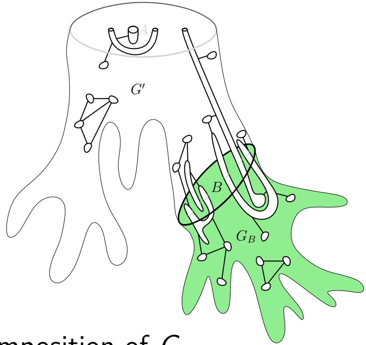

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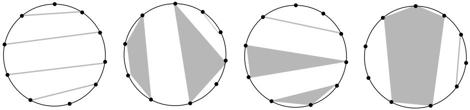

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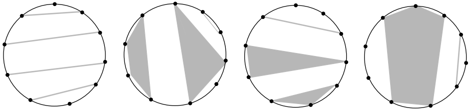

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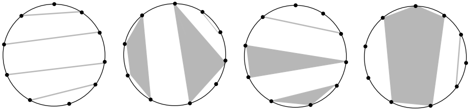

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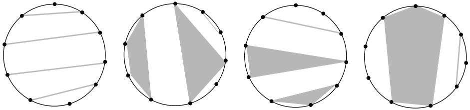

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. [Rué, S., Thilikos. 2014]

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- Consider families $\mathcal{F}$ containing disconnected graphs.

Gràcies!

LLIBERTAT
PRESOS POLÍTICS

FREEDOM FOR ALL CATALAN POLITICAL PRISONERS IN SPAIN