Reducing graph transversals via edge contractions

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Outline of the talk

Next section is...

Graph modification problems

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- $\pi =$ chromatic/independence/clique/matching/domination number
  [Bentz et al. 2010] [Costa et al. 2011] [Bazgan et al. 2011, 2015]
  [Diner et al. 2018] [Paulusma et al. 2019] [Fomin et al. 2020]

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Proposition (Galby, Lima, Ries. 2019)

Let π be a graph parameter such that

(i) *it is* NP-*hard to compute the π-number of a graph and*

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(i) *it is* NP*-hard to compute the π-number of a graph and*

(ii) *contracting an edge reduces π by at most one.*

Unless P=NP, *there exists no polynomial-time algorithm deciding whether contracting one given edge decreases the π-number of a graph.*

Next section is...

1. Introduction
2. Our results
3. Some proofs
4. Further research

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These three parameters satisfy the conditions of the previous Proposition.

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The $\textsc{Contraction}(\mathsf{vc})$ *problem can be solved on* n*-vertex graphs in time* $f(d) \cdot n^{2d}$ *for some computable function* f.

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Next section is...

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- STAR CONTRACTION $\equiv$ CONNECTED VERTEX COVER.
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- CONNECTED VERTEX COVER is NP-hard even if vc is polynomial (bipartite graphs). [Escoffier et al. 2010]

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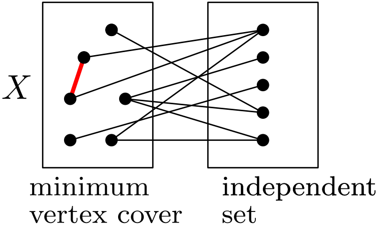

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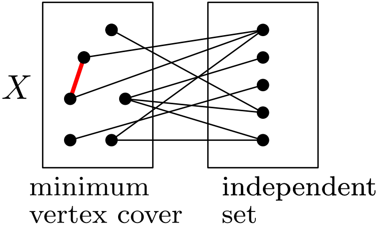

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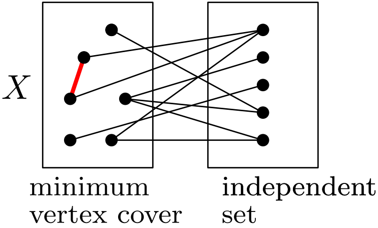

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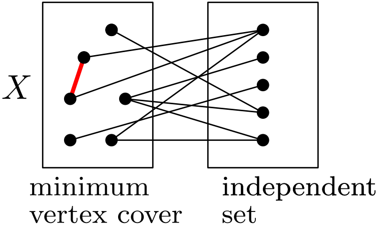

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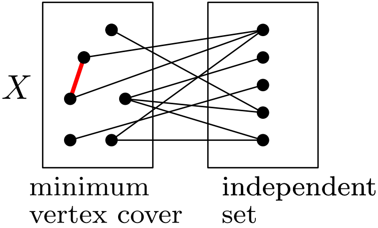

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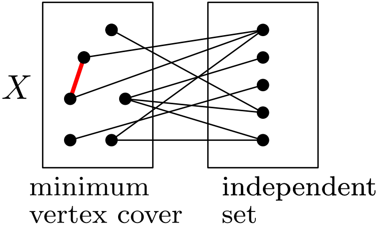

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  - For every edge $e \in E(G)$, we compute $\mathrm{vc}(G/e)$ in polynomial time.

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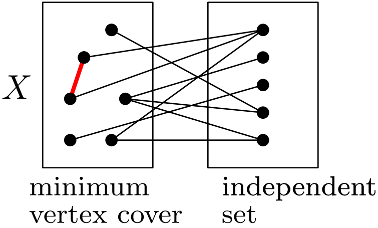

- Given G, consider a minimum vertex cover X of G.
- If G is not bipartite, then $G[X]$ contains some edge $\Rightarrow$ YES-instance!
- Otherwise, G is bipartite:
  - We first compute $\mathrm{vc}(G)$ in polynomial time.
  - For every edge $e \in E(G)$, we compute $\mathrm{vc}(G/e)$ in polynomial time.
  - We check whether $\mathrm{vc}(G/e) < \mathrm{vc}(G)$ for some edge $e \in E(G)$.

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      Then $G[X]$ contains at least d "good" edges

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       We solve the problem in time $f(d) \cdot n^{\mathcal{O}(1)}$ by expressing it with an MSO formula (Courcelle), and then using a simple dynamic programming algorithm to combine the solutions for the connected components.

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     - There exists a connected component C of G such that $\text{vc}(C) \geq d+1$.

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$\boxed{\text{Claim}}$ $\exists\ F \subseteq E(G)$ with $|F| \leq 2d$ such that $\mathrm{vc}(G/F) \leq \mathrm{vc}(G) - d$.

Sufficient: H connected, X minimum vertex cover of H, $|X| \geq 2$:
there exist $u, v \in X$ such that $\mathrm{dist}_H(u, v) \leq 2$.

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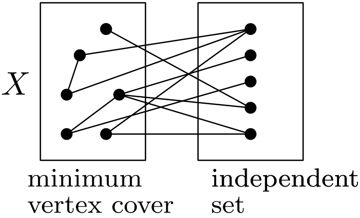

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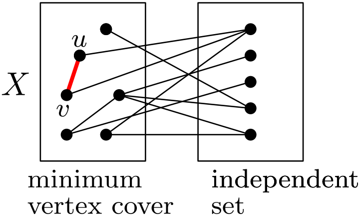

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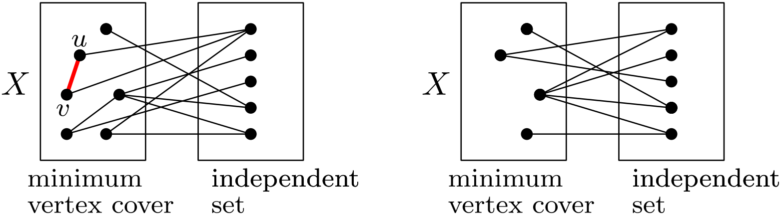

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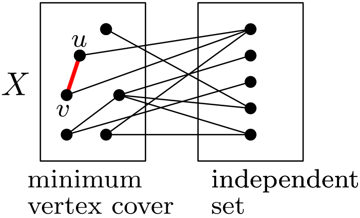

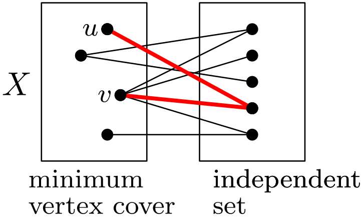

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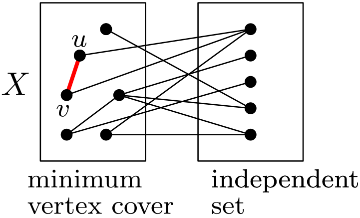

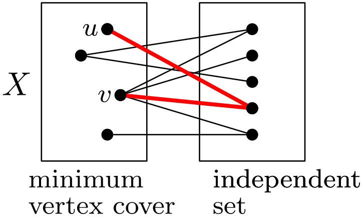

Since $\mathrm{vc}(C) \geq d+1$, iteratively contracting such pairs of vertices $u, v \in X$ gives the desired set $F \subseteq E(G)$ with $|F| \leq 2d$ s.t. $\mathrm{vc}(G/F) \leq \mathrm{vc}(G) - d$.

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      Set $B := V(L) \cup V_F$ (vertices resulting from the contraction of F).
  - Finally, check whether $\mathrm{vc}(G/F) < \mathrm{vc}(G) - d$ for some set $F \subseteq E(G)$.

Next section is...

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

- $\mathcal{H} =$ 2-connected graphs containing at least one non-complete graph, $\prec =$ (induced) subgraph or (topological) minor.
- $\mathcal{H} =$ cliques with at least three vertices, $\prec =$ (topological) minor.
- $\mathcal{H} = \{P_i\}$ with $i \geq 4$, $\prec =$ (induced) subgraph or (topological) minor.

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- $\mathcal{H} = \{K_h\}$ with $h \geq 3$ for $\prec$ = (induced) subgraph.

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- $\mathcal{H} = \{P_3\}$ for any $\prec$.
- $\mathcal{H} = \{T\}$ for a tree T, for any $\prec$.

We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

- $\mathcal{H}$ = 2-connected graphs containing at least one non-complete graph, $\prec$ = (induced) subgraph or (topological) minor.
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Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

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co-NP-hard cases: natural to parameterize $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ by $\tau_{\mathcal{H}}^{\prec}$.

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co-NP-hard cases: natural to parameterize $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ by $\tau_{\mathcal{H}}^{\prec}$.

- If $\prec$ = minor and $\mathcal{H}$ contains a planar graph, FPT param. by $\tau_{\mathcal{H}}^{\prec} + k$.

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We proved that $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ is co-NP-hard for fixed $k = d = 1$ if:

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Polynomial-time solvable for any fixed $d \geq 1$ if $\mathcal{H} = \{K_2\}$, for any $\prec$.

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co-NP-hard cases: natural to parameterize $\textsc{Contraction}(\tau_{\mathcal{H}}^{\prec})$ by $\tau_{\mathcal{H}}^{\prec}$.

- If $\prec =$ minor and $\mathcal{H}$ contains a planar graph, FPT param. by $\tau_{\mathcal{H}}^{\prec} + k$.
- In particular, $\textsc{Contraction}(\text{fvs})$ is FPT param. by $\text{fvs} + k$.
- What about non-planar collections $\mathcal{H}$?

Gràcies!