Introduction + Bidimensionality Theory

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Outline of the talk

1. Introduction, part II
   - Treewidth
   - Dynamic programming on tree decompositions
   - Structure of H-minor-free graphs
   - Some algorithmic issues
   - A few words on other containment relations

2. Bidimensionality
   - Some ingredients
   - An illustrative example
   - Meta-algorithms
   - Further extensions

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Parameterized complexity in one slide

- Idea given an NP-hard problem, fix one parameter of the input to see if the problem gets more "tractable".

  Example: the size of a Vertex Cover.

- Given a (NP-hard) problem with input of size n and a parameter k, a fixed-parameter tractable (FPT) algorithm runs in

$$f(k) \cdot n^{\mathcal{O}(1)}, \quad \text{for some function } f.$$

  Examples: k-Vertex Cover, k-Longest Path.

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- Subexponential FPT algorithm: $2^{o(k)} \cdot n^{O(1)}$

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Tree decompositions and treewidth

A *tree decomposition* of a graph G is a pair $D = (T, \mathcal{X})$ such that T is a tree and $\mathcal{X} = \{X_t \mid t \in V(T)\}$ is a collection of subsets of $V(G)$ such that:

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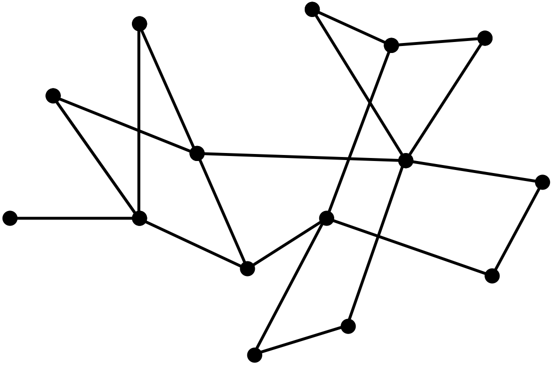
Invariant that measures the topological complexity of a graph.

Tree decompositions: example



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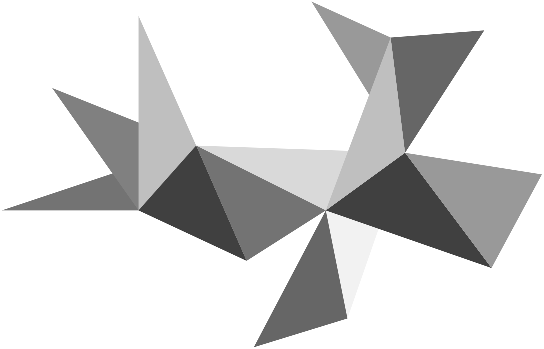

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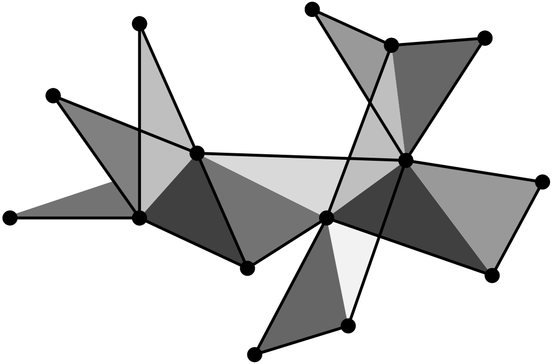

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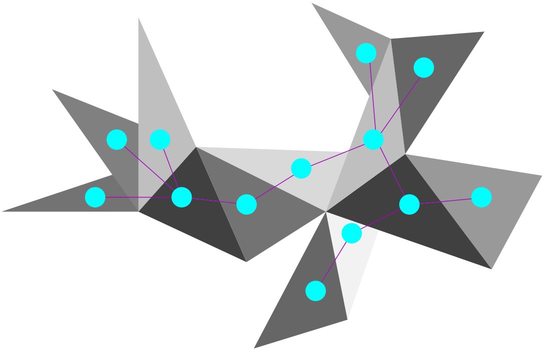

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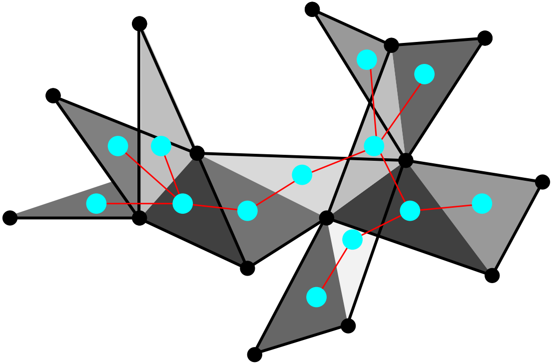

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- The tables (S_i, a_i) at node X_i can be computed from the tables of its children as follows. For each S_i independent set of G_i:
  - If X_j is a child of X_i, an independent set S_j of G_j is feasible for S_i if $X_j \cap S_i = X_i \cap S_j$.
  - For each children X_j of X_i, let S_j be feasible for S_i s.t. (S_j, a_j) is defined and a_j is maximized.
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  - We set $a_i := |S_i| + \sum_j (a_j - |S_i \cap S_j|)$.
- Running time for each node X_i: $O(2^{2|X_i|})$.
- Overall running time: $2^{O(\mathrm{tw}(G))} \cdot n$.

Courcelle's Theorem - algorithmic importance of treewidth

What we have seen with Max. Independent Set can be generalized to a wide family of problems:

Theorem (Courcelle. 1988)

Graph problems expressible in Monadic Second Order Logic (MSOL) can be solved in time $f(k) \cdot n^{O(1)}$ in graphs with n vertices and $\mathrm{tw} \leqslant k$.

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- Running time (tight):

$$2^{2^{2^{\cdots 2^k}}} \cdot n^{O(1)}.$$

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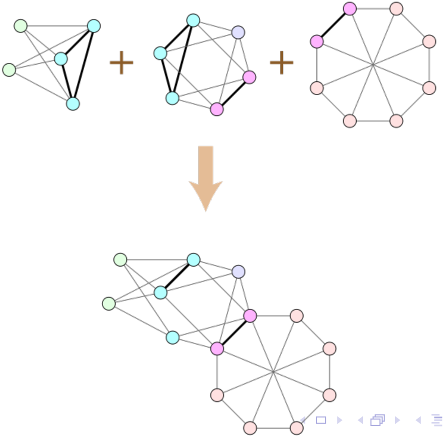
Structure of H-minor-free graphs: roughly "bidimensional"

- Some (simplified) preliminaries:
  - *h*-clique-sum of two graphs G_1 and G_2:
    choose cliques $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ with $|V(K_1)| = |V(K_2)| = h$, identify them, and possibly remove some edges of that clique.



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Proof. By the Graph Minors Theorem, there exists a finite list of minimal excluded minors $\{F_1, \ldots, F_\ell\}$ for the class $\mathcal{G}$. Then we can do minor testing and check whether $F_i \preceq_m G$ in time $O(n^2)$ for each $1 \leqslant i \leqslant \ell$.

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- But... what about the constant $h = f(|V(H)|)$??

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  > Fix a graph H. There exists a constant $h = f(|V(H)|)$ such that any H-minor-free graph G can be decomposed (in a tree-like way) into h-clique-sums from h-almost-embeddable graphs.

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★ Very recently, this constant has been made explicit and "reasonable"!

[Geelen, Huynh, and Richter. 2013]

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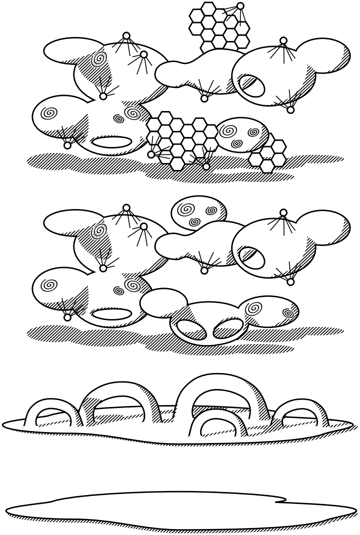
Structure of sparse graphs - a nice picture by Felix Reidl



H-topological-minor-free — treewidth-bounding

$\cup$

H-minor-free — bidimensional, separation property

$\cup$

bounded genus — quasi-compact

$\cup$

planar — “distance-property”

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A few representative problems

VERTEX COVER
Input: A graph $G = (V, E)$ and a positive integers k.
Parameter: k.
Question: Does there exist a subset $C \subseteq V$ of size at most k such that $G[V \setminus C]$ is an independent set?

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LONG PATH
Input: A graph $G = (V, E)$ and a positive integers k.
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Question: Does there exist a path P in G of length at least k?

A few representative problems (II)

FEEDBACK VERTEX SET
Input: A graph $G = (V, E)$ and a positive integers k.
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Input: A graph $G = (V, E)$ and a positive integers k.
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Minor closed parameters

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- We say that a parameter P is *closed under taking of minors/contractions* (or, briefly, *minor/contraction closed*) if for every graph H, $H \preceq_m G$ / $H \preceq_{cm} G$ implies that $P(H) \leqslant P(G)$.

Examples of minor/contraction closed parameters

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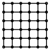
Grid Exclusion Theorem

- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_{\ell}$

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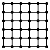
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  ℓ We have $\mathrm{tw}\left(H_{\ell,\ell}\right) = \ell$.

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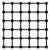
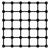
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geqslant \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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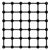
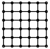
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\text{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor closed, if $_\ell \preceq_m G$, then $\text{tw}(G) \geqslant \text{tw}(H_{\ell,\ell}) = \ell$.
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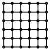
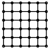
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\text{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor closed, if $_\ell \preceq_m G$, then $\text{tw}(G) \geqslant \text{tw}(H_{\ell,\ell}) = \ell$.
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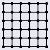
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geqslant c(\ell)$ contains $_\ell$ *as a minor.*

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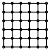
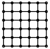
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\text{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor closed, if  $_\ell \preceq_m G$, then $\text{tw}(G) \geqslant \text{tw}(H_{\ell,\ell}) = \ell$.
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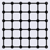
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geqslant c(\ell)$ contains  $_\ell$ as a minor.

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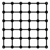
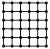
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor closed, if  $_\ell \preceq_m G$, then
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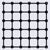
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geqslant c(\ell)$ contains  $_\ell$ *as a minor.*

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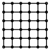
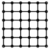
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor closed, if  $_\ell \preceq_m G$, then
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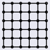
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geqslant c(\ell)$ contains  $_\ell$ *as a minor.*

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- Some improvement: $c(\ell) = 2^{O(\ell \log \ell)}$. [Leaf and Seymour. 2012]

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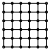
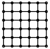
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geqslant \mathrm{tw}(H_{\ell,\ell}) = \ell$.
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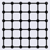
Theorem (Robertson and Seymour. 1986)

For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geqslant c(\ell)$ contains  $_\ell$ *as a minor.*

- Smallest possible function $c(\ell)$? $\Omega(\ell^2 \log \ell) \leqslant c(\ell) \leqslant 20^{2\ell^5}$
- Some improvement: $c(\ell) = 2^{O(\ell \log \ell)}$. [Leaf and Seymour. 2012]
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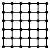
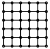
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As Treewidth is minor closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geqslant \mathrm{tw}(H_{\ell,\ell}) = \ell$.
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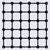
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geqslant c(\ell)$ contains  $_\ell$ *as a minor.*

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Important message grid-minors are the certificate of large treewidth.

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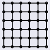
Every planar graph of treewidth $\geqslant 6 \cdot \ell$ *contains*  $_\ell$ *as a minor.*

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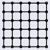
Every planar graph of treewidth $\geqslant 6 \cdot \ell$ contains  $_\ell$ *as a minor.*

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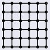
For every fixed g, there is a constant c_g such that every graph of genus g and of treewidth $\geqslant c_g \cdot \ell$ contains  $_\ell$ *as a minor.*

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Every planar graph of treewidth $\geqslant 6 \cdot \ell$ contains $\boxplus_\ell$ as a minor.

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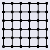
Every planar graph of treewidth $\geqslant 6 \cdot \ell$ *contains*  $_\ell$ *as a minor.*

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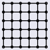
For every fixed g*, there is a constant* c_g *such that every graph of genus* g *and of treewidth* $\geqslant c_g \cdot \ell$ *contains*  $_\ell$ *as a minor.*

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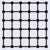
For every fixed graph H*, there is a constant* c_H *such that every* H*-minor-free graph of treewidth* $\geqslant c_H \cdot \ell$ *contains*  $_\ell$ *as a minor.*

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In sparse graphs: linear dependency between treewidth and grid-minors

How to use Grid Theorems algorithmically?

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   - Structure of H-minor-free graphs
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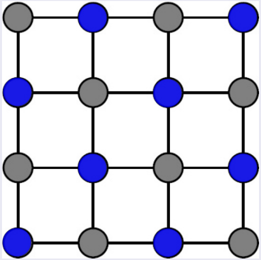
Example: FPT algorithm for Planar Vertex Cover

A vertex cover C of a graph G, $\mathbf{vc}(G)$, is a set of vertices such that every edge of G has at least one endpoint in C.



Example: FPT algorithm for Planar Vertex Cover

INPUT: Planar graph G on n vertices, and an integer k.

OUTPUT: Either a vertex cover of G of size $\leqslant k$, or a proof that G has no such a vertex cover.

RUNNING TIME: $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Objective: subexponential FPT algorithm for PLANAR VERTEX COVER.

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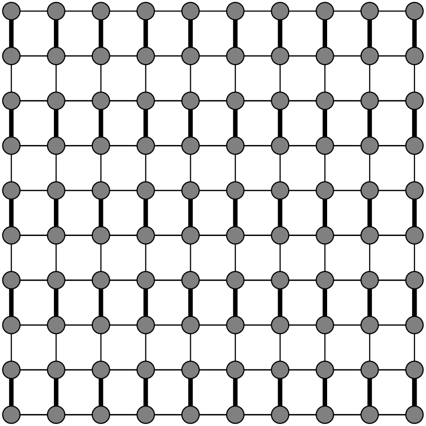

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- The size of any vertex cover of $H_{\ell,\ell}$ is at least $\ell^2/2$.
- Recall that VERTEX COVER is a minor closed parameter.
- Since $H_{\ell,\ell} \preceq_m G$, it holds that $\mathbf{vc}(G) \geqslant \mathbf{vc}(H_{\ell,\ell}) \geqslant \ell^2/2$.

We are already very close to an algorithm...

Recall:

- k is the parameter of the problem.
- We have that $\mathrm{tw}(G) = 6 \cdot \ell$ and ℓ is the size of a grid-minor of G.
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This gives a subexponential FPT algorithm!

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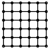
minor-closed $+$ **vc**( $_k$) $= \Omega(k^2)$.

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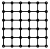
minor-closed $+$ **vc**( $_k$) $= \Omega(k^2)$.

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★ Only the linear Grid Theorem!

Arguments go through up to H-minor-free graphs.

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Minor Bidimensionality:

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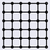
2. **p** ( $_k$) $= \Omega(k^2)$.

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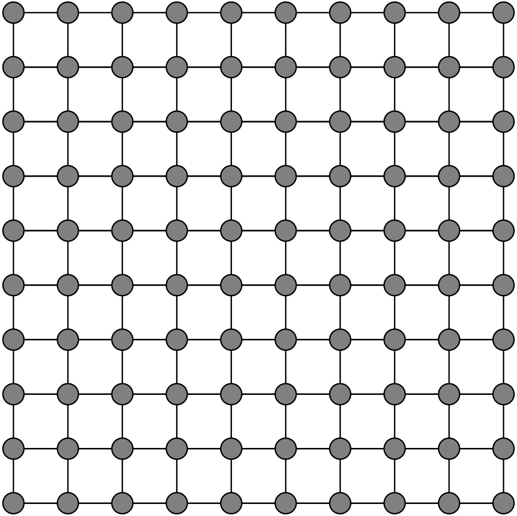

$H_{r,r}$ for $r = 10$

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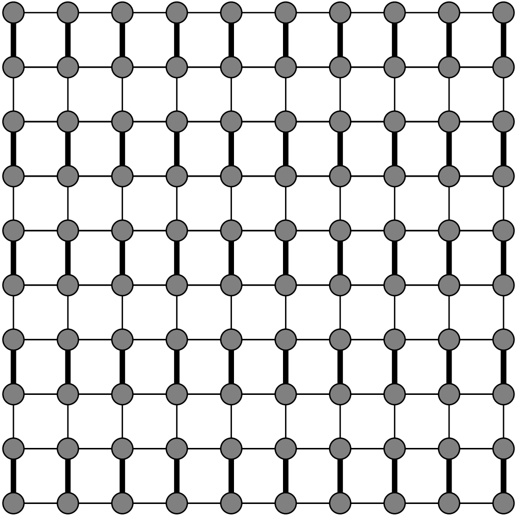

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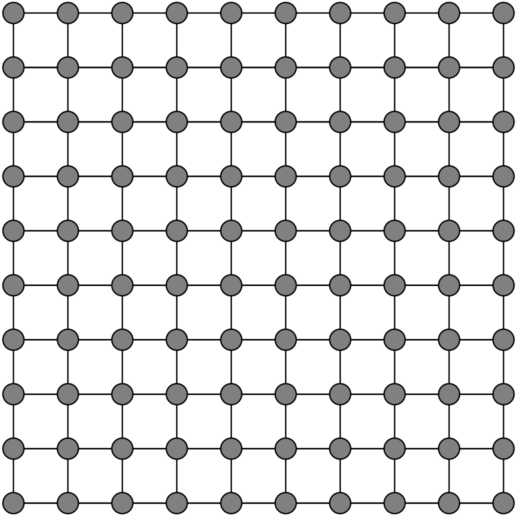

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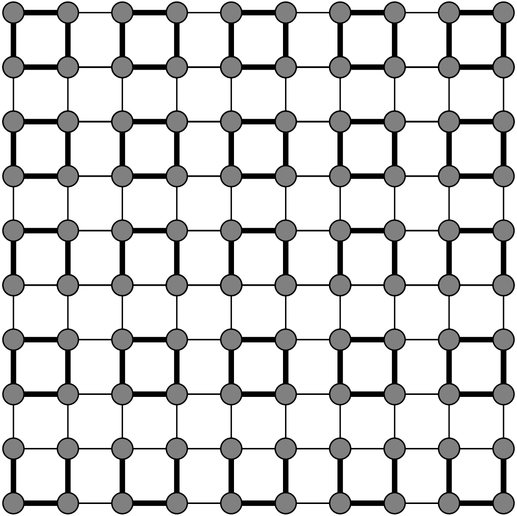

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- If we have a DP algorithm for bounded treewidth running in time c^t or t^t, then it implies $2^{O(\sqrt{k})}$ or $2^{O(\sqrt{k}\log k)}$ algorithm.

Piecing everything together

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Let G be an H-minor-free graph, and let $\mathbf{p}$ be a minor bidimensional graph parameter computable in time $2^{O(\mathbf{tw}(G))} \cdot n^{O(1)}$.
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Minor Bidimensionality provides a meta-algorithm

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- What about contraction closed parameters??

  DOMINATING SET, CONNECTED VERTEX COVER, r-DOMINATING SET, ...

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What is a $(k \times k)$-grid-like graph...?

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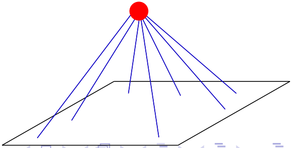
- ★ For apex-minor-free graphs, this is a ($k \times k$)-augmented grid, i.e., partially triangulated grid augmented with additional edges such that each vertex is incident to $O(1)$ edges to non-boundary vertices of the grid.

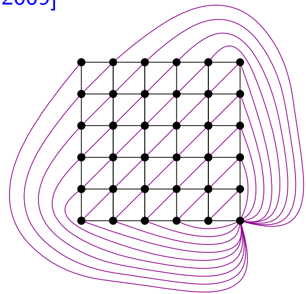
  [Demaine, Fomin, Hajiaghayi, Thilikos. 2005]

H is an *apex graph* if $\exists v \in V(H)$: $H - v$ is planar



Contraction bidimensionality: new definition

Finally, the right "($k \times k$)-grid-like graph" was found:
[Fomin, Golovach, Thilikos. 2009]



k

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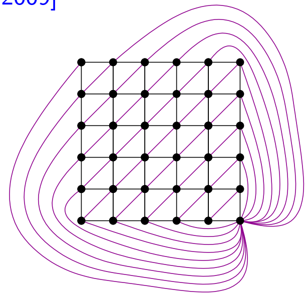
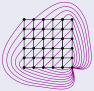


k

Definition

A parameter **p** is *contraction bidimensional* if the following hold:

1. **p** is contraction closed, and
2. **p**( $_k$) $= \Omega(k^2)$.

Meta-algorithms for contraction bidimensional parameters

Theorem

Let H be a fixed apex *graph, let G be an* H-minor free graph*, and let* $\mathbf{p}$ *be a* contraction bidimensional *parameter computable in* $2^{O(\mathbf{tw}(G))} \cdot n^{O(1)}$. *Then deciding* $\mathbf{p}(G) = k$ *can be done in time* $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

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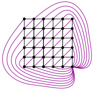
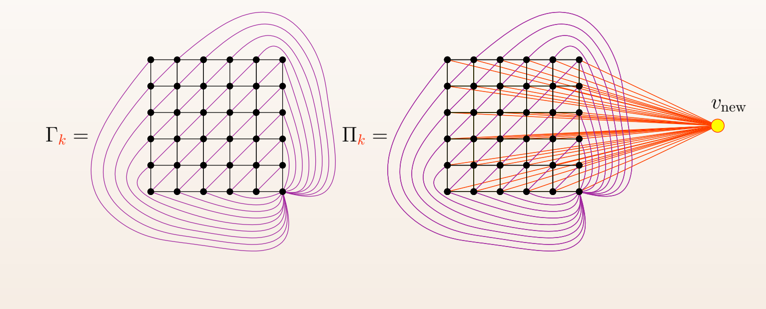
As for minor bidimensionality, we need to prove that

- If $\mathbf{tw}(G) = \Omega(k)$ then G contains  $_k$ as a contraction.

Two important grid-like graphs

Two pattern graphs Γ_k and Π_k:



$\Pi_k = \Gamma_k +$ a new universal vertex v_{new}.

The "contraction-certificates" for large treewidth

Theorem (Fomin, Golovach, Thilikos. 2009)

For any integer $\ell > 0$, there is c_ℓ such that every connected graph of treewidth at least c_ℓ, contains K_ℓ, Γ_ℓ, or Π_ℓ as a contraction.

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Further applications of Bidimensionality

1. Bidimensionality + DP $\Rightarrow$ Subexponential FPT algorithms

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4. Bidimensionality + new Grid Theorems ⇒ Geometric graphs

   [Fomin, Lokshtanov, Saurabh. 2012]
   [Grigoriev, Koutsonas, Thilikos. 2013]

Gràcies!