Introduction to logic in graphs and algorithmic meta-theorems

Ignasi Sau

LIRMM, Université de Montpellier, CNRS, France

20th JCALM

LIRMM, Montpellier, December 13-14, 2023

20èmes JCALM

JCALM: Journées de Combinatoire et d'Algorithmes du Littoral Méditerranéen.

Take place once or twice a year since 2006.

Involved research teams:

- DALGO and ACRO teams at LIS in Marseille.
- AIGCo team (among others) at LIRMM in Montpellier.
- COATI common project at I3S and INRIA Sophia-Antipolis.
- GAPCOMB team at UPC in Barcelona.

Current JCALM

Topic: Logic and graph algorithms

Programme

Wednesday December 13th

- 09:30–10:20: Café+croissants et accueil des participant-e-s (cafétéria du LIRMM, bâtiment 4)
- 10:20–11:20: **Ignasi Sau**, Introduction to logic in graphs and algorithmic meta-theorems
- 11:30–12:30: **Hugo Jacob**, First-order model-checking on sparse graph classes
- 12:30–14:00: Repas (cafétéria du LIRMM, bâtiment 4)
- 14:00–15:00: **Eunjung Kim**, First-order model-checking on graphs of bounded twin-width
- 15:10–16:10: **Dimitrios Thilikos**, Logic and algorithms for graph minors
- 16:10–16:40: Café+croissants
- 16:40–17:30: Open problems
- Soirée: dîner en centre ville de Montpellier

Thursday December 14th

- 09:30–10:20: Café+croissants (cafétéria du LIRMM, bâtiment 4)
- 10:20–11:20: **Matthieu Rosenfeld**, Monadic second-order logic and treewidth
- 11:30–12:30: **Giannos Stamoulis**, Elementary first-order model-checking
- 12:00–14:00: Repas (cafétéria du LIRMM, bâtiment 4)

Algorithmic meta-theorems

Typical statement of an algorithmic meta-theorem (AMT):

> Every computational problem that can be formalized in a given logic $\mathcal{L}$ can be solved efficiently on every class $\mathcal{C}$ of structures (typically, graphs) satisfying certain (typically, combinatorial) conditions.

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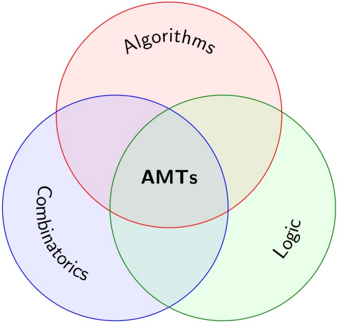
- Logical component: given by a logic $\mathcal{L}$ (such as first-order or second-order logic).
- Structural (combinatorial) component: given by a class $\mathcal{C}$ (such as planar graphs, or graphs of bounded degree).



Outline of this introductory talk

★ Strongly inspired from the survey of Stephan Kreutzer (2011):

"Algorithmic Meta-Theorems"

1. Introduction to logic (in graphs)
2. AMTs for monadic second-order logic
3. AMTs for first-order logic

Next section is...

1. Introduction to logic (in graphs)
2. AMTs for monadic second-order logic
3. AMTs for first-order logic

Basics on logic

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Example of a relation symbol: "edge" relation in graphs, with arity 2.

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▷ $\mathrm{MSO}[\sigma]$: class of formulas of monadic second-order logic over σ:

- Additional rule: if X is a second-order variable and $\varphi \in \mathrm{MSO}[\sigma]$, then $\exists X \varphi \in \mathrm{MSO}[\sigma]$ and $\forall X \varphi \in \mathrm{MSO}[\sigma]$.

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▷ Abbreviations:

- $x \neq y$, instead of $\neg x = y$.
- $\varphi \rightarrow \psi$ instead of $(\neg\varphi \vee \psi)$.

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- If we deal with (non-annotated) graphs: $\sigma = E$ (i.e., the edge relation).

Examples of FO formulas in graphs

$$\varphi_k := \exists x_1 \ldots \exists x_k \bigwedge_{1 \le i < j \le k} \left(x_i \neq x_j \wedge \neg E x_i x_j \right)$$

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A pair (G, U), where $U \subseteq V(G)$, satisfies $(G, U) \models \varphi$ if and only if U is a dominating set in G (every vertex not in U has a neighbor in U).

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Expresses that a graph contains a dominating set of size k.

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This suggests that the DOMINATING SET problem might be harder than the INDEPENDENT SET problem, as we shall see later...

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$$\tau := \exists C_1 \exists C_2 \exists C_3 (\forall x \bigvee_{i=1}^{3} C_i x) \wedge \forall x \forall y (Exy \to \bigwedge_{i=1}^{3} \neg (C_i x \vee C_i y))$$

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A graph $G \models \tau$ if and only if G is 3-colorable.

The model-checking problem

Let $\mathcal{L}$ be a fixed logic. We define the Model-Checking problem of $\mathcal{L}$:

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Related problem:

SATISFIABILITY($\mathcal{L}$)
Input: A sentence $\varphi \in \mathcal{L}$.
Question: Does there exist a structure A such that $A \models \varphi$?

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Is it the end of the story?

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  Idea: *try all possible choices for the first-order variables appearing in the formula, and check whether it is satisfied. Running time:* $|A|^{f(\varphi)}$.

The hardness depends on whether the formula is part of the input or not.

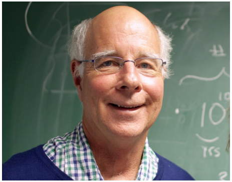
Is it the end of the story?

We need to parameterize the problem!

The area of parameterized complexity

Idea Measure the complexity of an algorithm in terms of the input size and an additional integer parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established area with hundreds of articles published every year in the most prestigious TCS journals and conferences.

Parameterized problems

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$,
where Σ is a fixed, finite alphabet.

For an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, k is called the parameter.

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- k-VERTEX COVER: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, containing at least an endpoint of every edge?
- INDEPENDENT SET: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \geq k$, of pairwise non-adjacent vertices?
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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- Independent Set: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
- Vertex k-Coloring: NP-hard for fixed $k = 3$.

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Working hypothesis of paramet. complexity: **k-Indep. Set is not FPT**

(in classical complexity: SAT cannot be solved in poly-time)

How to transfer hardness among parameterized problems?

Let A, B be two parameterized problems.

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W[i]-hard: strong evidence of not being FPT. Hypothesis: FPT $\neq$ W[1]

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The "right" problem to consider is the following :

$\mathrm{MC}(\mathcal{L})$
Input: A structure A and a sentence $\varphi \in \mathcal{L}$.
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The “right” problem to consider in graphs is the following :

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k-Indep. Set is W[1]-hard $\Rightarrow$ $\mathrm{MC}(\mathrm{FO})$ is W[1]-hard (and XP).

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$f(|\varphi|, \mathbf{p}(G)) \cdot |G|^{\mathcal{O}(1)}$

Next section is...

1. Introduction to logic (in graphs)
2. AMTs for monadic second-order logic
3. AMTs for first-order logic

MSO model-checking

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Input: A graph G and a sentence $\varphi \in$ MSO.
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As some of you may imagine: we take $\mathbf{p} =$ treewidth.

Crucial notion: treewidth

- 1972: Bertelè and Brioschi (dimension).
- 1976: Halin (S-functions of graphs).
- 1984: Arnborg and Proskurowski (partial k-trees).
- 1984: Robertson and Seymour (treewidth).

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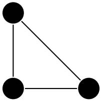
Treewidth measures the (topological) similarity of a graph with a forest.

Treewidth via k-trees

Example of a 2-tree:



[Figure by Julien Baste]

For $k \geq 1$, a k-tree is a graph that can be built starting from a $(k+1)$-clique and then iteratively adding a vertex connected to a k-clique.

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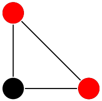

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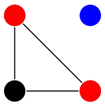

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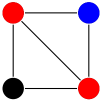

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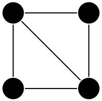

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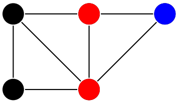

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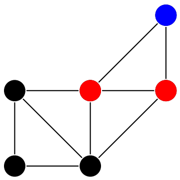

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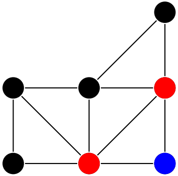

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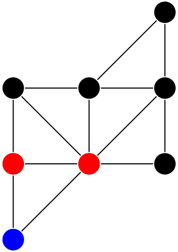

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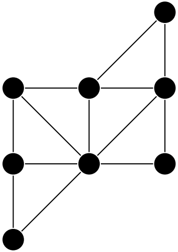

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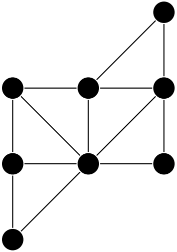

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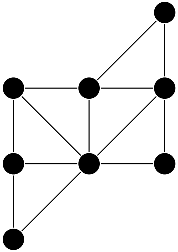

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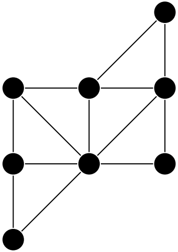

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{X_t \mid t \in V(T)\})$, where
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- Treewidth of a graph G, $\mathrm{tw}(G)$: minimum width of a tree decomposition of G.

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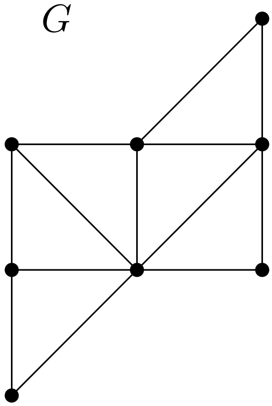

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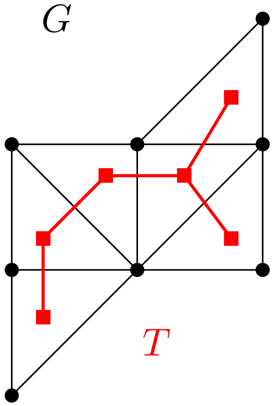

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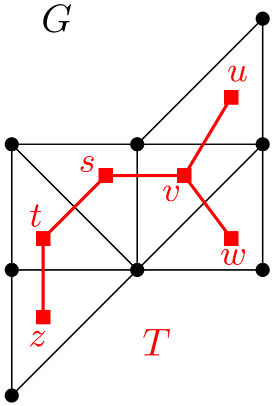

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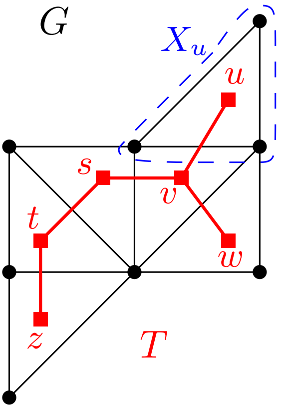

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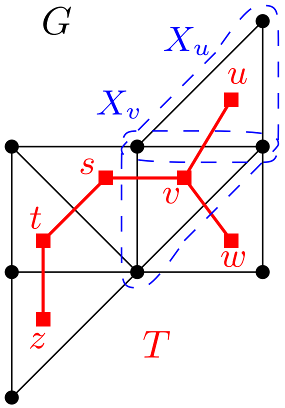

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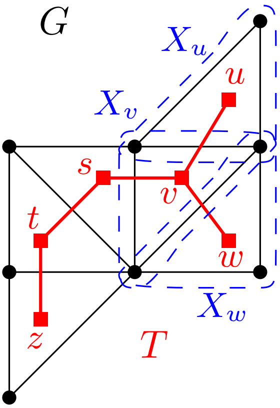

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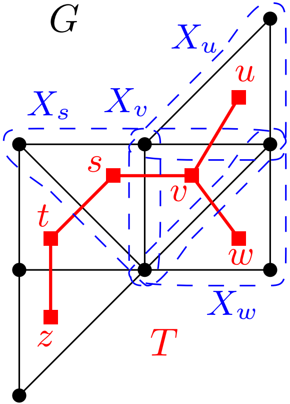

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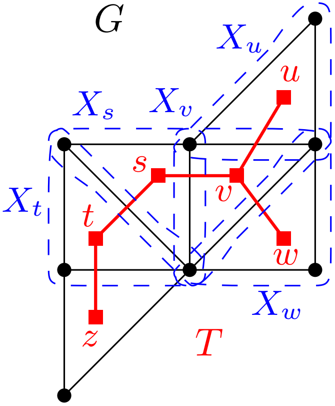

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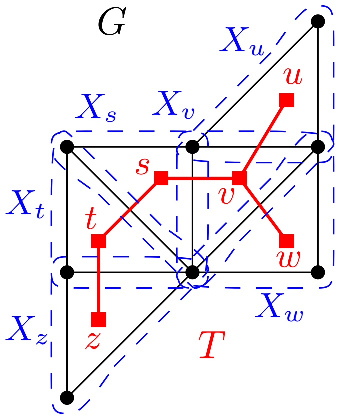

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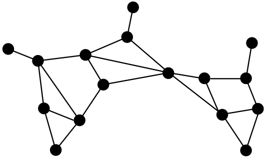

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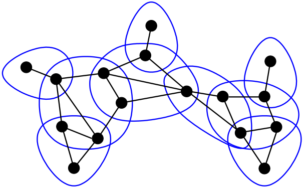

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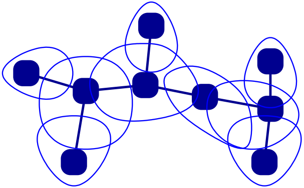

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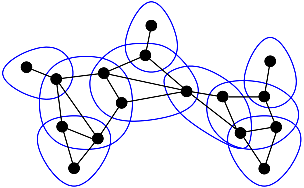

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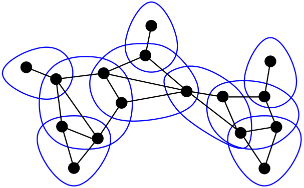

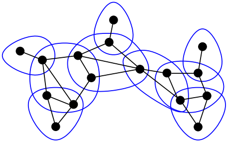

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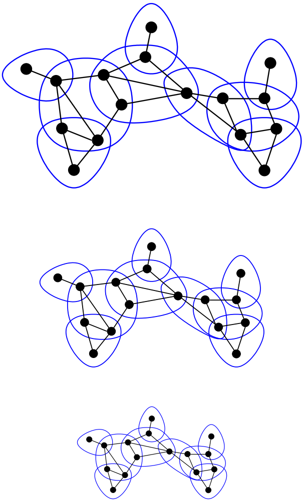

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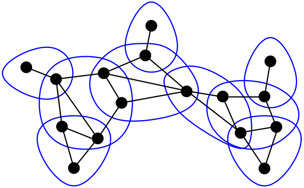

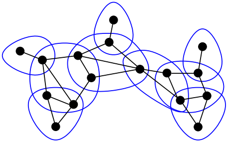

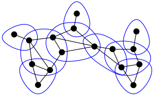

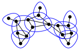

Courcelle's theorem

(Credit also goes to Arnborg, Lagergreen, and Seese.)

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The following problem is fixed-parameter tractable:

$\mathrm{MC}(\mathsf{MSO}, \mathsf{tw})$
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Courcelle's theorem and its extensions

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- Also applies to the extension of MSO by modular counting: CMSO. [Courcelle. 1990]
- Can be generalized to optimize a linear function of free second-order variables. [Arnborg, Lagergreen, Seese. 1991]

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For Courcelle's theorem: interpret "tw $\leq k$" into the class of labeled trees.

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Small cliquewidth does not mean "being tree-like" (such as small treewidth), but having a structure with a "tree-like decomposition".

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Is the above theorem strictly more general than Courcelle's theorem?

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Edge subdivisions do not preserve cliquewidth:

- ▷ Is it possible that $\mathrm{MC}(\mathsf{MSO}_2, \mathsf{cw})$ is FPT?

The limits of cliquewidth

EDGE DOMINATING SET, HAMILTONIAN CYCLE, and GRAPH COLORING are MSO_2-definable and W[1]-hard parameterized by cliquewidth. [Fomin, Golovach, Lokshtanov, Saurabh. 2010]

The limits of cliquewidth and MSO

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For MSO we cannot go really further than bounded treewidth/cliquewidth:

3-Colorability is NP-complete on planar graphs of degree at most 4. [Garey, Johnson, Stockmeyer. 1974]

Next section is...

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$\mathrm{MC}(\mathsf{FO}, \mathcal{C})$
Input: A graph $G \in \mathcal{C}$ and a sentence $\varphi \in \mathsf{FO}$.
Parameter: $|\varphi|$.
Question: $G \models \varphi$?

Question: for which (parameterized) graph classes $\mathcal{C}$ is $\mathrm{MC}(\mathsf{FO}, \mathcal{C})$ FPT?

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▷ A first-order formula $\varphi(x)$ on graphs is r-local if, for every graph G and every $v \in V(G)$,

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▷ A first-order formula $\varphi(x)$ on graphs is local if it is r-local for an $r \in \mathbb{N}$.

▷ A basic local sentence is a first-order sentence of the form

$$\exists x_1 \ldots \exists x_k \left(\bigwedge_{1 \leq i < j \leq k} \mathrm{dist}(x_i, x_j) > 2r \;\wedge\; \bigwedge_{i=1}^{k} \psi(x_i) \right),$$

where $\psi(x_i)$ is a local first-order formula.

Gaifman's theorem

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This translation may involve a non-elementary blow-up in the size of the sentence.

[Dawar, Grohe, Kreutzer, Schweikardt. 2007]

Example: k-Dominating Set

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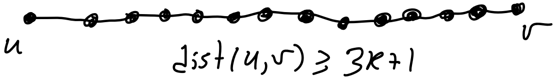

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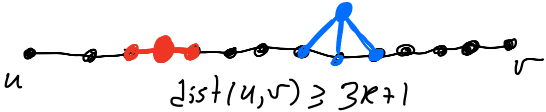

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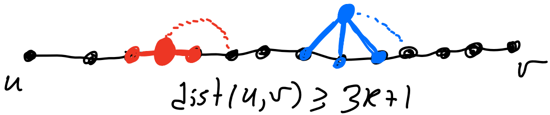

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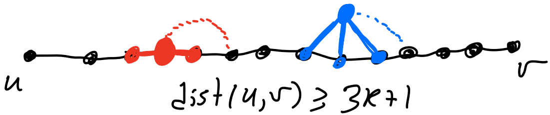

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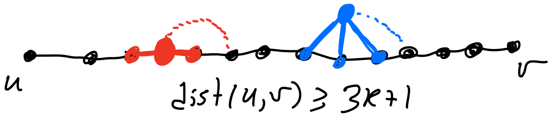

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2. $\exists x \chi(x)$, where $\chi(x)$ is the $(3k + 1)$-local formula

$$\exists y_1 \in N_{3k+1}(x) \ldots \exists y_k \in N_{3k+1}(x) \forall z \in N_{3k+1}(x) \left(\bigvee_{i=1}^{k} (y_i = z \vee Ezy_i) \right)$$

FO model-checking on graphs of bounded degree

Theorem (Seese. 1996)

Let $d \in \mathbb{N}$ and let $\mathcal{C}_d$ be the class of graphs of degree bounded by d. Then $\mathrm{MC}(\mathsf{FO}, \mathcal{C}_d)$ is FPT.

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- Finally, we greedily try to find k such "good" vertices far apart. □

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This has triggered a lot of research in the last 20 years...

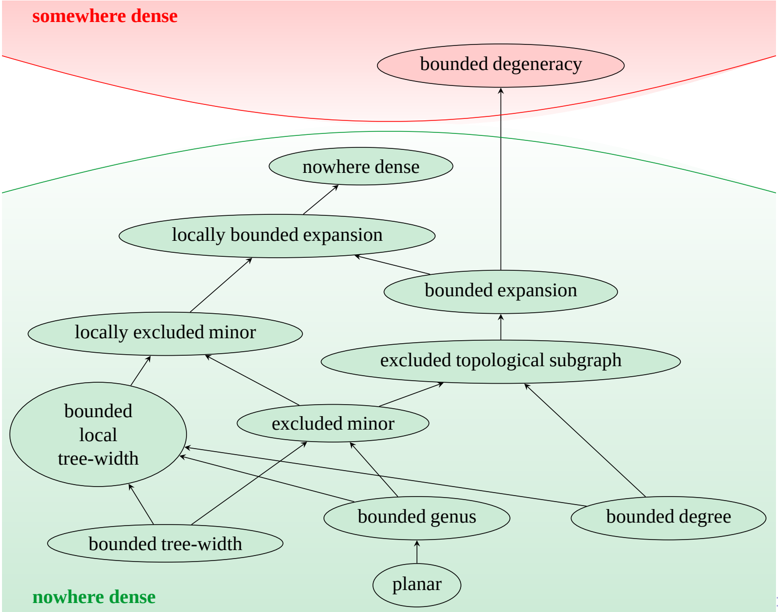
AMTs for MSO and FO

bounded treewidth [Courcelle,1990] [Arnborg, Lagergren, Seese, 1991] [Borie, Parker, Tovey, 1992]
bounded cliquewidth. [Courcelle, Makowski, Rotics, 2000] [Oum & Seymour, 2006] **MSO**

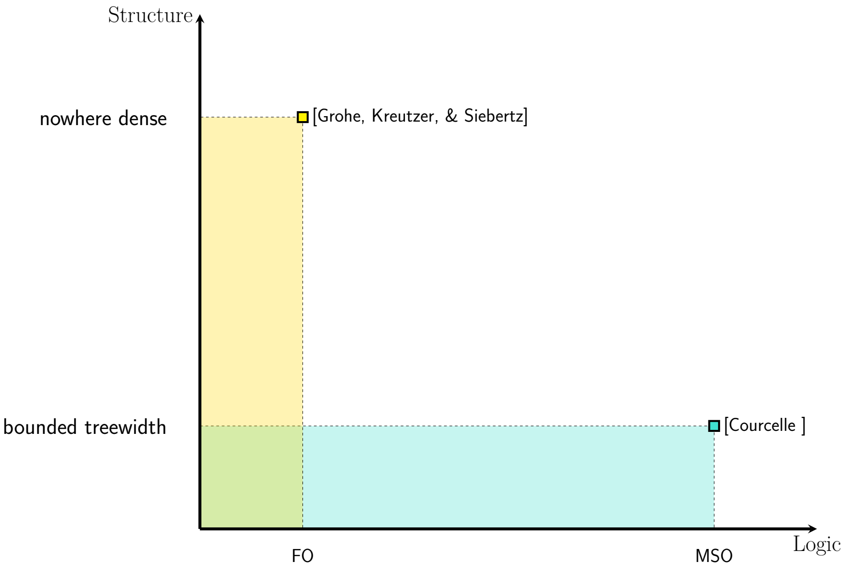
---

bounded degree [Seese, 1996] **FO**
locally bounded treewidth [Frick & Grohe, 2001]
excluding a minor [Flum & Grohe, 2001]
locally excluding a minor [Dawar, Grohe, Kreutzer, 2007]
bounded expansion [Dvořák, Kráľ, Thomas, 2011]
nowhere dense [Grohe, Kreutzer, Siebertz, 2017]
bounded twinwidth [Bonnet, Kim, Thomassé, Watrigant, 2022]
structurally bounded degree [Gajarský, Hliněný, Lokshtanov, Obdržálek, Ramanujan, 2016]
structurally bounded expansion [Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Mi. Pilipczuk, Siebertz, Toruńczyk, 2018]
structurally nowhere dense [Dreier, Mählmann, Siebertz, 2023]
structurally bounded local cliquewidth [Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, Toruńczyk, 2022]
monadically stable [Dreier, Eleftheriadis, Mählmann, McCarty, Mi. Pilipczuk, Toruńczyk, 2023]
monadically NIP/dependent ?

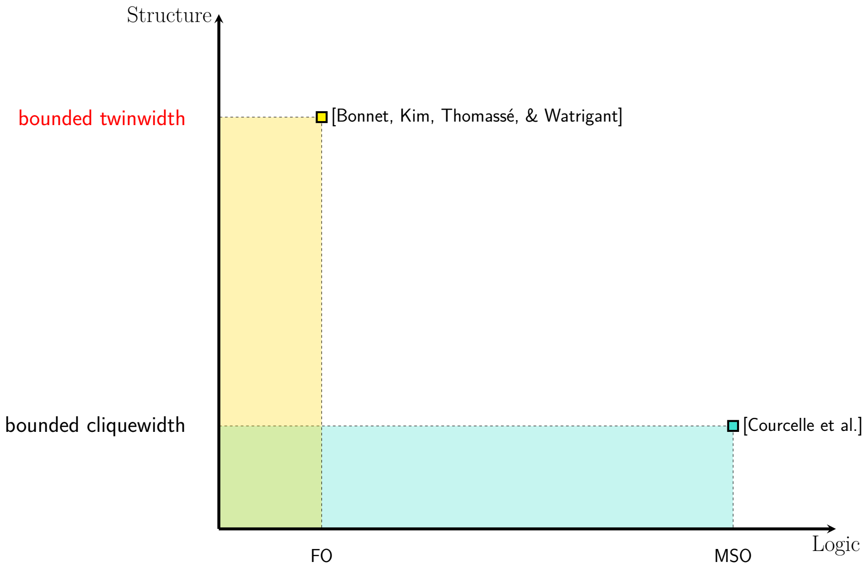
FO model-checking on sparse graph classes



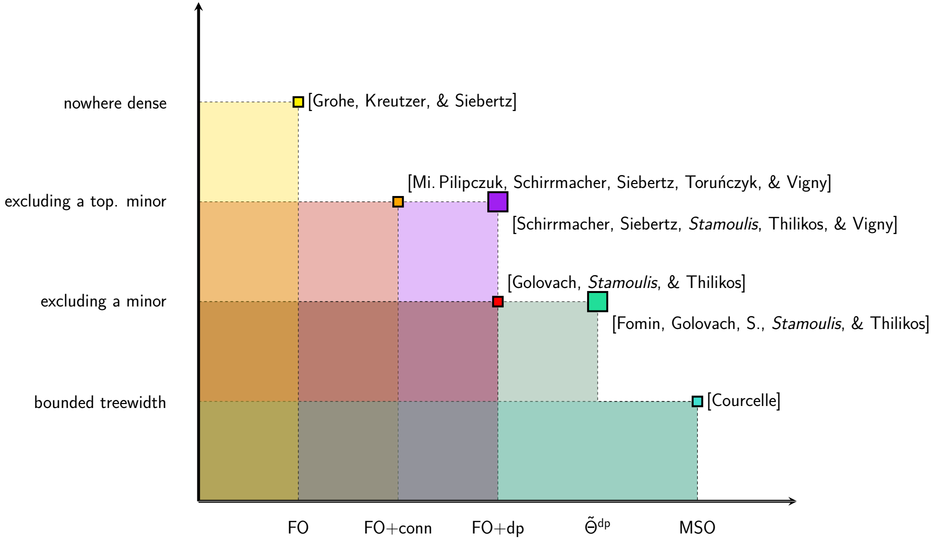
Simplified picture for monotone graph classes



Simplified picture for hereditary graph classes



A lot of interesting stuff between FO and MSO



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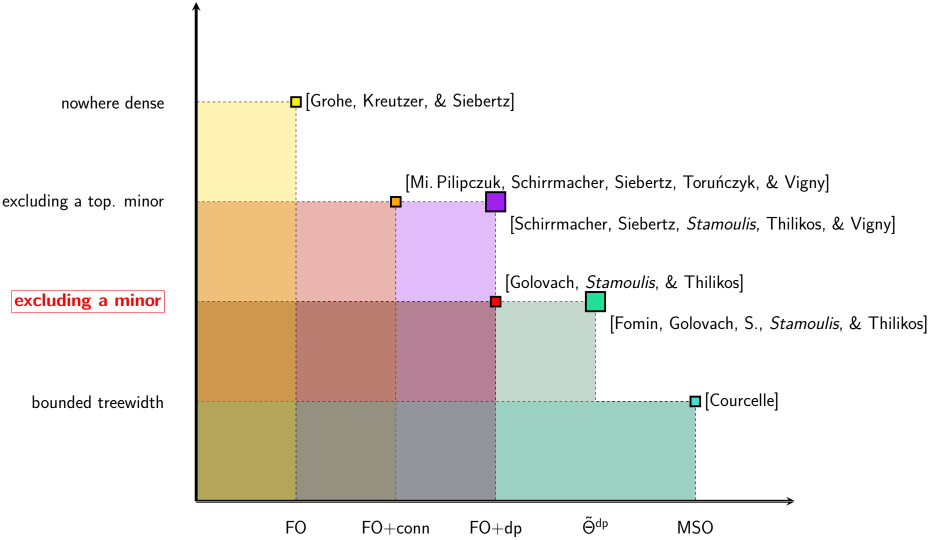
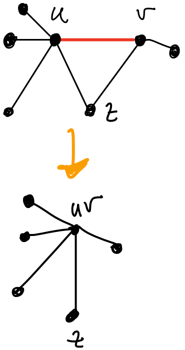
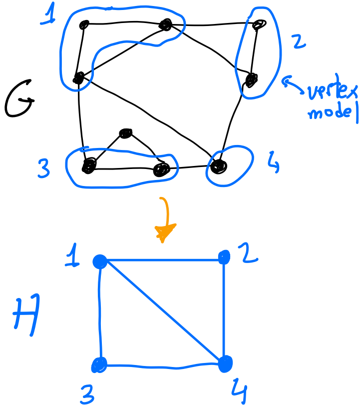


Graph minors

A graph H is a minor of a graph G, denoted by $H \leqslant_m G$, if H can be obtained by a subgraph of G by contracting edges.





Minor-closed graph classes

A graph class $\mathcal{C}$ is minor-closed (or closed under minors) if

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Examples of minor-closed graph classes:

- Independent sets.
- Forests.
- Subgraphs of series-parallel graphs.
- Planar graphs.
- Graphs embeddable in a fixed surface.
- Linklessly embeddable graphs.
- Knotlessly embeddable graphs.
- ...

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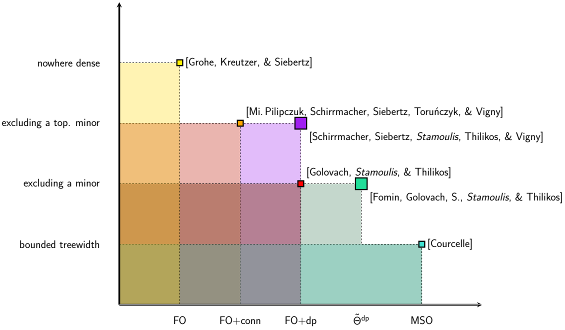
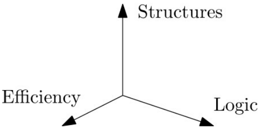
Theorem (Robertson, Seymour. 1983-2004)

Every minor-closed graph class $\mathcal{C}$ can be characterized by a finite list of excluded minors.

Missing axis: efficiency dimension





$$f(|\varphi|, \mathbf{p}(G)) \cdot |G|^{\mathcal{O}(1)}$$

Gràcies!