Traffic Grooming in Bidirectional WDM Rings

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Colloquim JCB, 31 mars 2011

Outline

Next section is...

Introduction

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s
- Idea:
  Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  $\longrightarrow$ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)

- Objectives:
  - Better use of bandwidth
  - Reduce the equipment cost (mostly given by electronics)

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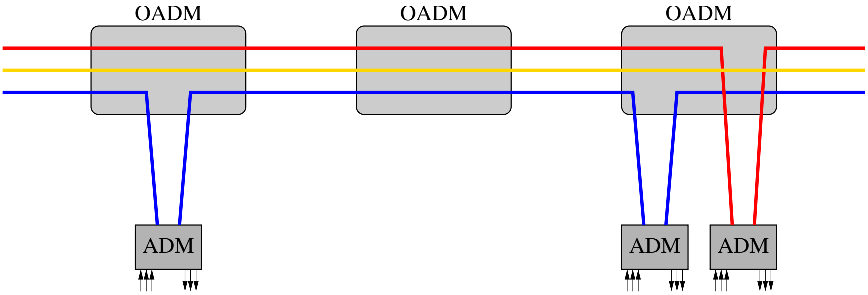
ADM and OADM

- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



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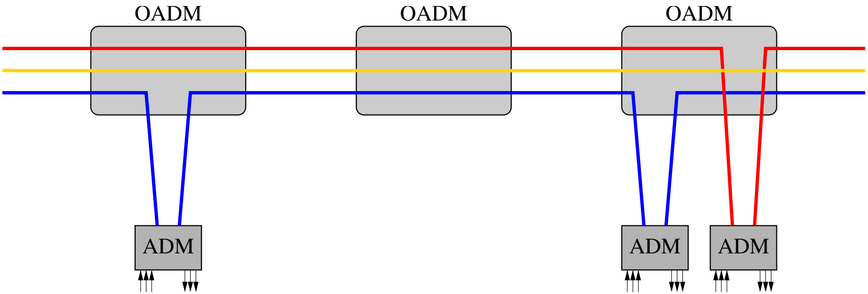

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Definitions

- **Request** (i,j): two vertices (i,j) that want to exchange (low-speed) traffic

- **Grooming factor** C:

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

Example:

Capacity of one wavelength $= 2400\ Mb/s$
Capacity used by a request $= 600\ Mb/s$ $\quad \Rightarrow \quad C = 4$

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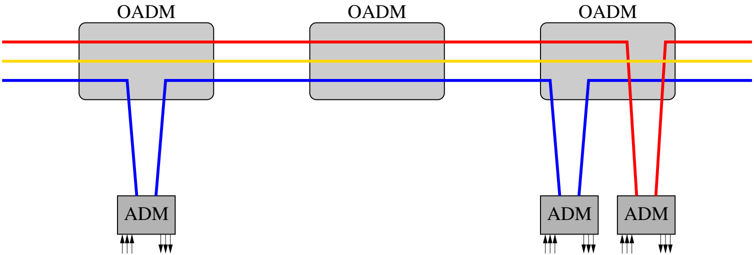

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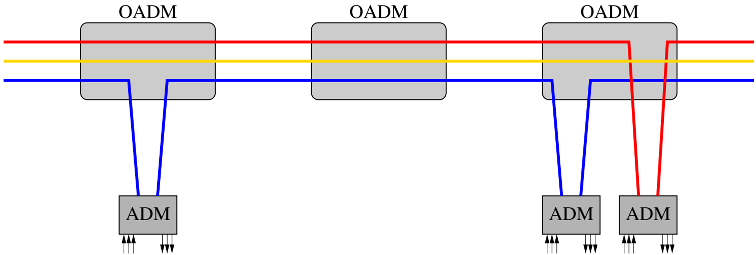

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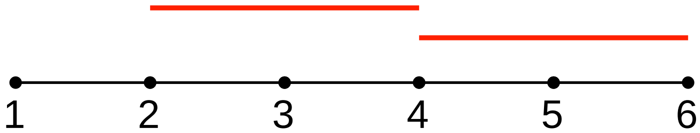

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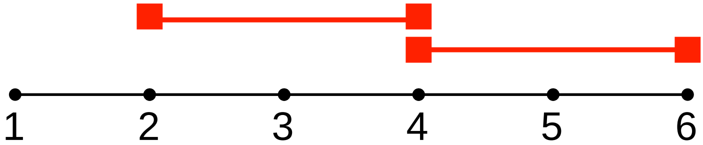

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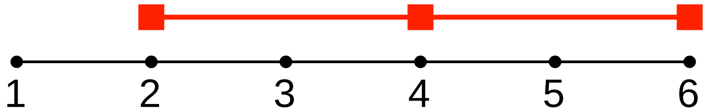
With grooming, C=1



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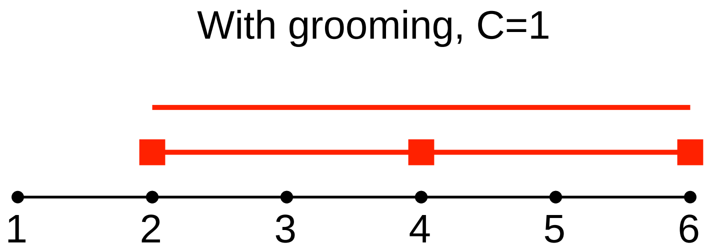

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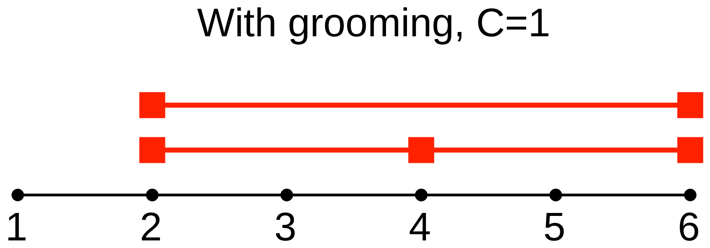

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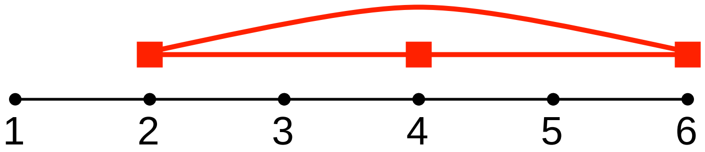
With grooming, C=2



Model

Topology	$\rightarrow$	(di)graph G
Request set	$\rightarrow$	(di)graph R
Grooming factor	$\rightarrow$	integer C
Requests in a **wavelength**	$\rightarrow$	arcs in a **subgraph** of R
ADM in a wavelength	$\rightarrow$	vertex in a subgraph of R

⋆ **Important case:** $G = \overrightarrow{C}_n$, with symmetric requests

[**J.-C. Bermond and D. Coudert**. Traffic Grooming in Unidirectional WDM Ring Networks using Design Theory. *IEEE ICC, 2003*]

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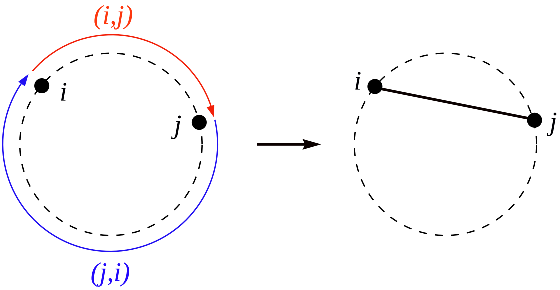
Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: we have both (i, j) and (j, i).



- W.l.o.g. requests (i, j) and (j, i) are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ grooming factor $C \Leftrightarrow$ each subgraph has $\leq C$ edges.

- C-edge-partition of a graph G:
  partition of $E(G)$ into subgraphs with at most C edges each.

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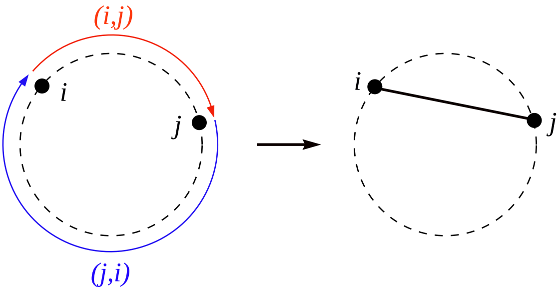

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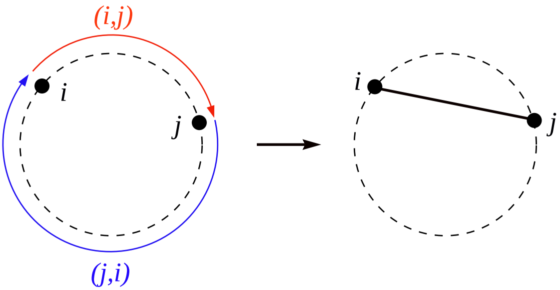

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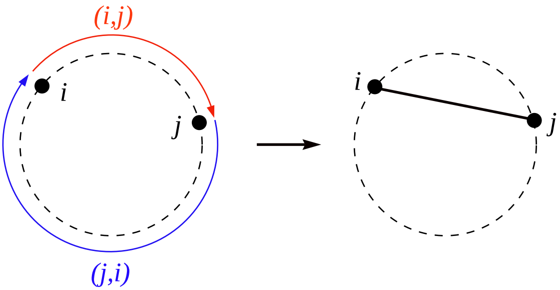

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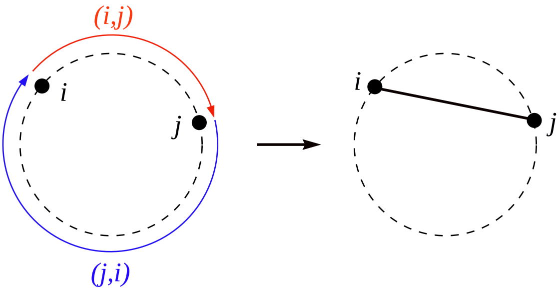

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- *C*-edge-partition of a graph G:
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Statement of the problem in unidirectional rings

Traffic Grooming in Unidirectional Rings

Input A cycle C_n on n nodes (network);
An *undirected* graph R on n nodes (request set);
A grooming factor C.

Output A C-edge-partition of R into subgraphs $R_1, \ldots, R_W$.

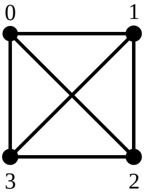
Objective Minimize $\sum_{\omega=1}^{W} |V(R_\omega)|$.

Example (unidirectional ring with symmetric requests)

$n = 4$

$R = K_4$

$C = 3$



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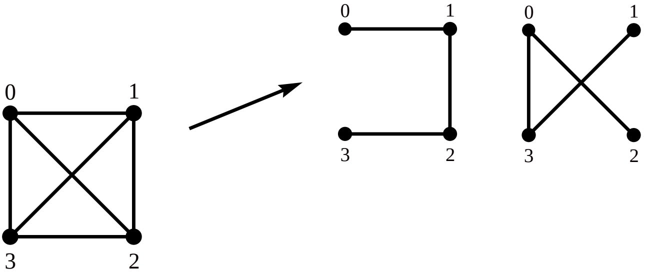

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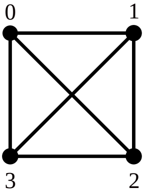

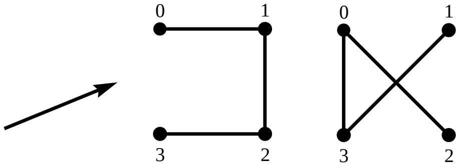

8 ADMs

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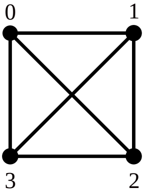

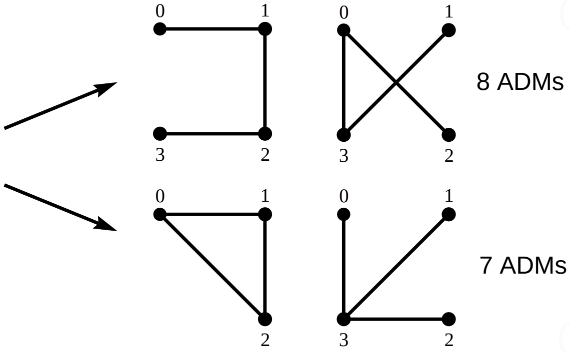

Next section is...

1. Motivation: traffic grooming
2. Jean-Claude's contribution
3. The bidirectional ring
    - Preliminaries
    - Lower bounds
    - Upper bounds

Jean-Claude, the “traffic groomer”

[1] J.-C. Bermond and S. Ceroi. Minimizing SONET ADMs in unidirectional WDM rings with grooming ratio 3. *Networks, 2003.*

[2] J.-C. Bermond, C. Colbourn, A. Ling, and M.-L. Yu. Grooming in unidirectional rings: $K_4 - e$ designs. *Discrete Mathematics, 2004.*

[3] J.-C. Bermond, C. Colbourn, D. Coudert, G. Ge, A. Ling, and X. Muñoz. Traffic Grooming in Unidirectional WDM Rings With Grooming Ratio $C = 6$. *SIAM J. on Discr. Math., 2005.*

[4] J.-C. Bermond and D. Coudert. Handbook of Combinatorial Designs, chapter VI.27: Grooming. *Chapman & Hall-CRC Press, 2006.*

[5] J.-C. Bermond, L. Braud, and D. Coudert. Traffic Grooming on the Path. *Theoretical Computer Science, 2007.*

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[7] J.-C. Bermond, C. Colbourn, L. Gionfriddo, G. Quattrocchi, and I. Sau. Drop Cost and Wavelength Optimal Two-Period Grooming with Ratio 4. *SIAM J. on Discr. Math., 2010.*

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Case under study

We focus on the following particular case:

- The topology is given by a bidirectional ring.
- There is an all-to-all traffic.
- The routing uses shortest paths.
- The routing is symmetric
  (makes sense only if the size of the ring is even).

⋆ Simplification: we consider the requests *clockwise* and *counterclockwise* independently.

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Statement of our problem

Traffic Grooming in Bidirectional Rings

Input

- A unidirectional cycle $\vec{C}_n$;
- A grooming factor C;
- A digraph of requests consisting of a "clockwise" tournament T_n.

Output

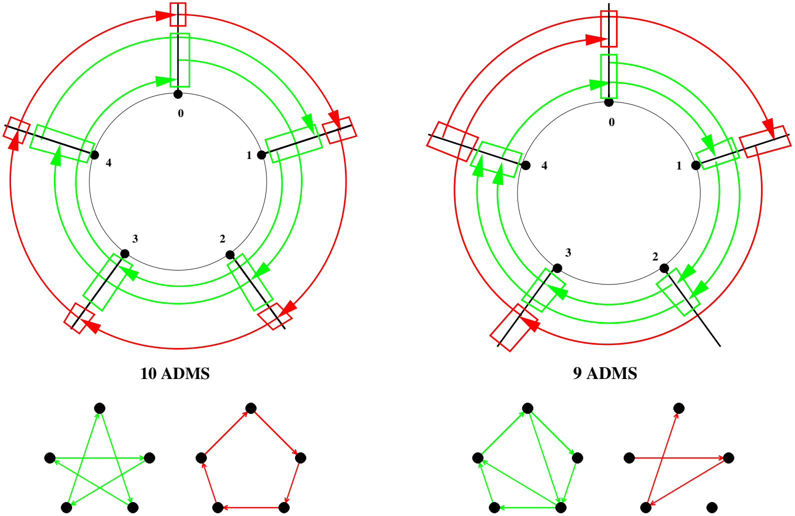
A partition of $E(T_n)$ into digraphs B_ω, $1 \leq \omega \leq W$, such that for each arc $e \in E(\vec{C}_n)$, $\text{load}(B_\omega, e) \leq C$.

Objective

Minimize $\sum_{\omega=1}^{W} |V(B_\omega)| \quad =: A(C, n)$.

Example: $n = 5$ and $C = 2$

Here we partition T_5 in two ways, both using two wavelengths (colors):



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An **embedded** digraph B_ω is C-admissible if $\text{load}(B_\omega, e) \leq C$ for each arc $e \in E(\vec{C}_n)$.

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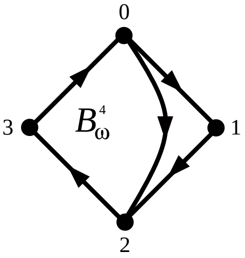

(a)

(a) A (non-embedded) digraph B^4_ω.

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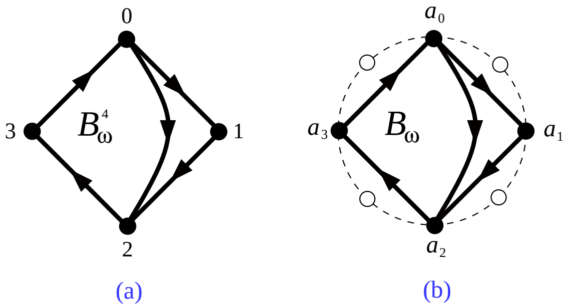

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(b) An embedded digraph B_ω, which is 2-admissible.

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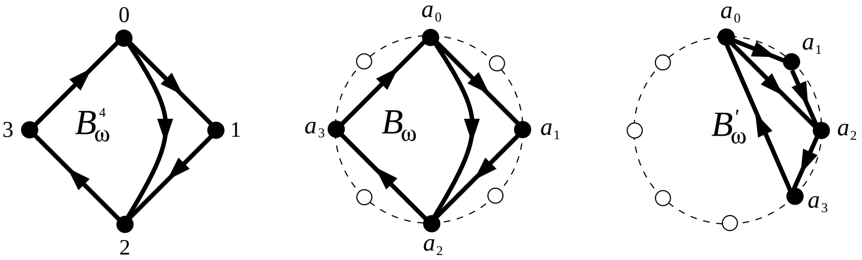

(a) (b) (c)

(a) A (non-embedded) digraph B_ω^4.

(b) An embedded digraph B_ω, which is 2-admissible.

(c) An embedded digraph B'_ω, which is NOT 2-admissible.

The parameter $\gamma(C, p)$

Definition

$\gamma(C,p) = \max\{|E(B_\omega)| : B_\omega$ C-admissible digraph with $|V(B_\omega)| = p\}$.

Is $\gamma(C, p)$ achieved using the requests of shortest length?

In the path, it is NOT the case! For instance, take $p = 11$ and $C = 10$:

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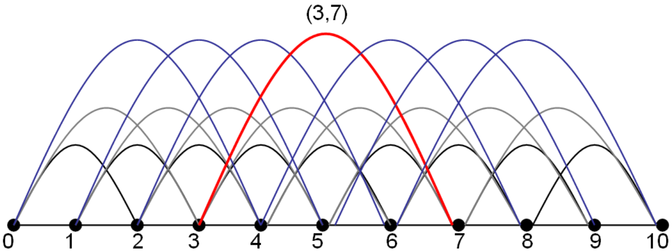
34 requests



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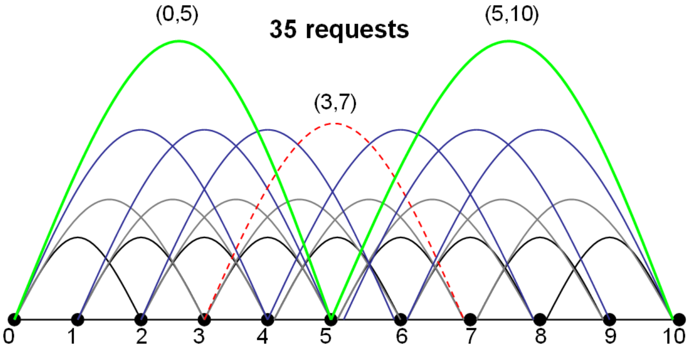

In bidirectional rings: the shorter, the better

Proposition

Let $C = \frac{k(k+1)}{2} + r$, *with* $0 \leq r \leq k$. *Then*

$$\gamma(C,p) = \begin{cases} \frac{p(p-1)}{2} & \text{, if } p \leq 2k+1 \text{, or } p = 2k+2 \text{ and } r \geq \frac{k+2}{2} \\ kp + 2r - 1 & \text{, if } p = 2k+2 \text{ and } 1 \leq r < \frac{k+2}{2} \\ kp + \left\lfloor \frac{rp}{k+1} \right\rfloor & \text{, otherwise} \end{cases}$$

The graphs achieving $\gamma(C,p)$ *are either the tournament* T_p *if* p *is small (namely, if* $p \leq 2k+1$ *or* $p = 2k+2$ *and* $r \geq \frac{k+2}{2}$*), or subgraphs of a circulant digraph containing all the arcs of length* $1, 2, \ldots, k$*, plus some arcs of length* $k+1$ *if* $r > 0$.

General lower bound

Definition

$$\rho(C) = \max_{p \geq 2} \left\{ \frac{\gamma(C,p)}{p} \right\} = k + \frac{r}{k+1}.$$

Theorem (General lower bound)

Let $C = \frac{k(k+1)}{2} + r$, with $0 \leq r \leq k$. The number of ADMs required in a bidirectional ring with n nodes and grooming factor C satisfies

$$A(C,n) \geq \left\lceil \frac{n(n-1)}{2 \cdot \rho(C)} \right\rceil = \left\lceil \frac{n(n-1)}{2} \frac{k+1}{k(k+1)+r} \right\rceil.$$

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Idea of proof: equations of the problem

Given a valid solution of the problem, let

- a_p: $\#$ of subgraphs of the partition with exactly p vertices;
- A: total $\#$ of ADMs in the solution; and
- W: $\#$ of subgraphs in the partition.

$$\sum_{p=2}^{n} p \cdot a_p = A$$

$$\sum_{w=1}^{W} |E(V_\omega)| = |E(T_n)| = \frac{n(n-1)}{2}$$

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$$A \; \geq \; \frac{n(n-1)}{2 \cdot \rho(C)}$$

Next subsection is...

1. Motivation: traffic grooming
2. Jean-Claude's contribution
3. The bidirectional ring
    - Preliminaries
    - Lower bounds
    - Upper bounds

Optimal constructions for $C = 3$

- If $C = 1 + \ldots + k$, then $\rho(C) = k + \frac{r}{k+1} = k$, so

$$A(C, n) \geq \frac{n(n-1)}{2 \cdot \rho(C)} = \frac{n(n-1)}{2 \cdot k}.$$

- For $C = 3$, we have $3 = 1 + 2$, so

$$A(3, n) \geq \frac{n(n-1)}{4}.$$

Proposition

For $n \equiv 1, 5 \pmod{12}$,

$$A(3, n) = \frac{n(n-1)}{4}.$$

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Sketch of proof

- ⋆ Steiner triple system of order N: partition of $E(K_N)$ into K_3's.
- They exist if and only if $N \equiv 1, 3 \pmod 6$.
- Let $N \equiv 1, 3 \pmod 6$ and $n = 2N - 1$, with
  - $V(K_N) = \{\infty, 1, \ldots, N-1\}$.
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- Consider a Steiner triple system for K_N.
- Then we transform each triangle as follows:

- # of vertices: $5 \cdot \left(\frac{N-1}{2}\right) + 6 \cdot \left(\frac{N(N-1)}{6} - \frac{N-1}{2}\right) = \frac{n(n-1)}{4}$.

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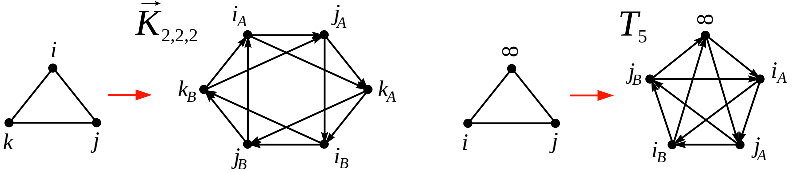

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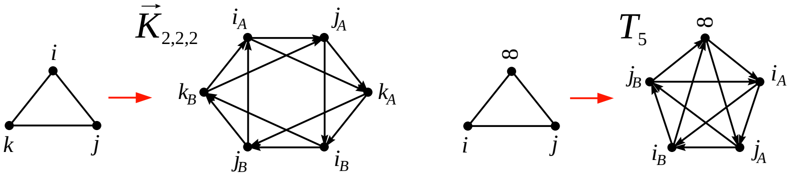

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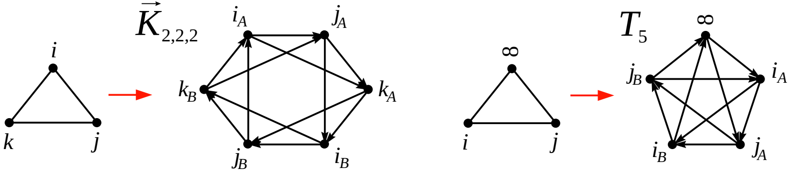

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Other optimal constructions

Theorem

If $K_{k\times q}$ can be partitioned into K_{k+1}'s, then there exists an optimal admissible partition of T_{2kq+1} for $C = \frac{k(k+1)}{2}$ with $\frac{n(n-1)}{2\cdot k}$ ADMs.

Corollary

If $C = 6$ and $n \equiv 1$ or $7 \pmod{24}$, $A(6,n) = \frac{n(n-1)}{6}$.
If $C = 10$ and $n \equiv 1$ or $9 \pmod{40}$, $A(10,n) = \frac{n(n-1)}{8}$.
If $C = 15$ and $n \equiv 1$ or $11 \pmod{30}$, $A(15,n) = \frac{n(n-1)}{10}$.
If $C = 21$ and $n \equiv 1$ or $13 \pmod{84}$, $A(21,n) = \frac{n(n-1)}{12}$.
If $C = 28$ and $n \equiv 1$ or $15 \pmod{112}$, $A(28,n) = \frac{n(n-1)}{14}$.
If $C = 36$ and $n \equiv 1$ or $17 \pmod{144}$, $A(36,n) = \frac{n(n-1)}{16}$.

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Asymptotically optimal solutions

Simple necessary conditions for K_v to be edge-partitioned into subgraphs isomorphic to a given graph H:

- $|E(H)|$ divides $\binom{v}{2}$.
- $\gcd\{$degree sequence of $H\}$ divides $v-1$.

Theorem (Wilson'75)

For v large enough, the above necessary conditions are also sufficient.

Corollary

If $C = \frac{k(k+1)}{2}$, then $A(C,n) = \frac{n(n-1)}{2\cdot k}$ for $n \equiv 1$ or $2k+1 \pmod{4C}$ large enough.

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Merci, Jean-Claude !!