Efficient algorithms on graphs of bounded treewidth

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Outline of the talk

1. Introduction
   - Parameterized complexity
   - Treewidth
2. FPT algorithms parameterized by treewidth
3. The $\mathcal{F}$-Deletion problem
4. Conclusions

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Some history of complexity: NP-completeness

- Cook-Levin Theorem (1971): the SAT problem is NP-complete.
- Karp (1972): list of 21 *important* NP-complete problems.
- Nowadays, literally thousands of problems are known to be NP-hard: unless P = NP, they cannot be solved in polynomial time.

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- Karp (1972): list of 21 *important* NP-complete problems.
- Nowadays, literally thousands of problems are known to be NP-hard: unless $\mathsf{P} = \mathsf{NP}$, they cannot be solved in polynomial time.
- But what does it mean for a problem to be NP-hard?

  No algorithm solves all instances optimally in polynomial time.

Are all instances really hard to solve?

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- VLSI design: the number of circuit layers is usually ≤ 10.
- Computational biology: Real instances of DNA chain reconstruction usually have treewidth ≤ 11.
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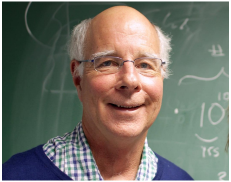
Message In many applications, not only the total size of the instance matters, but also the value of an additional parameter.

The area of parameterized complexity

Idea Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established area with hundreds of articles published every year in the most prestigious TCS journals and conferences.

Parameterized problems

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- k-Vertex Cover: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, containing at least an endpoint of every edge?
- k-Clique: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \geq k$, of pairwise adjacent vertices?
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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- k-Clique: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
- Vertex k-Coloring: NP-hard for fixed $k = 3$.

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- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n)) = f(k) \cdot n^{\mathcal{O}(1)}$.

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Working hypothesis of parameterized complexity: $\boxed{k\text{-Clique is not FPT}}$

(in classical complexity: 3-SAT cannot be solved in poly-time)

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W[i]-hard: strong evidence of not being FPT. Hypothesis: $\boxed{\text{FPT} \neq \text{W[1]}}$

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Do all FPT problems admit polynomial kernels? **NO!**

Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)

Deciding whether a graph has a PATH *with $\geq k$ vertices is* FPT *but does not admit a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

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Parameterized problem L

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k-Vertex Cover

k-Path

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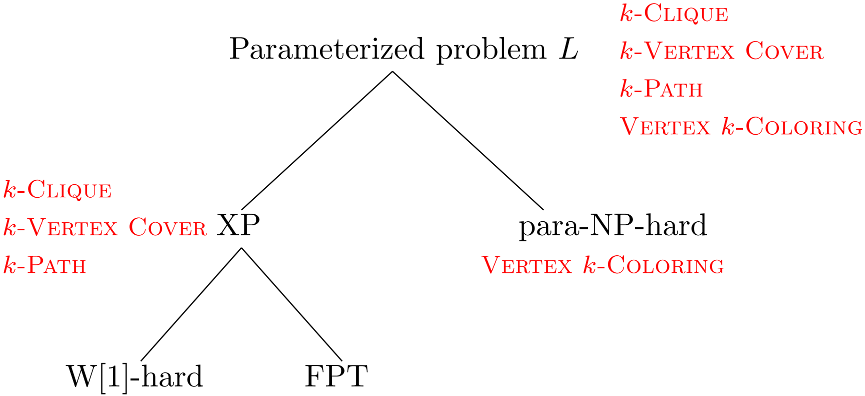

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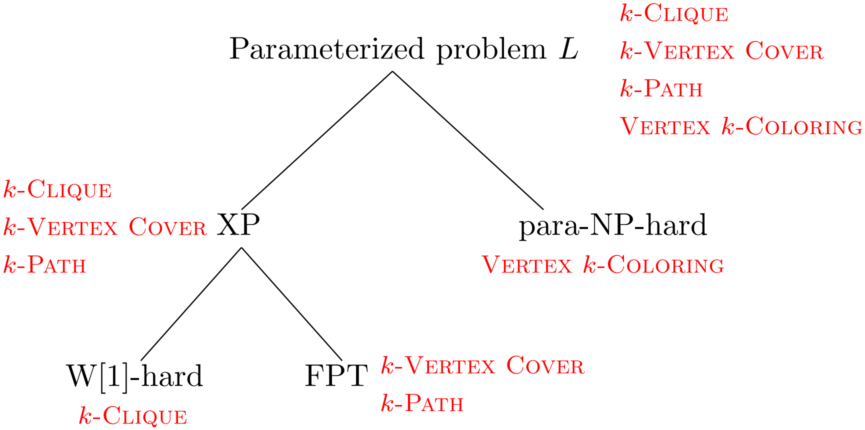

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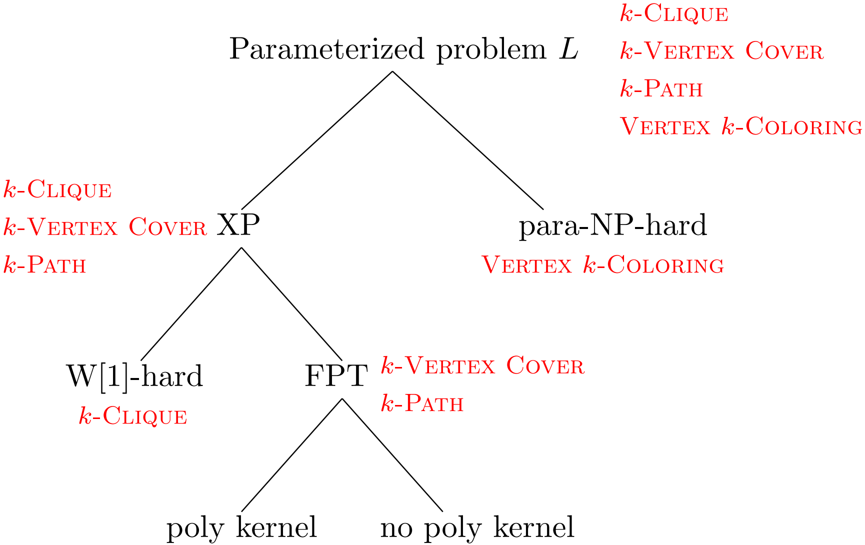

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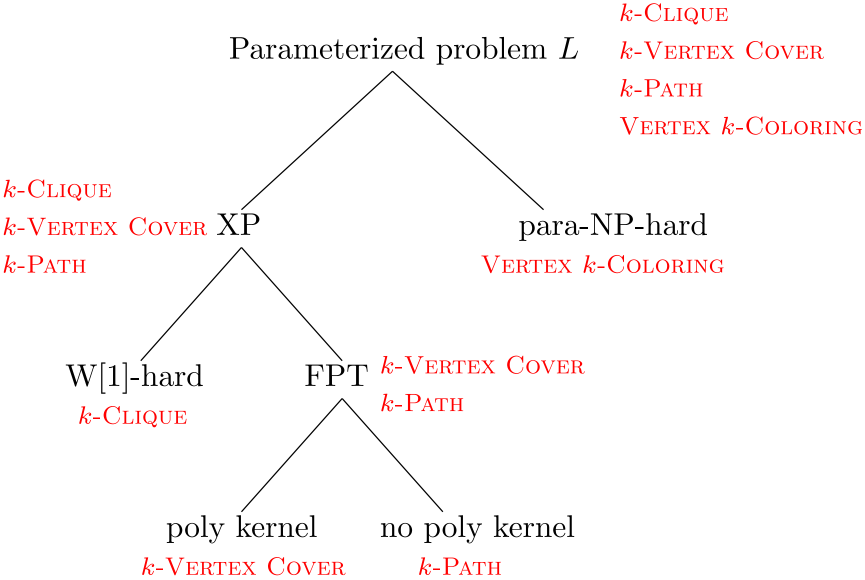

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Treewidth via k-trees

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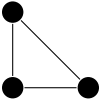

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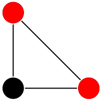

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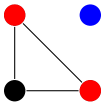

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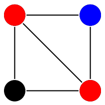

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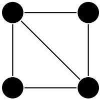

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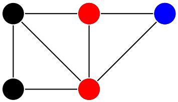

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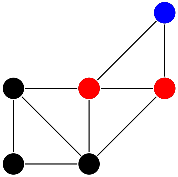

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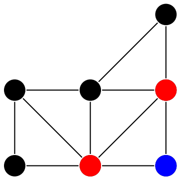

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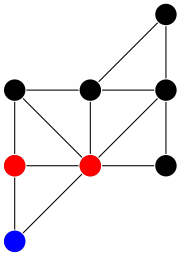

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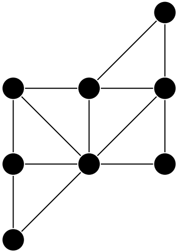

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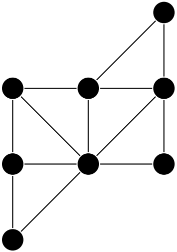

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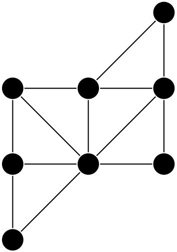

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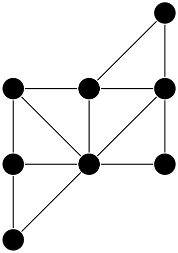

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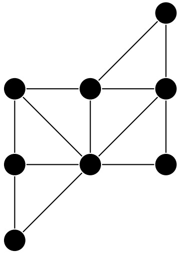

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{B_t \mid t \in V(T)\})$, where

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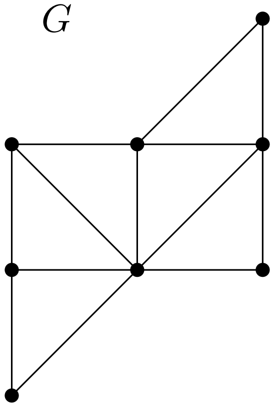

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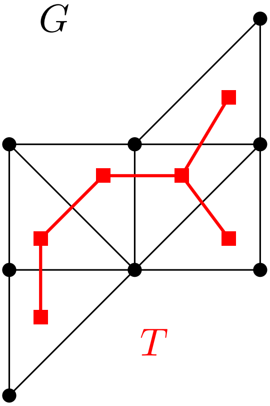

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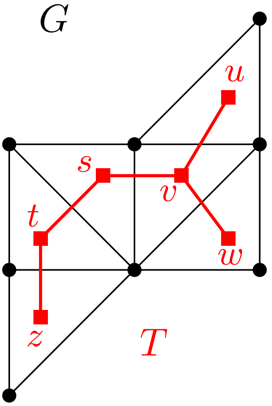

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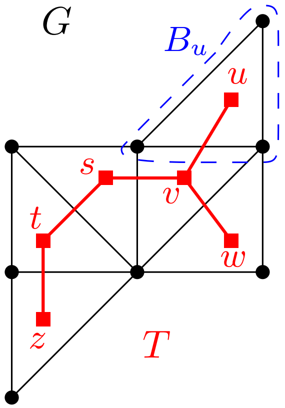

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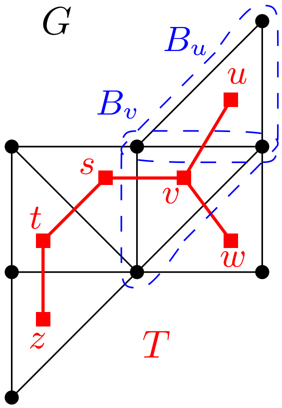

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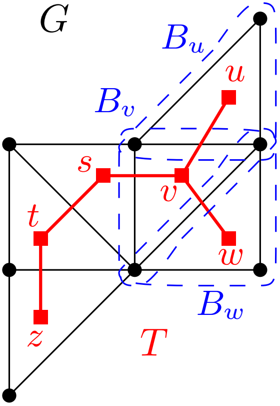

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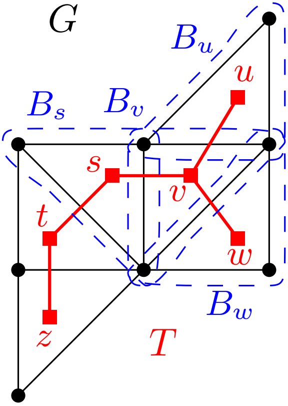

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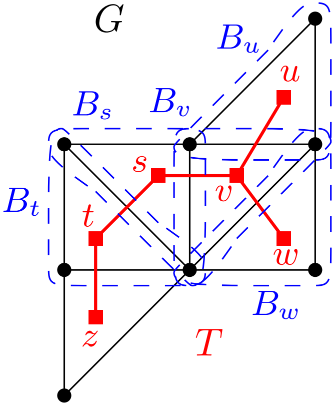

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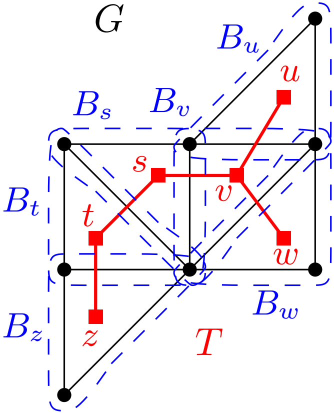

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2. Treewidth behaves very well algorithmically, and algorithms parameterized by treewidth appear very often in FPT algorithms.
3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

Next section is...

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Graph logic that allows quantification over sets of vertices and edges.

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   Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

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Remark: Algorithms parameterized by treewidth appear very often as a "black box" in all kinds of parameterized algorithms.

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Typical statements:

ETH $\Rightarrow$ k-Vertex Cover cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$.

ETH $\Rightarrow$ Planar k-Vertex Cover cannot in time $2^{o(\sqrt{k})} \cdot n^{O(1)}$.

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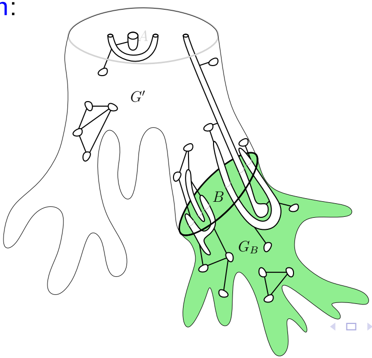
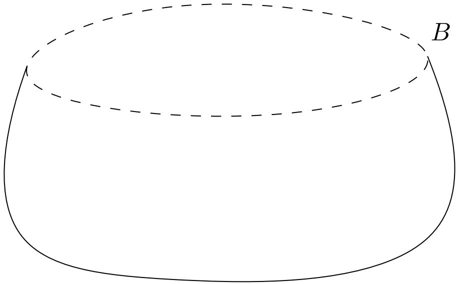
- Starting from the leaves of the tree decomposition, a set of appropriately defined partial solutions is computed recursively until the root, where a global solution is obtained.
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Two behaviors for problems parameterized by treewidth

Local problems: Vertex Cover, Dominating Set, Clique, Independent Set, q-Coloring for fixed q.



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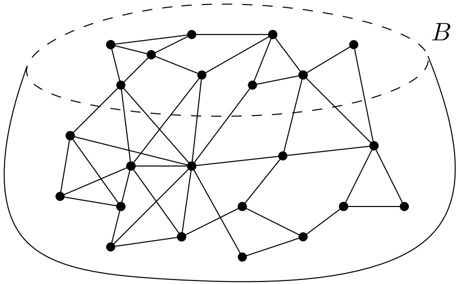

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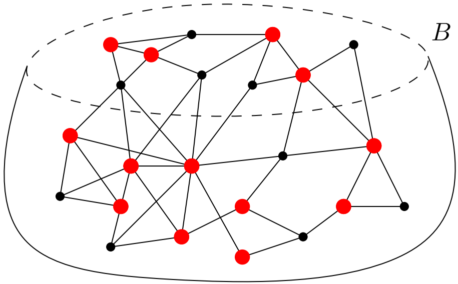

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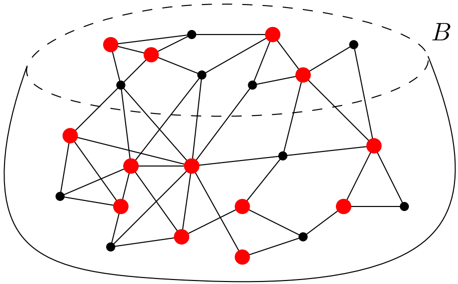

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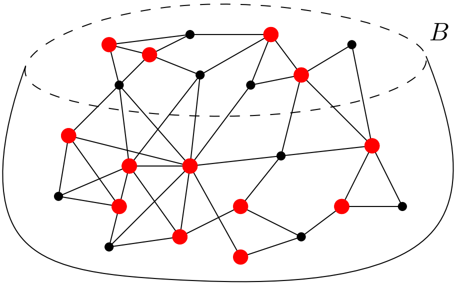

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- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

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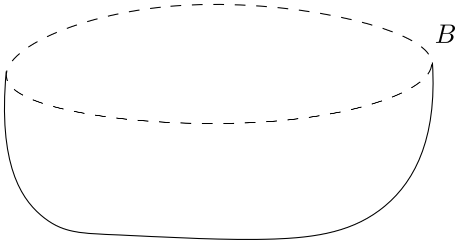
$\boxed{\text{Connectivity problems}}$ Hamiltonian Cycle, Longest Path, Steiner Tree, Connected Vertex Cover.



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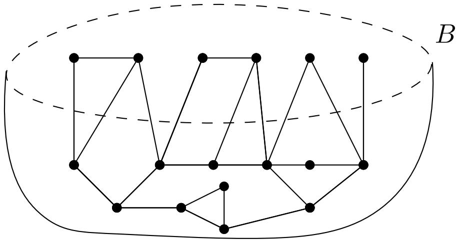

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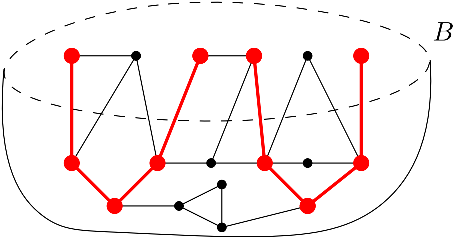

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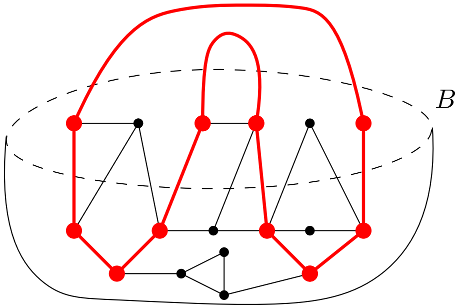

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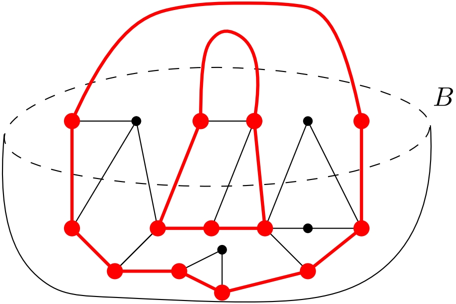

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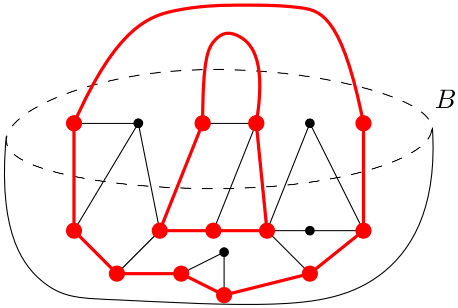

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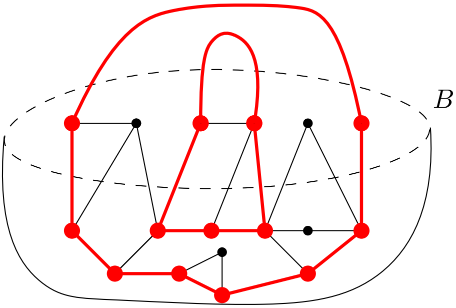

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- The "natural" DP algorithms provide only time $2^{\mathcal{O}(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$.

Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

  $$\boxed{2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

  Vertex Cover, Dominating Set, ...

- Connectivity problems:

  $$\boxed{2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

  Longest Path, Steiner Tree, ...

Single-exponential algorithms on sparse graphs

On $\boxed{\text{topologically structured}}$ graphs (planar, surfaces, minor-free), it is possible to solve connectivity problems in time $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$:

- Planar graphs: [Dorn, Penninkx, Bodlaender, Fomin. 2005]
- Graphs on surfaces: [Dorn, Fomin, Thilikos. 2006] [Rué, S., Thilikos. 2010]
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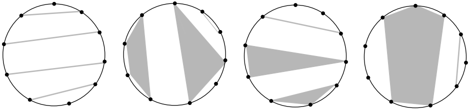
$\boxed{\text{Main idea}}$ special type of decomposition with nice topological properties:

partial solutions $\Longleftrightarrow$ non-crossing partitions



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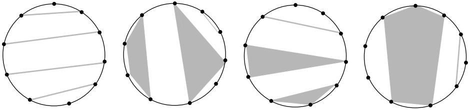

$$\text{CN}(k) \; = \; \frac{1}{k+1}\binom{2k}{k} \; \sim \; \frac{4^k}{\sqrt{\pi}k^{3/2}} \; \leq \; 4^k.$$

The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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1. Relax the connectivity requirement by considering a set of cuts that contain the relevant (connected) solutions.
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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

End of the story?

Do all connectivity problems admit single-exponential algorithms (on general graphs) parameterized by treewidth?

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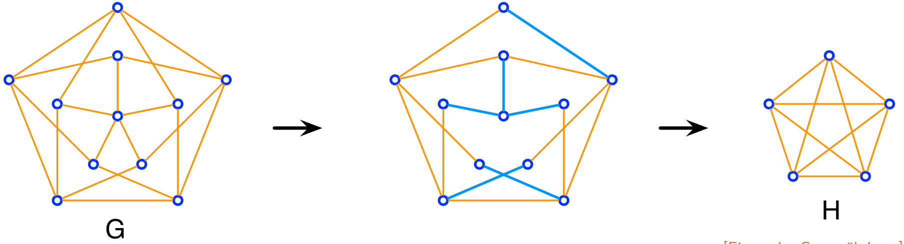
There are other examples of such problems...

Next section is...

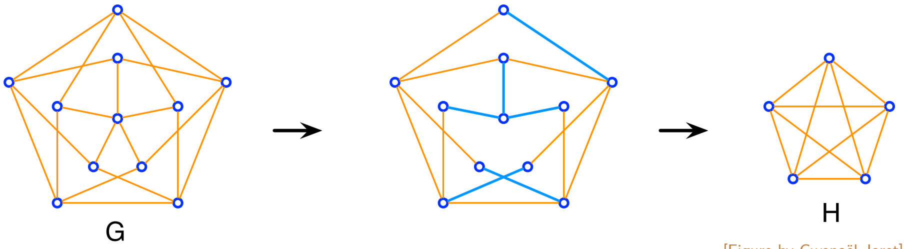
Minors and topological minors



[Figure by Gwenaël Joret]

- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

Minors and topological minors



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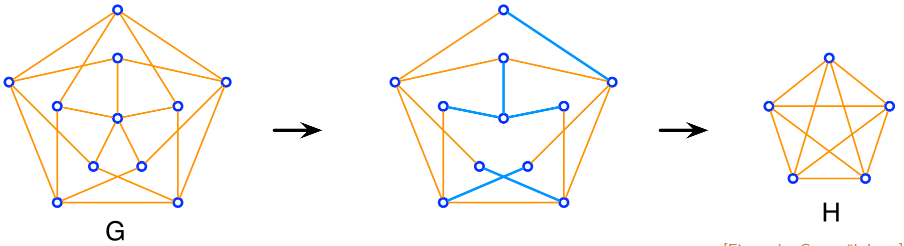

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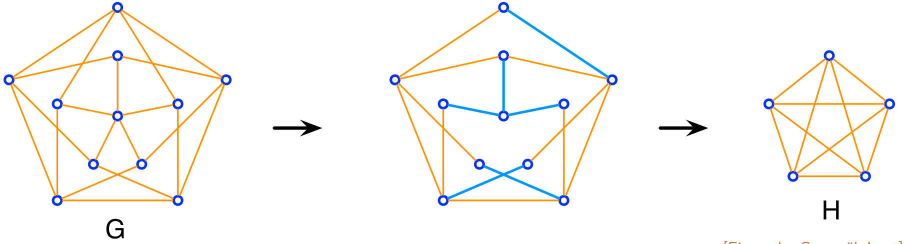

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- Therefore: $\boxed{H \text{ topological minor of } G \not\Leftarrow H \text{ minor of } G}$

The $\mathcal{F}$-M-Deletion problem

Let $\mathcal{F}$ be a fixed finite collection of graphs.

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Covering topological minors

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- $\boxed{\text{Goal}}$ find smallest function $f_{\mathcal{F}}$ s.t. $\exists$ algo in time $f_{\mathcal{F}}(\mathbf{tw}) \cdot n^{\mathcal{O}(1)}$.

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

[1] Connected collection $\mathcal{F}$: all the graphs are connected.
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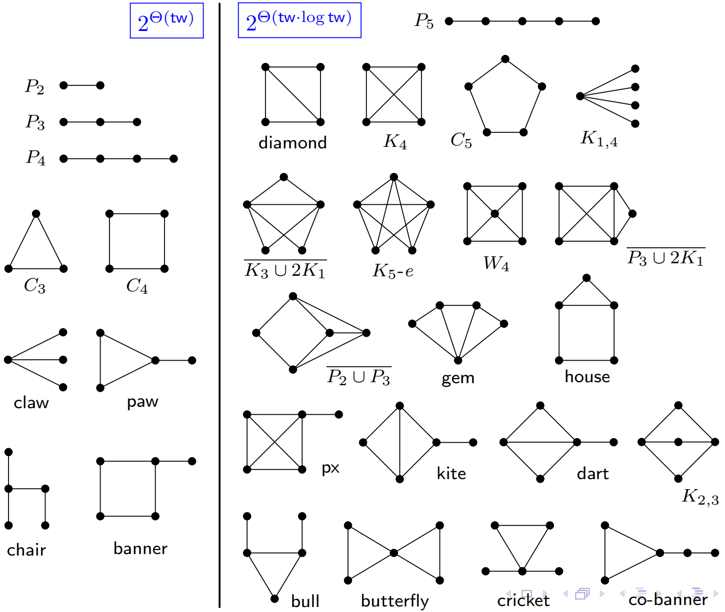
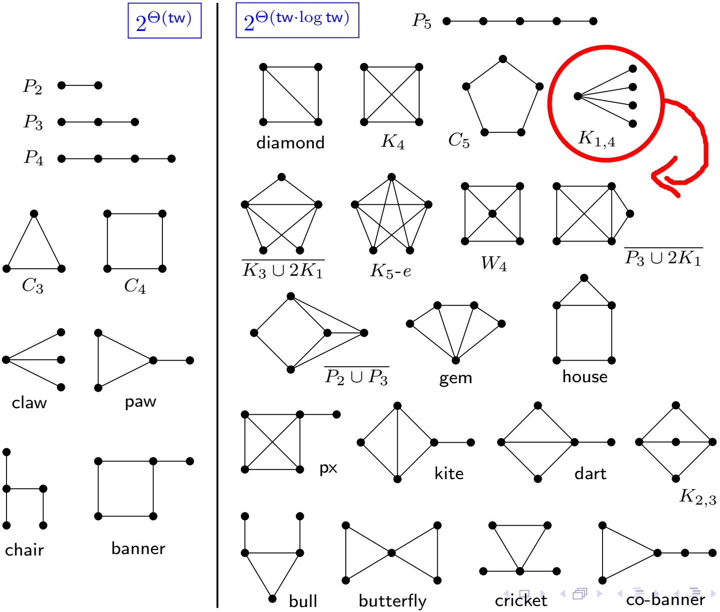
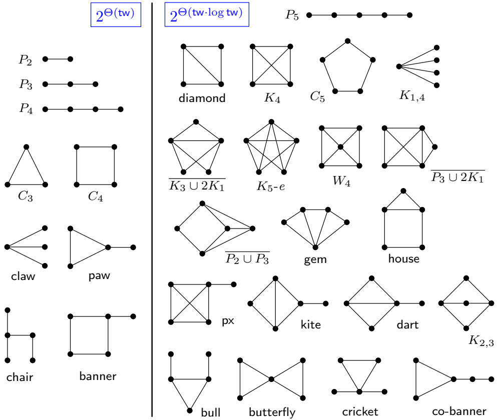
Complexity of hitting small planar minors H



For topological minors, there is only one change

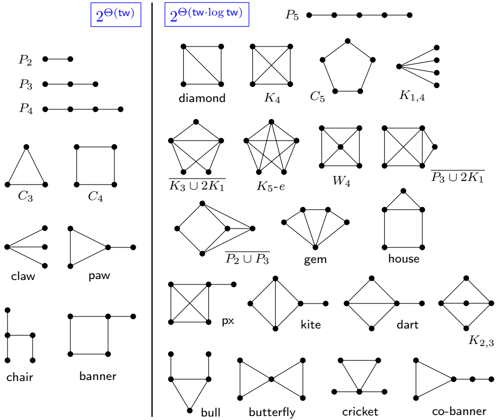


A compact statement for small planar minors



All these cases can be succinctly described as follows:

A compact statement for small planar minors



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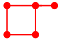
- All the graphs on the left are minors of 

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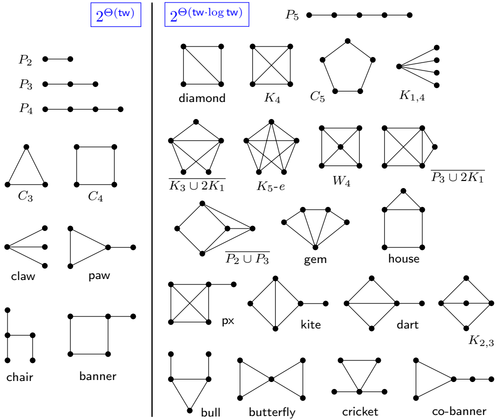

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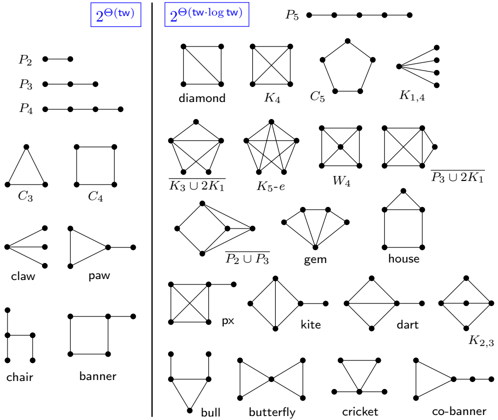
- All the graphs on the right are not minors of 

A compact statement for small planar minors



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- All the graphs on the left are minors of 
- All the graphs on the right are not minors of  ... except P_5.

A dichotomy for hitting connected planar minors

We can prove that any connected planar H with $|V(H)| \geq 6$ is "hard".

A dichotomy for hitting connected planar minors

We can prove that any connected planar H with $|V(H)| \geq 6$ is "hard".

Theorem (Baste, S., Thilikos. 2018)

Let H be a connected planar graph.

A dichotomy for hitting connected planar minors

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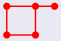
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{m}}$  *and* $H \neq P_5$.

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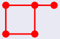
- $2^{\mathcal{O}(\mathsf{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathsf{m}}$  *and* $H \neq P_5$.
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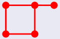
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{m}}$  *and* $H \neq P_5$.
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In both cases, the running time is asymptotically optimal under the ETH.

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     [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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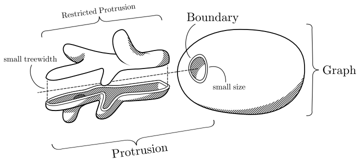
We build on the machinery of boundaried graphs and representatives:



[Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh, Thilikos. 2009]

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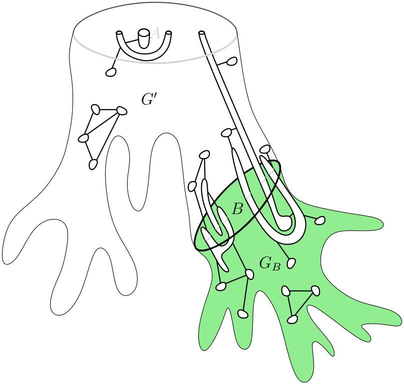

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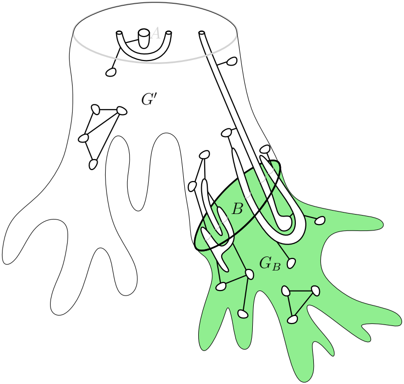

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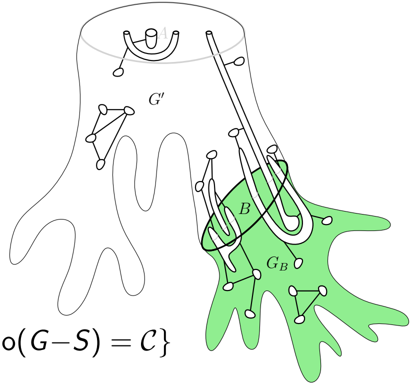

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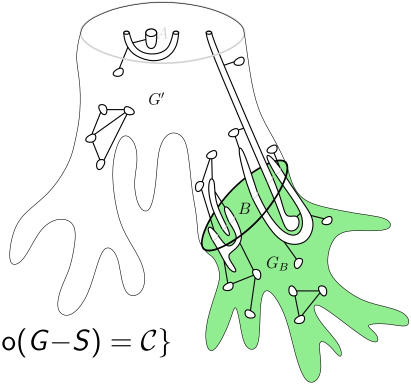

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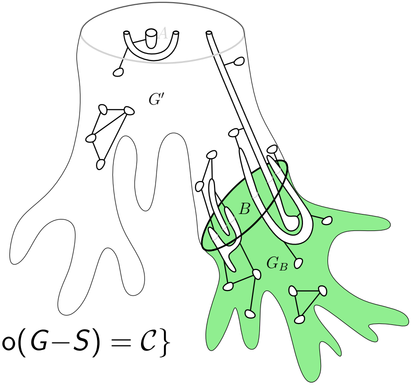

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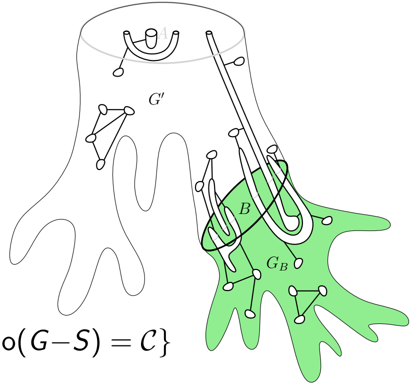

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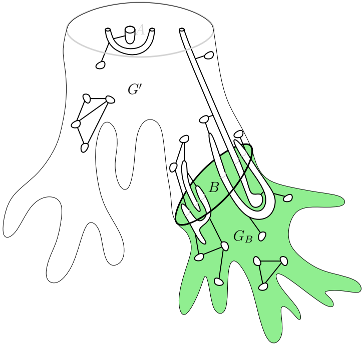

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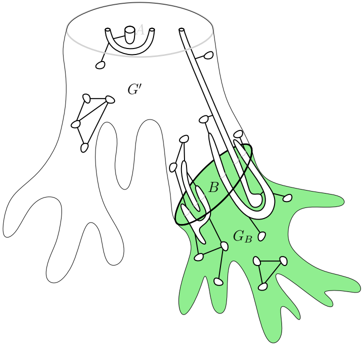

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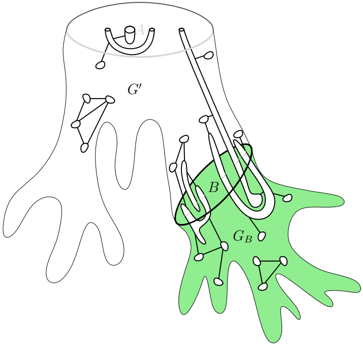

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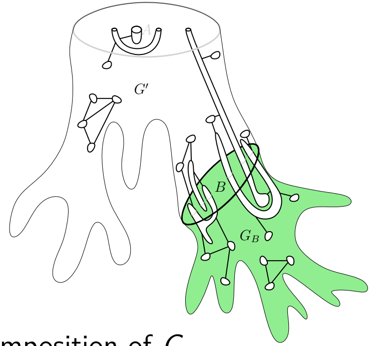

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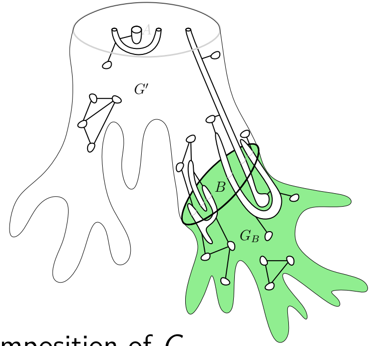

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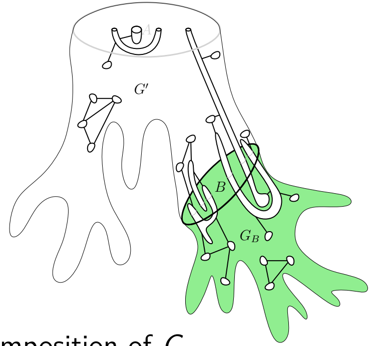

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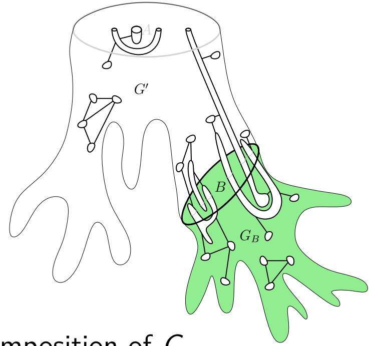

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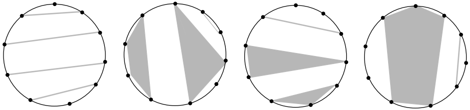

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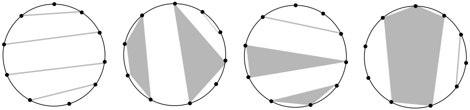

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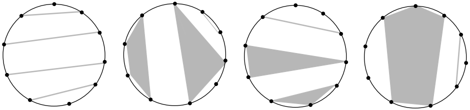

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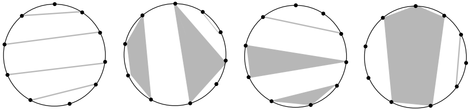

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. [Rué, S., Thilikos. 2014]

Next section is...

1. Introduction
   - Parameterized complexity
   - Treewidth

2. FPT algorithms parameterized by treewidth

3. The $\mathcal{F}$-Deletion problem

4. Conclusions

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  We think that $\{K_5\}$-Deletion is solvable in time $2^{\Theta(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$.
- We do not even know if there exists some $\mathcal{F}$ such that $\mathcal{F}$-Deletion cannot be solved in time $2^{o(\mathrm{tw}^2)} \cdot n^{\mathcal{O}(1)}$ under the ETH.

  Deletion to genus at most g: $2^{\mathcal{O}_g(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$. [Kociumaka, Pilipczuk. 2017]
- $\boxed{\text{Conjecture}}$ For every connected family $\mathcal{F}$, the $\mathcal{F}$-Deletion problem is solvable in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$.
- Consider families $\mathcal{F}$ containing disconnected graphs.

Gràcies!

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