Efficient algorithms parameterized by treewidth

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Outline of the talk

1. Introduction
   - Parameterized complexity
   - Treewidth
2. FPT algorithms parameterized by treewidth
3. The $\mathcal{F}$-Deletion problem

Next section is...

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Next subsection is...

Crucial notion in complexity theory: NP-completeness

- Cook-Levin Theorem (1971): the SAT problem is NP-complete.
- Karp (1972): list of 21 *important* NP-complete problems.
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- Nowadays, literally thousands of problems are known to be NP-hard:
  unless $\mathsf{P} = \mathsf{NP}$, they cannot be solved in polynomial time.
- But what does it mean for a problem to be NP-hard?

  No algorithm solves all instances optimally in polynomial time.

Are all instances really hard to solve?

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- VLSI design: the number of circuit layers is usually ≤ 10.
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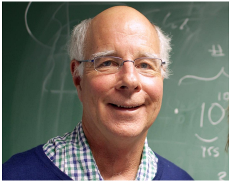
Message In many applications, not only the total size of the instance matters, but also the value of an additional parameter.

The area of parameterized complexity

Idea Measure the complexity of an algorithm in terms of the input size and an additional integer parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established area with hundreds of articles published every year in the most prestigious TCS journals and conferences.

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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

1. k-Vertex Cover: solvable in time $2^k \cdot n^2$
2. k-Clique: solvable in time $k^2 \cdot n^k$
3. Vertex k-Coloring: NP-hard for every fixed $k \geq 3$

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   $\boxed{\text{The problem is para-NP-hard}}$

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Working hypothesis of parameterized complexity: $\boxed{k\text{-Clique is not FPT}}$

(in classical complexity: 3-SAT cannot be solved in poly-time)

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Do all FPT problems admit polynomial kernels? $\boxed{\text{NO!}}$

Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)

Deciding whether a graph has a PATH *with* $\geq k$ *vertices* is FPT *but does not admit a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

Typical approach to deal with a parameterized problem

Parameterized problem L

k-Clique

k-Vertex Cover

k-Path

Vertex k-Coloring

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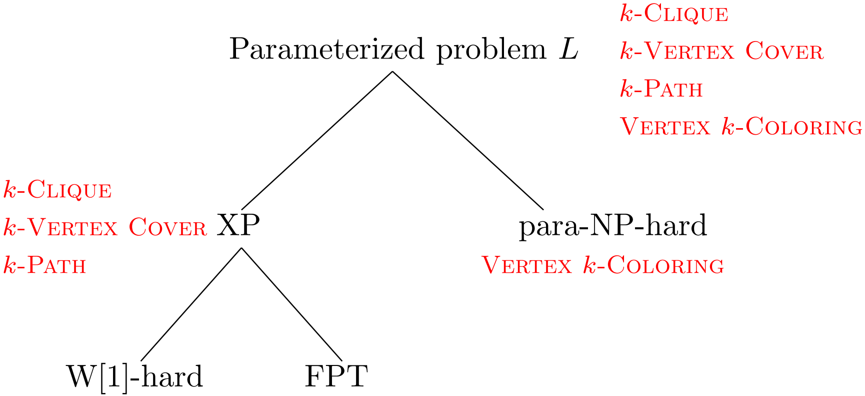

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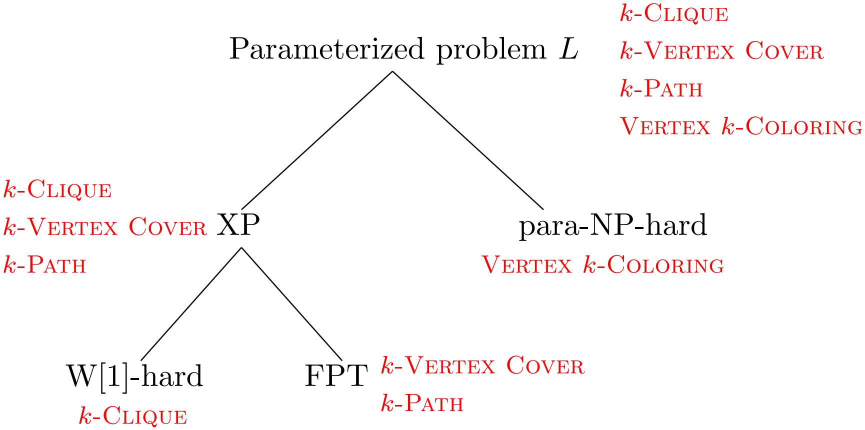

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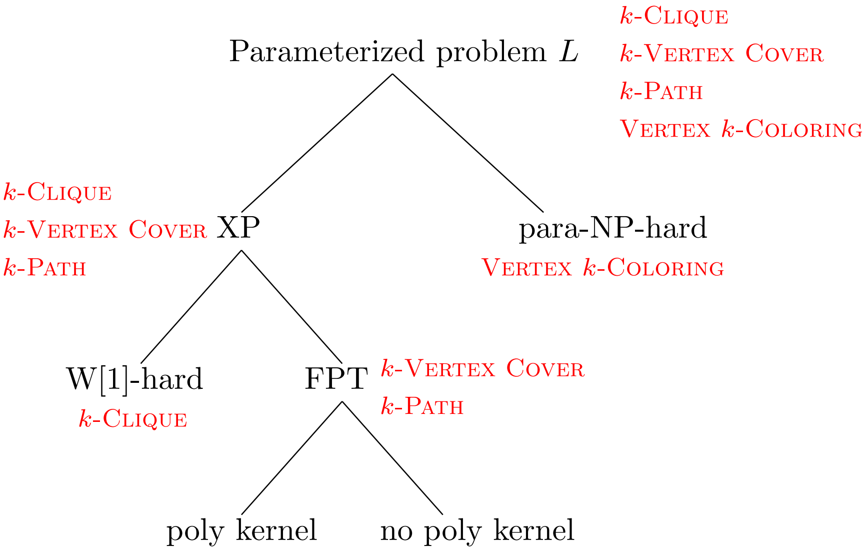

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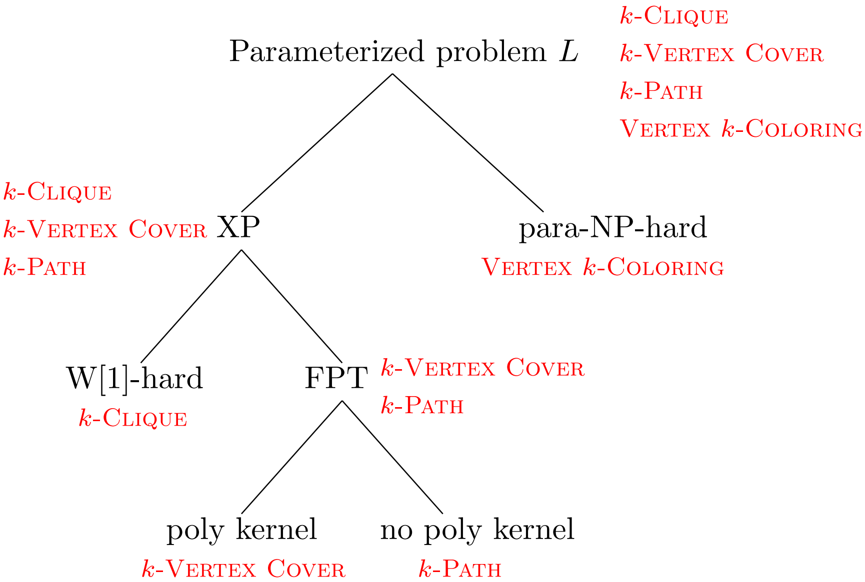

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Treewidth via k-trees

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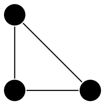

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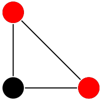

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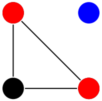

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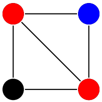

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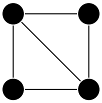

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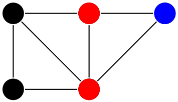

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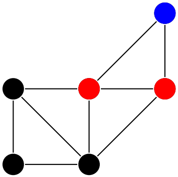

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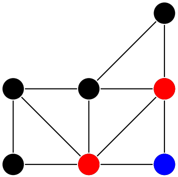

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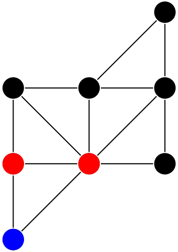

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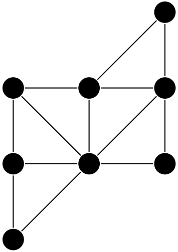

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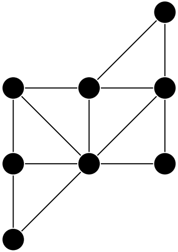

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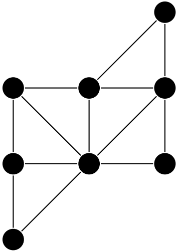

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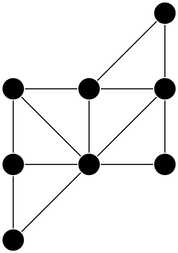

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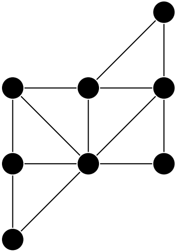

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{B_t \mid t \in V(T)\})$, where
  T is a tree, and
  $B_t \subseteq V(G) \;\; \forall t \in V(T)$ (bags),

  satisfying the following:

  - $\bigcup_{t \in V(T)} B_t = V(G)$,
  - $\forall \{u, v\} \in E(G)$, $\exists t \in V(T)$ with $\{u, v\} \subseteq B_t$.
  - $\forall v \in V(G)$, bags containing v define a connected subtree of T.

- Width of a tree decomposition:
  $\max_{t \in V(T)} |B_t| - 1$.

- Treewidth of a graph G:
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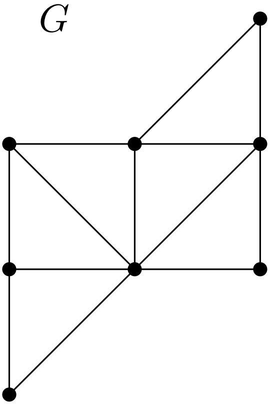

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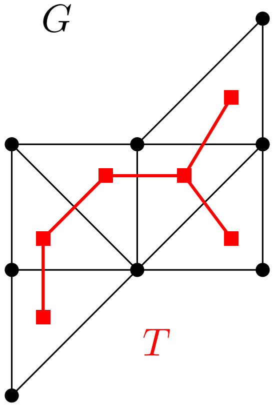

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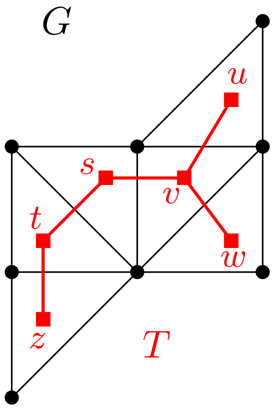

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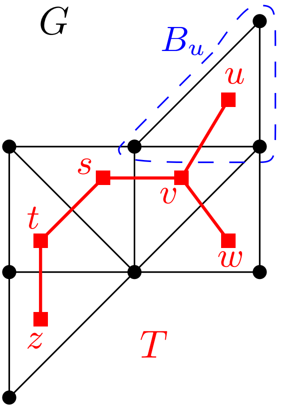

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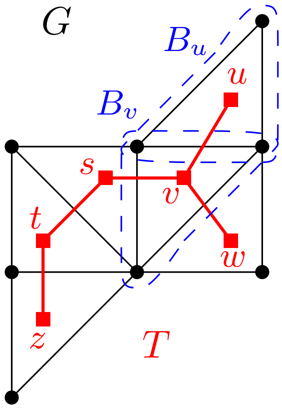

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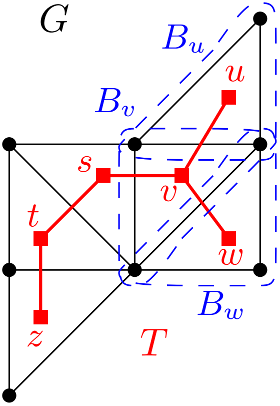

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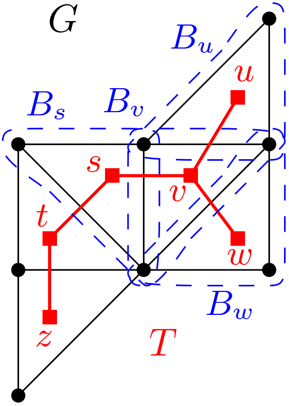

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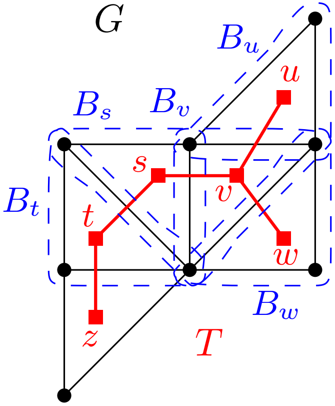

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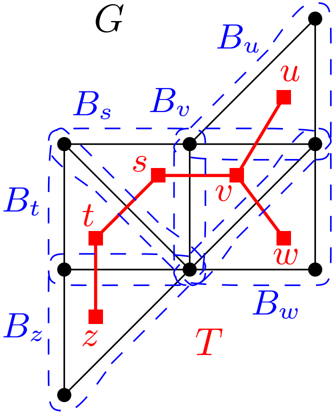

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2. Treewidth behaves very well algorithmically, and algorithms parameterized by treewidth appear very often in FPT algorithms.
3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

Next section is...

1. Introduction
   - Parameterized complexity
   - Treewidth

2. FPT algorithms parameterized by treewidth

3. The $\mathcal{F}$-Deletion problem

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   Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

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Typical statements:

ETH $\Rightarrow$ k-Vertex Cover cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$.

ETH $\Rightarrow$ Planar k-Vertex Cover cannot in time $2^{o(\sqrt{k})} \cdot n^{O(1)}$.

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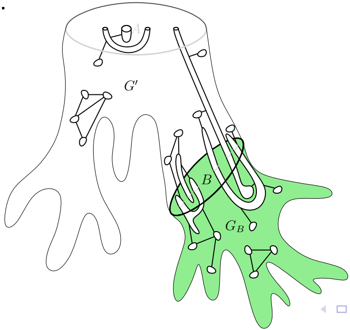
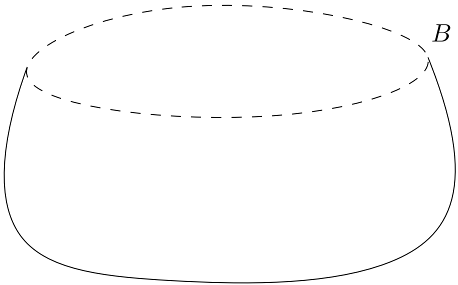
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Two behaviors for problems parameterized by treewidth

Local problems: Vertex Cover, Dominating Set, Clique, Independent Set, q-Coloring for fixed q.



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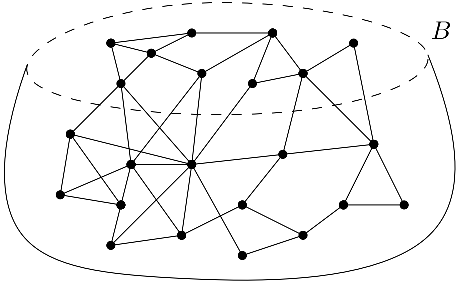

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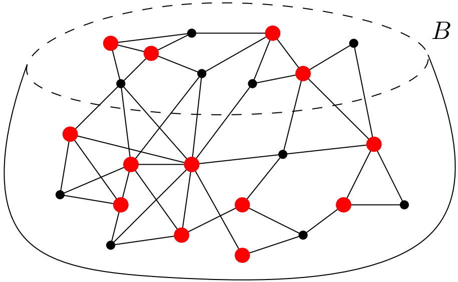

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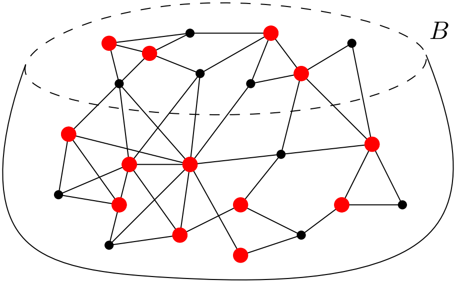

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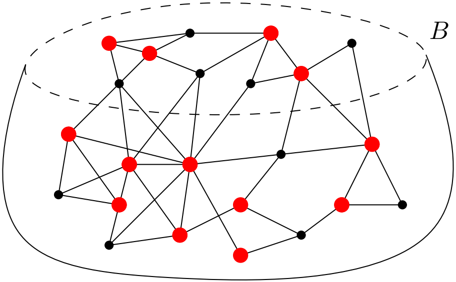

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- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

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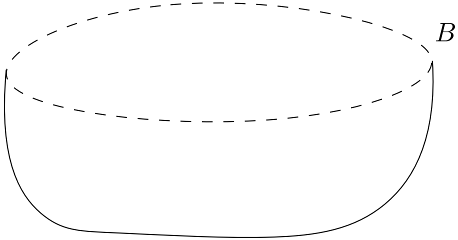
$\boxed{\text{Connectivity problems}}$ Hamiltonian Cycle, Longest Path, Steiner Tree, Connected Vertex Cover.



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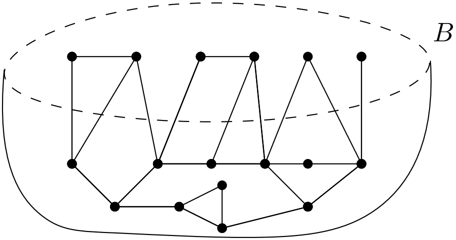

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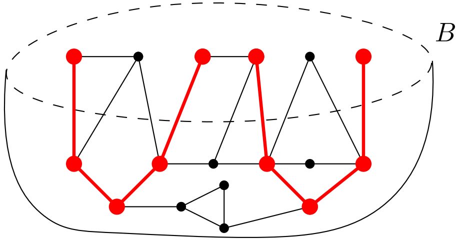

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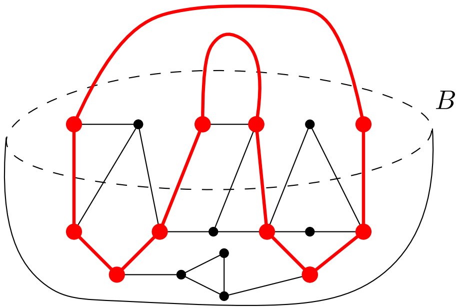

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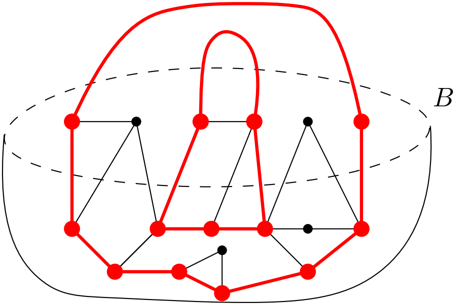

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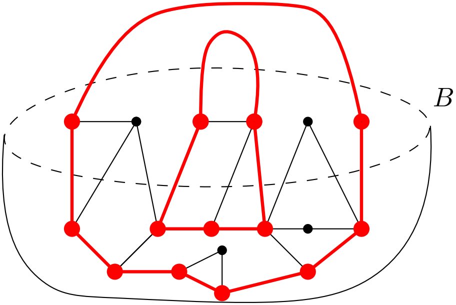

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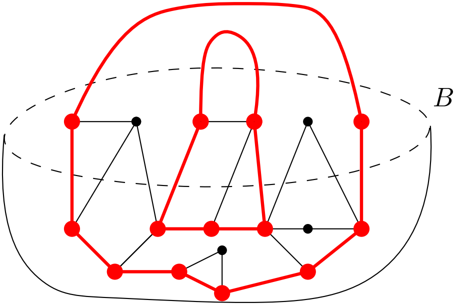

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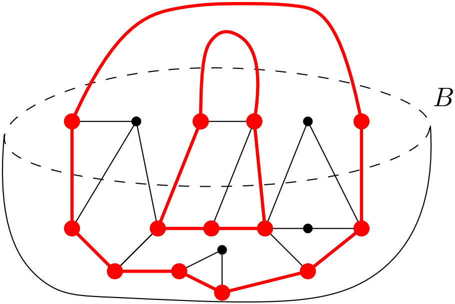

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Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

  $$\boxed{2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

  Vertex Cover, Dominating Set, ...

- Connectivity problems:

  $$\boxed{2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

  Longest Path, Steiner Tree, ...

The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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There are other examples of such problems...

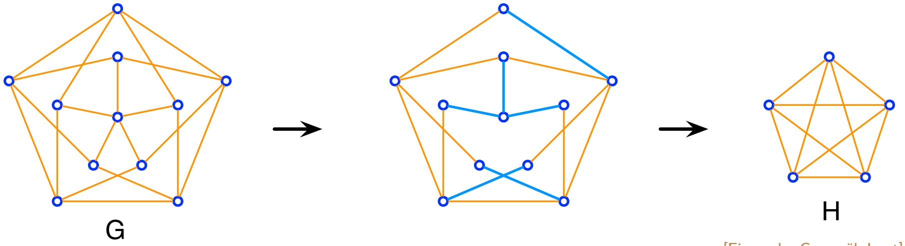
Next section is...

1. Introduction
   - Parameterized complexity
   - Treewidth

2. FPT algorithms parameterized by treewidth

3. The $\mathcal{F}$-DELETION problem

Minors and topological minors



[Figure by Gwenaël Joret]

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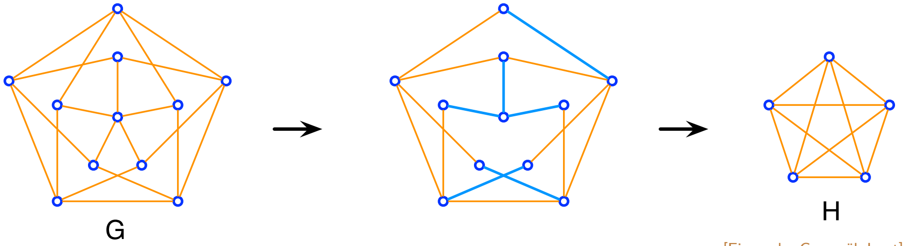
Minors and topological minors



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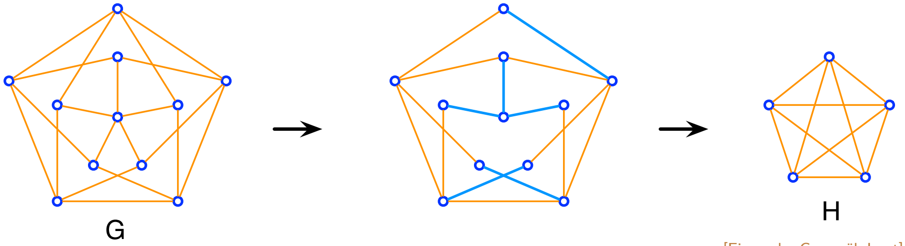

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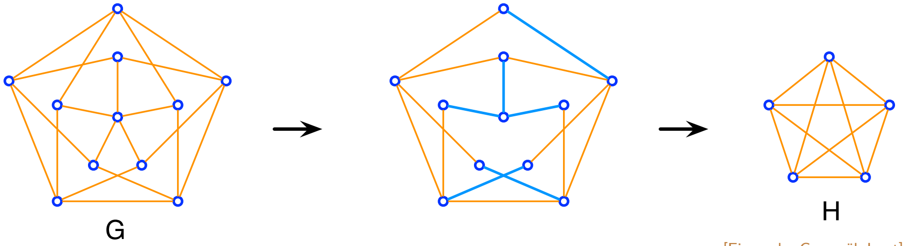

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The $\mathcal{F}$-M-DELETION problem

Let $\mathcal{F}$ be a fixed finite collection of graphs.

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FPT by Courcelle's Theorem.

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-Deletion/$\mathcal{F}$-TM-Deletion can be solved in time

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- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

Summary of our results (Baste, S., Thilikos)

[1] Connected collection $\mathcal{F}$: all the graphs are connected.

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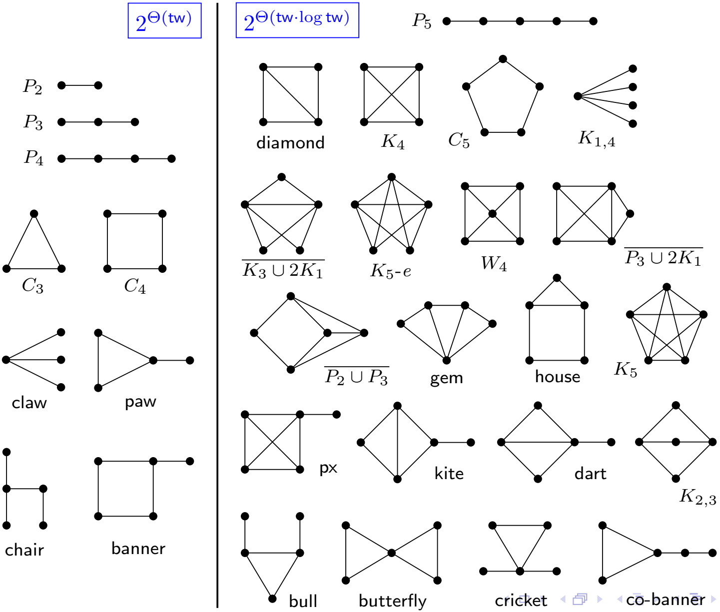
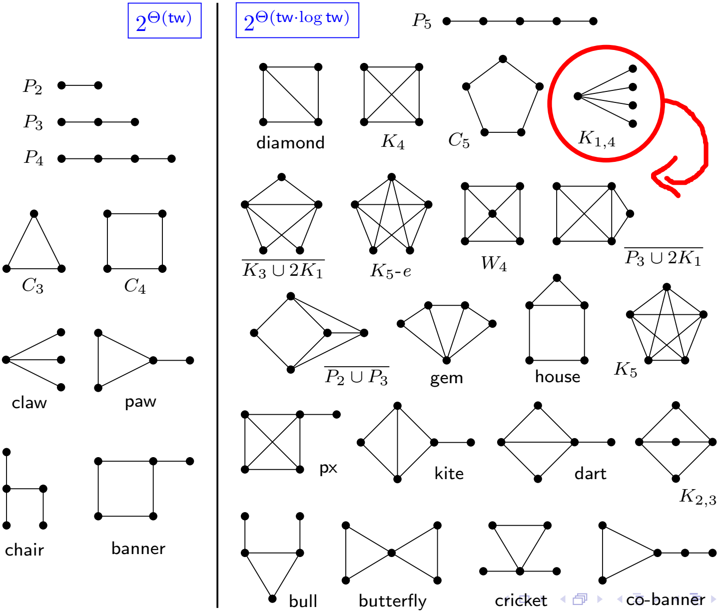
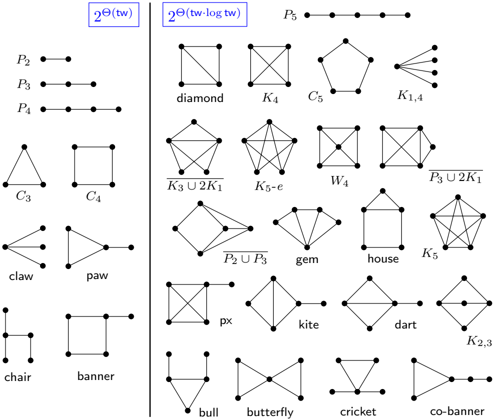
Complexity of hitting a single connected minor H



For topological minors, there is (at least) one change



A compact statement for a single connected graph



All these cases can be succinctly described as follows:

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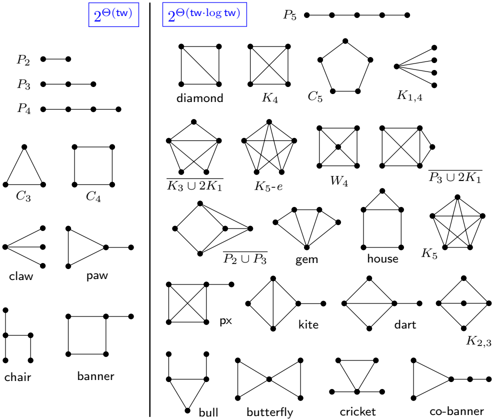
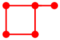


All these cases can be succinctly described as follows:

- All the graphs on the left are minors of  (called the banner)

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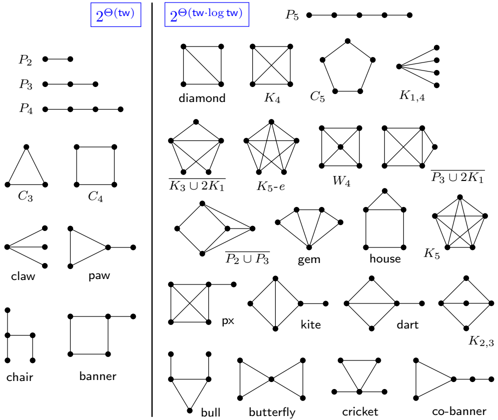

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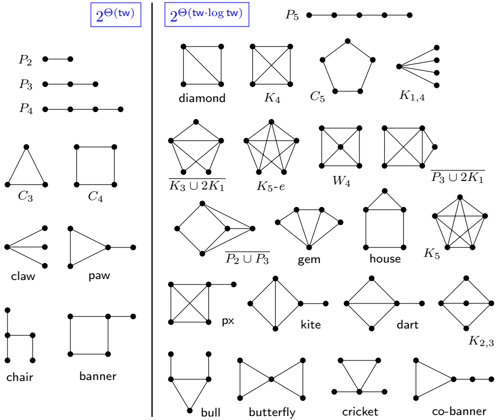
- All the graphs on the right are not minors of 

A compact statement for a single connected graph



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- All the graphs on the right are not minors of  ... except P_5.

A dichotomy for hitting connected minors

We can prove that any connected H with $|V(H)| \geq 6$ is $\boxed{\text{hard}}$:
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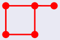
- $2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\text{m}}$  *and* $H \neq P_5$.

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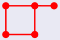
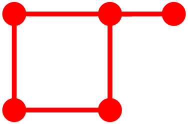
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{m}}$  *and* $H \neq P_5$.
- $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

In both cases, the running time is asymptotically optimal under the ETH.

skip

Why the banner??



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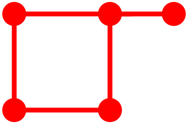

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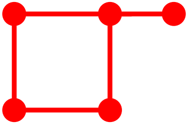

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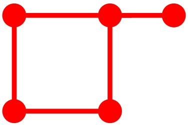

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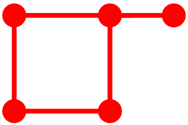

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- If the characterization of the allowed connected components is enriched in some way, such as restricting the length of the allowed cycles or forbidding certain degrees, the problem becomes harder.

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3. Lower bounds under the ETH

   - $2^{o(\mathrm{tw})}$ is "easy".
   - $2^{o(\mathrm{tw} \cdot \log \mathrm{tw})}$ is much more involved and we get ideas from:

   [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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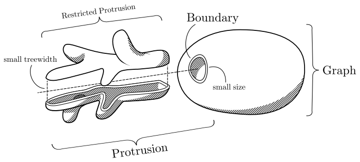

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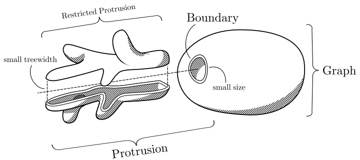

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  Extra: Bidimensionality, irrelevant vertices, protrusion decompositions...

skip

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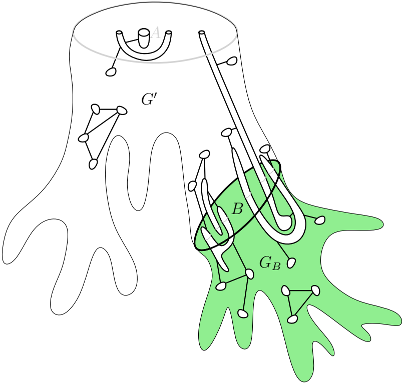

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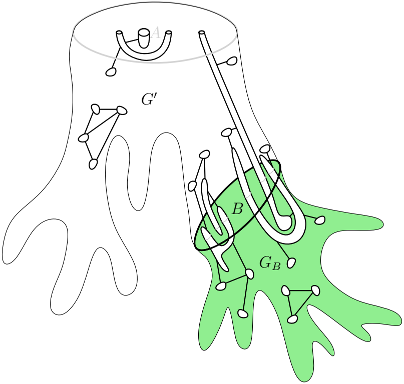

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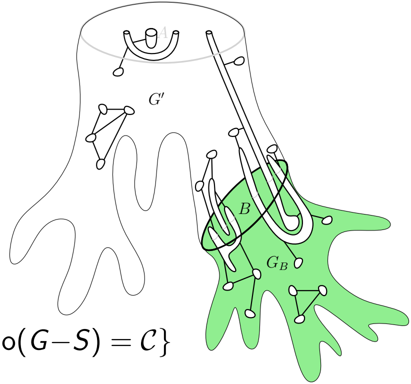

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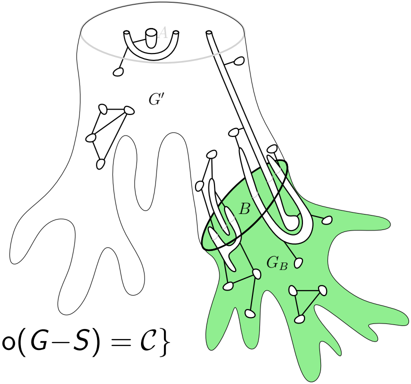

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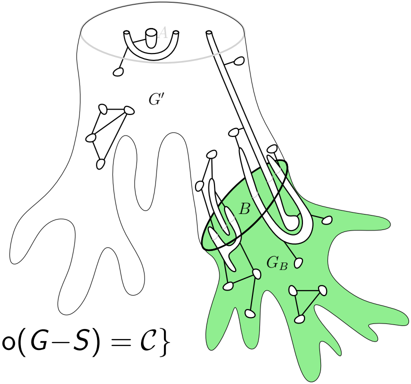

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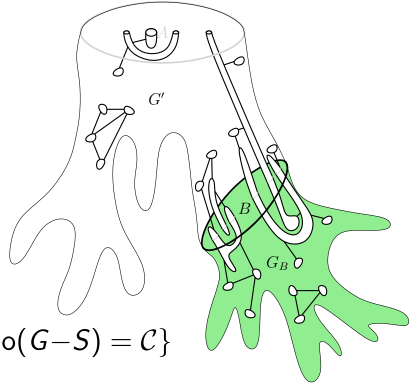

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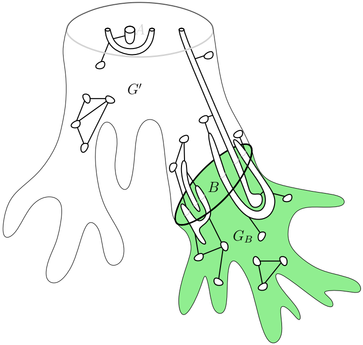

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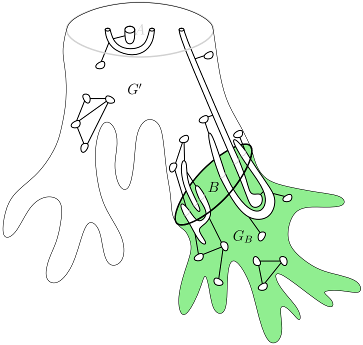

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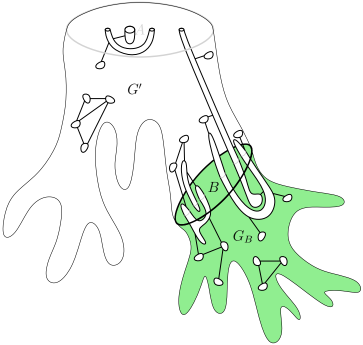

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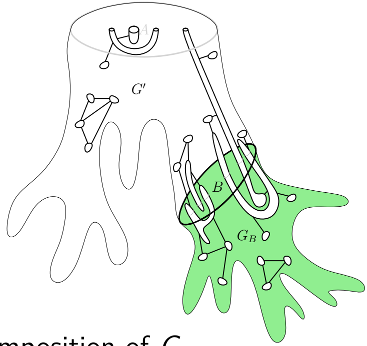

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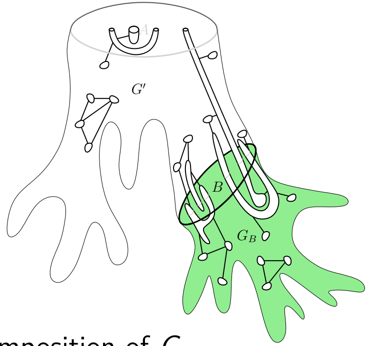

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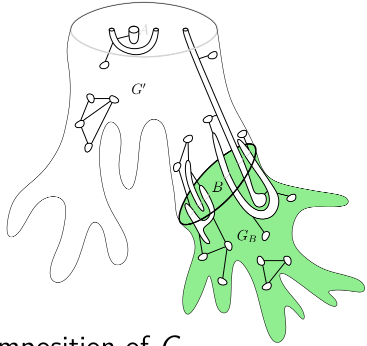

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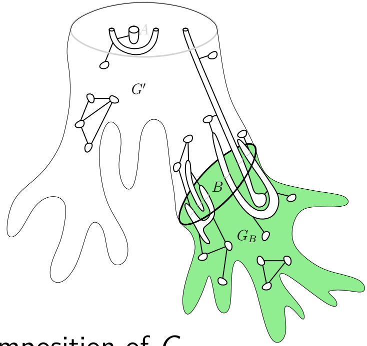

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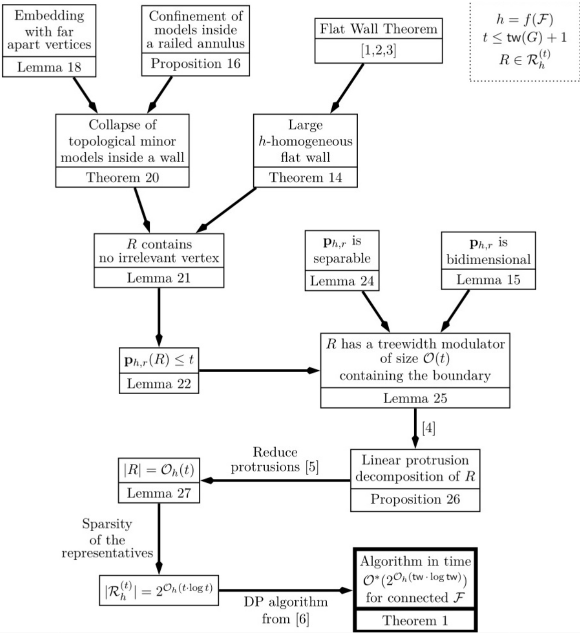

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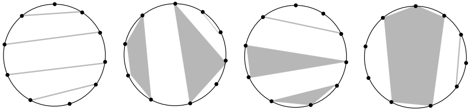

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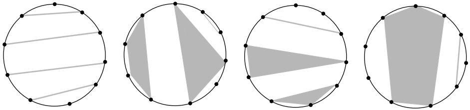

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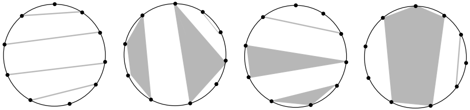

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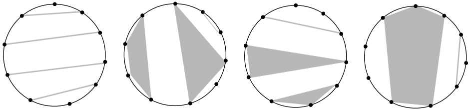

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. [Rué, S., Thilikos. 2014]

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Gràcies!