Compound Logics for Modification Problems

Fedor V. Fomin, Petr A. Golovach, **Ignasi Sau**,
Giannos Stamoulis, and Dimitrios M. Thilikos

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Thanks Dimitrios for most of the slides!!

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Let $\mathcal{M}$ be a set of allowed graph modification operations
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Input: A graph G and an integer k ("amount of modification").
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Want: algorithms in time $\boxed{f(\varphi) \cdot n^{\mathcal{O}(1)}}$, where $n = |V(G)|$.

Algorithmic Meta-Theorems (AMTs)

For some logic L *and some class* $\mathcal{C}$ *of combinatorial structures, every algorithmic problem* Π *that is expressible in* L, *there is an* efficient *algorithm solving* Π *for inputs that belong in* $\mathcal{C}$.

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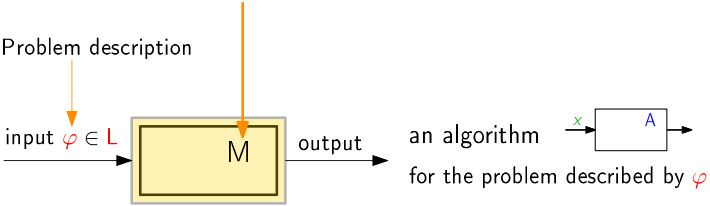

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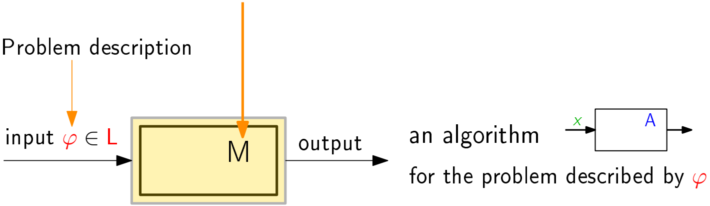
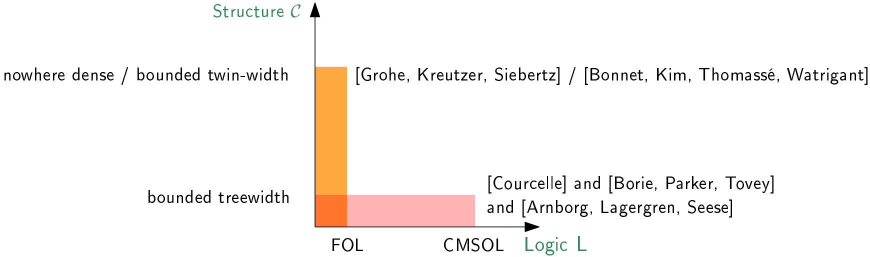


Two main logics for φ:

- FOL: First Order Logic
  - ▶ quantification on vertices or edges
- CMSOL: Counting Monadic Second Order Logic
  - ▶ quantification on sets of vertices or edges

Famous AMTs for model-checking in time FPT



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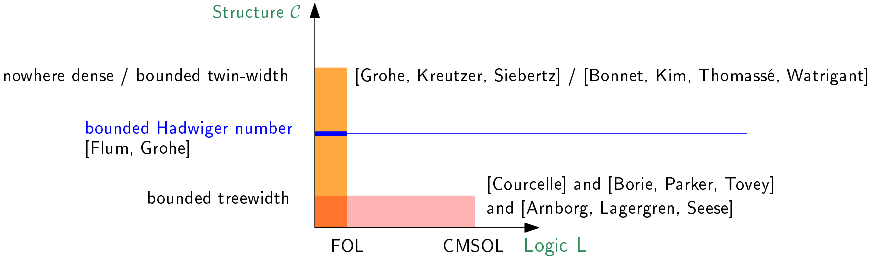

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Hadwiger number: $\mathbf{hw}(G) =$ max clique-minor of the graph G

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Modulator: $X = \{x_1, \ldots, x_k\}$

Target property: minor-exclusion of $\mathcal{H} = \{K_5, K_{3,3}\}$

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Topological minor exclusion:

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Vertex Deletion to Planarity + more
Given G and k, is there an $X \subseteq V(G)^{\leq k}$ such that $G \setminus X$ is planar+more?

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- Extensions to general minor-closed target classes $\mathcal{G}$?

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- More general modification operations do not seem to be captured...

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- Can we define successive target properties?
- Hierarchical clustering?
- Multi-level modification?
- Consider different modification scenarios?
- We may demand target conditions to be satisfied by the connected components (or even the blocks) of $G \setminus X$ (CMSOL-demand).
- Multiway Cut or Multicut to some target property $\mathcal{G}$.
- We may demand vertex/edge removals with prescribed adjacencies.
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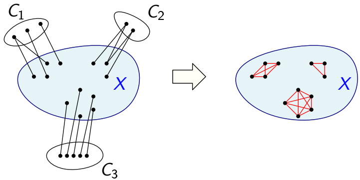

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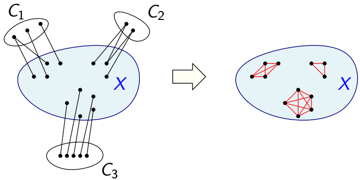

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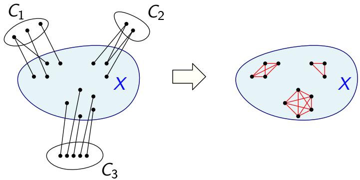

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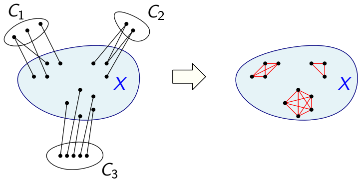

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- **p=bridge-depth**: $\mathcal{G}$-bridge-depth:

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- or just $G[X] \models \beta_k$ for some $\beta_k \in \mathsf{CMSOL}^{\mathsf{tw}}$?

  - $\mathsf{CMSOL}^{\mathsf{tw}}[\mathtt{E}, \mathtt{X}]$ (on annotated graphs): every $\beta \in \mathsf{CMSOL}[\mathtt{E}, \mathtt{X}]$ for which there exists some c_β such that the torsos of all the models of β have treewidth at most c_β.

Is there **one** meta-theorem that deals with **all** these cases?

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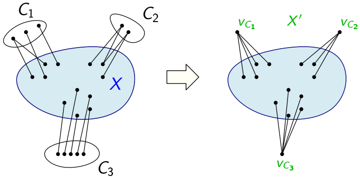

Compound logic We define $\beta \triangleright \gamma$ so that

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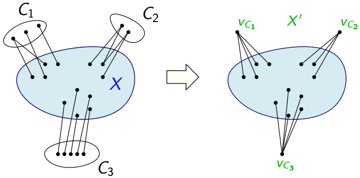

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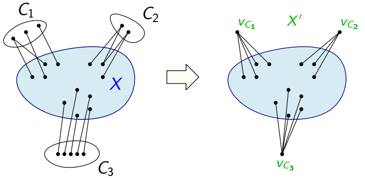

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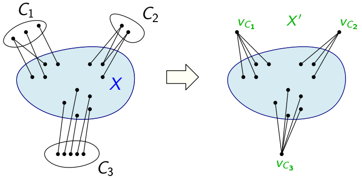

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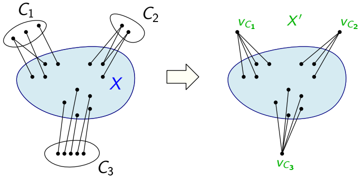

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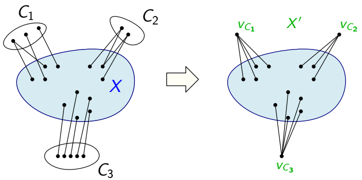

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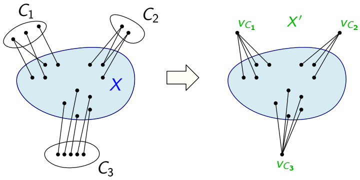

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Theorem (our result, in a less simple form)

For every $\beta \in \text{CMSOL}^{\text{tw}}$ and every $\gamma \in \Theta_0^{(\text{c})}$, there is an algorithm deciding $\text{Mod}(\beta \triangleright \gamma)$ in quadratic time.

- for $\varphi \in \text{CMSOL}$, define $\varphi^{(\text{c})}$: $G \models \varphi^{(\text{c})}$ if $\forall C \in \text{cc}(G), C \models \varphi$.
- for $\mathsf{L} \subseteq \text{CMSOL}$, define $\mathsf{L}^{(\text{c})} = \mathsf{L} \cup \{\varphi^{(\text{c})} \mid \varphi \in \mathsf{L}\}$.

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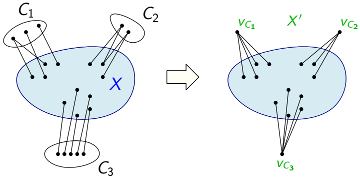

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Theorem (our result, in a simple form)

For every $\beta \in \mathsf{CMSOL}^{\mathsf{tw}}$ and every $\gamma \in \mathbf{MB}(\Theta_0^{(c)})$, there is an algorithm deciding $\mathrm{Mod}(\beta \triangleright \gamma)$ in quadratic time.

- $\mathbf{MB}(\mathsf{L})$: all monotone Boolean combinations of sentences in L.

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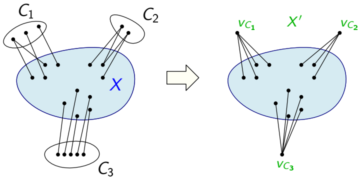

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*For every $\beta \in \text{CMSOL}^{\text{tw}}$ and every $\gamma \in$ **MB**$(\Theta_0^{(c)})$, there is an algorithm deciding $\text{Mod}(\beta \triangleright \gamma)$ in quadratic time.*

- This automatically implies algorithms in all aforementioned directions, beyond the applicability of the theorems of Courcelle and Grohe and Flum.

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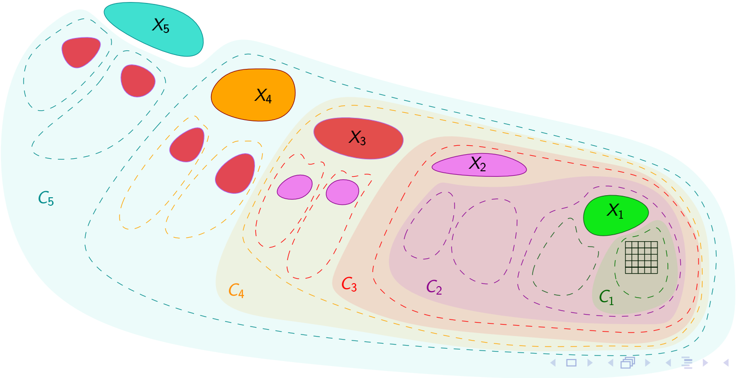

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Our results are **constructive**:

Theorem

There is a **Meta-Algorithm M** *that,*
with input a sentence $\theta \in \Theta$ and an upper bound c_θ on $\mathbf{hw}(\mathrm{Mod}(\theta))$,
returns as output the algorithm $\mathbf{A}_\theta$.

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For every $\tilde{\theta} \in \tilde{\Theta}$, there is an algorithm deciding $\mathrm{Mod}(\tilde{\theta})$ in quadratic time on graphs of fixed Hadwiger number.

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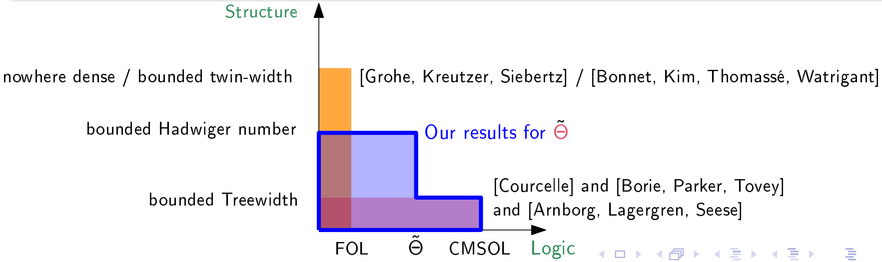

Generalization to extensions of FOL

First-Order Logic with Connectivity Operators

[Schirrmacher, Siebertz, Vigny, CSL 2022] + [Bojańczyk, 2021]
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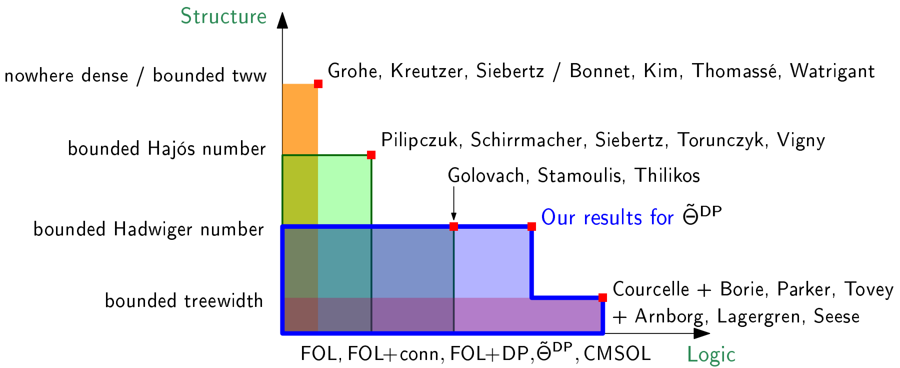
The current meta-algorithmic landscape



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- The combinatorial/algorithmic results in

1. Ken-ichi Kawarabayashi, Robin Thomas, and Paul Wollan. A new proof of the flat wall theorem. *Journal of Combinatorial Theory, Series B*, 129:204–238, 2018.
2. Ignasi Sau, Giannos Stamoulis, and Dimitrios M. Thilikos. A more accurate view of the Flat Wall Theorem, 2021. arXiv:2102.06463.
3. Petr A. Golovach, Giannos Stamoulis, and Dimitrios M. Thilikos. Hitting topological minor models in planar graphs is fixed parameter tractable. In *Proc. of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms, (SODA)*, pages 931–950, 2020.
4. Julien Baste, Ignasi Sau, and Dimitrios M. Thilikos. A complexity dichotomy for hitting connected minors on bounded treewidth graphs: the chair and the banner draw the boundary. In *Proc. of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 951–970, 2020.
5. Ignasi Sau, Giannos Stamoulis, and Dimitrios M. Thilikos. k-apices of minor-closed graph classes. I. Bounding the obstructions. Transactions on Algorithms 2022.
6. Fedor V. Fomin, Petr A. Golovach, Giannos Stamoulis, and Dimitrios M. Thilikos. An algorithmic meta-theorem for graph modification to planarity and FOL. In *Proc. of the 28th Annual European Symposium on Algorithms (ESA)*, volume 173 of *LIPIcs*, pages 51:1–51:17, 2020.

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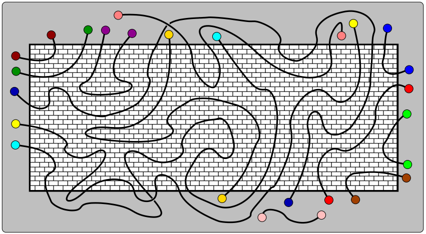
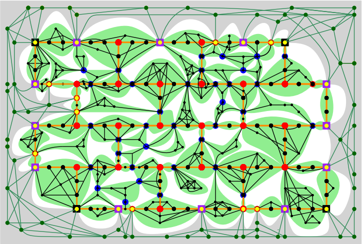
Irrelevant Vertex Technique

($>$ 1200 citations and used in $>$ 120 papers)

- Neil Robertson and Paul D. Seymour. Graph minors. XIII. The disjoint paths problem. *Journal of Comb. Theory, Ser. B*, 63(1):65–110, 1995.





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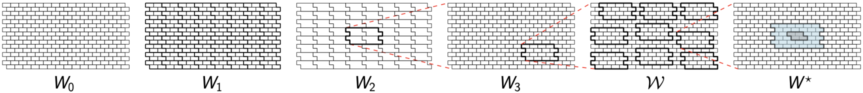

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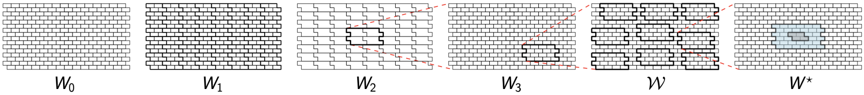
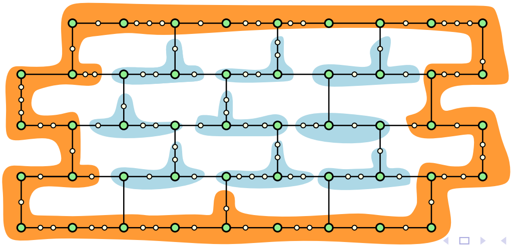


Crucial fact: the fact that the modulator sentence $\beta \in \text{CMSOL}^{\text{tw}}$ allows to prove that the removal of the modulator X does not destroy a flat wall too much.



Necessity of the ingredients of our logic

Theorem (our result, in its simplest form)

For every $\beta \in \mathsf{CMSOL}^{\mathsf{tw}}$ *and every* $\gamma \in \Theta_0$, *there is an algorithm deciding* $\mathrm{Mod}(\beta \triangleright \gamma)$ *in quadratic time.*

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- $\mathsf{CMSOL}^{\mathsf{tw}}[\mathtt{E}, \mathtt{X}]$ (on annotated graphs): every $\beta \in \mathsf{CMSOL}[\mathtt{E}, \mathtt{X}]$ for which there exists some c_β such that the torsos of all the models of β have treewidth at most c_β.
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1. Why bounded treewidth of torso of modulator? $\beta \in \mathsf{CMSOL}^{\mathsf{tw}}$.
2. Why the target sentence $\sigma \in \mathsf{FOL}$ (or extensions)?
3. Why the target sentence μ expresses minor-exclusion?

skip

- $\mathsf{CMSOL}^{\mathsf{tw}}[\mathtt{E},\mathtt{X}]$: every $\beta \in \mathsf{CMSOL}[\mathtt{E},\mathtt{X}]$ for which $\exists c_\beta$ such that the torsos of all the models of β have treewidth at most c_β.
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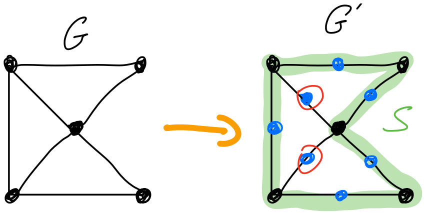

- G Hamiltonian $\Leftrightarrow$ G' has a vertex set S such that $G'[S]$ is a cycle and $G' \setminus S$ is edgeless.
- $\text{tw}(G'[S]) = 2$ but $\text{tw}(\text{torso}(G', S)) = \text{tw}(G)$ unbounded.

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Expressing whether a graph G contains a clique on k vertices is FOL-expressible, while k-Clique is W[1]-hard on general graphs
(again, consider a void modulator).

Some final remarks

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- Limitations
  - are torsos really necessary?
  - which are the optimal combinatorial assumptions on FOL+CMSOL?
- Extensions
  - irrelevant friendliness (bipartiteness)
  - other modification operations (blocks, contractions, ...)
- Open problems
  - constants hidden in $\mathcal{O}_{|\theta|}(n^2)$
  - is the Θ-hierarchy proper?
  - Is quadratic time improvable?
  - Further than minor-exclusion?