# Graph Partitioning and Traffic Grooming with Bounded Degree Request Graph

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# Outline of the talk

1. Traffic grooming
2. Statement of the problem
3. The parameter $M(C, \Delta)$
4. Previous work (Muñoz and S., WG 2008)
5. Our results
   - Case $\Delta = 3$, $C = 4$
   - Case $\Delta \geq 4$ even
   - Case $\Delta \geq 5$ odd
   - Improved lower bound when $\Delta \equiv C \pmod{2C}$
6. Conclusions

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# Introduction

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s
- Idea:
  Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  $\longrightarrow$ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)
- Objectives:
  - Better use of bandwidth
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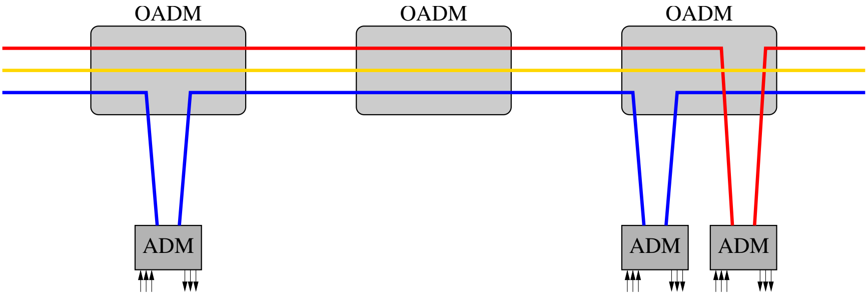
- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength

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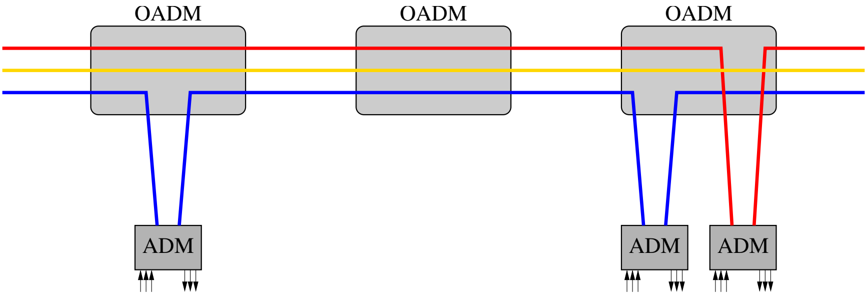


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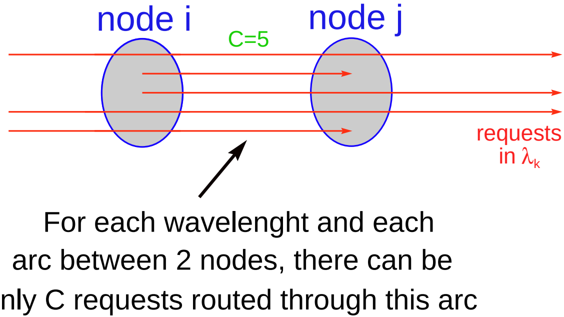


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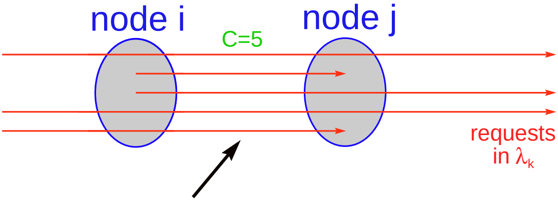


For each wavelenght and each arc between 2 nodes, there can be only C requests routed through this arc

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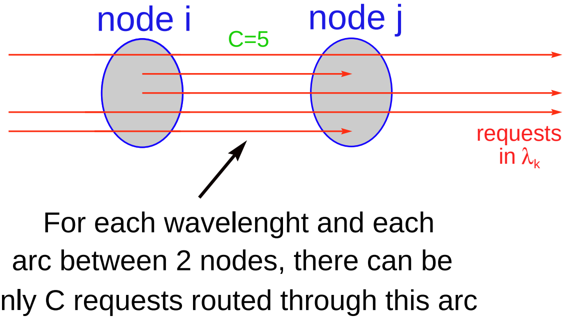


- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

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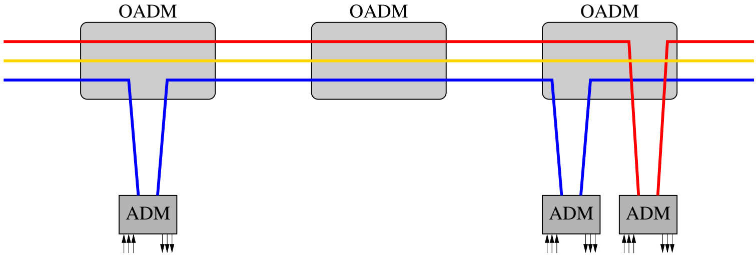


- **Idea:** Use an **ADM only at the endpoints of a request** (lightpaths) in order to save as many ADMs as possible

# To fix ideas...

- Model:

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| Topology | $\longrightarrow$ | graph $G$ |
| Request set | $\longrightarrow$ | graph $R$ |
| Grooming factor | $\longrightarrow$ | integer $C$ |
| Requests in a **wavelength** | $\longrightarrow$ | edges in a **subgraph** of $R$ |
| ADM in a wavelength | $\longrightarrow$ | node in a subgraph of $R$ |

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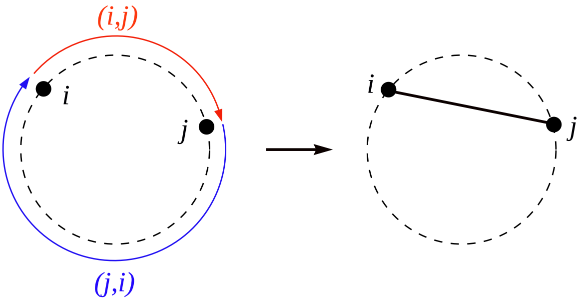
# Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: whenever there is the request $(i, j)$, there is also the request $(j, i)$.




- W.l.o.g. requests $(i, j)$ and $(j, i)$ are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ **grooming factor** $C \Leftrightarrow$ **each subgraph has** $\leq C$ **edges.**

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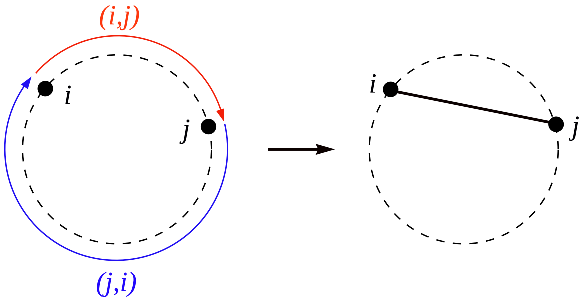


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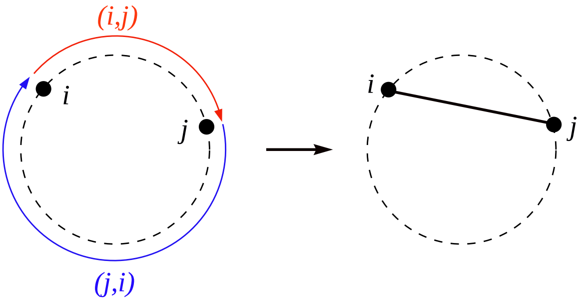


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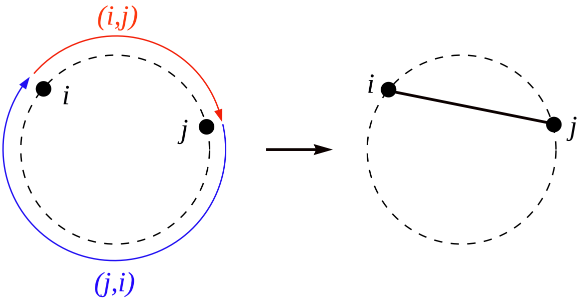


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# Statement of the ”old” problem

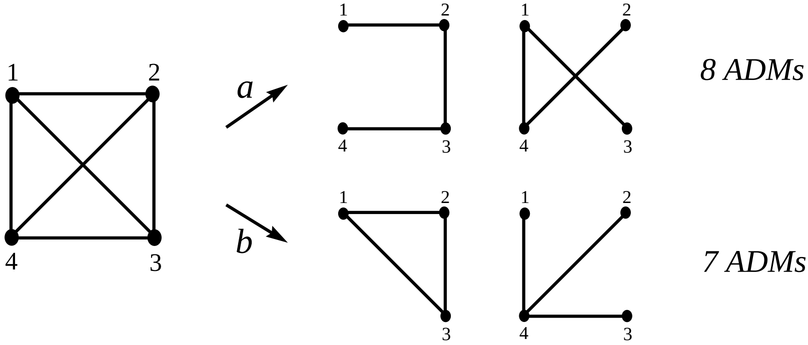
**Traffic Grooming in Unidirectional Rings**

**Input** A cycle $C_n$ on $n$ nodes (network);
An *undirected* graph $R$ on $n$ nodes (request set);
A grooming factor $C$.

**Output** A partition of $E(R)$ into subgraphs
$R_1, \ldots, R_W$ with $|E(R_i)| \leq C$, i=1. . . ,W.

**Objective** Minimize $\sum_{\omega=1}^{W} |V(R_\omega)|$.

# Example: $n = 4$, $R = K_4$, and $C = 3$



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- In all of them: place ADMs at nodes for a **fixed request graph**.
  $\rightarrow$ placement of ADMs **a posteriori**.

- **New model** [Muñoz and S., WG 2008]: place the ADMs at nodes such that the network can support **any request graph with maximum degree at most** $\Delta$.
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# Statement of the ”new” problem

## Traffic Grooming in Unidirectional Rings with Bounded-Degree Request Graph

**Input** An integer $n$ (size of the ring);
An integer $C$ (grooming factor);
An integer $\Delta$ (maximum degree).

**Output** An assignment of $A(v)$ ADMs to each $v \in V(C_n)$, in such a way that **for any graph** $R$ on $n$ nodes with **maximum degree at most** $\Delta$, it exists a partition of $E(R)$ into subgraphs $R_1, \ldots, R_W$ s.t.:

($i$) $|E(B_i)| \leq C$ for all $i = 1, \ldots, W$; and
($ii$) each $v \in V(C_n)$ is in $\leq A(v)$ subgraphs.

**Objective** Minimize $\sum_{v \in V(C_n)} A(v)$, and the optimum is denoted $A(n, C, \Delta)$.

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# $M(C, \Delta)$

**Definition**

Let $M(C, \Delta)$ be the least positive number $M$ such that, for all $n \geq 1$, the inequality $A(n, C, \Delta) \leq Mn$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** $v$, except for a subset whose size is **independent of** $n$.
- $M(C, \Delta)$ is always an **integer**.
- Equivalently:

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# More formally...

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most $\Delta$.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of partitions of $E(G)$ into subgraphs with at most $C$ edges.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left( \min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- If the request graph is restricted to belong to a subclass of graphs $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding positive integer is denoted by $M(C, \Delta, \mathcal{C})$.

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# Some properties of $M(C, \Delta)$ [Muñoz and S., WG 2008]

- W.l.o.g. we can assume that $R$ has **regular degree** $\Delta$.

- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.

- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

**Proposition (Lower Bound)**

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

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- W.l.o.g. we can assume that $R$ has **regular degree** $\Delta$.
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# Known values of $M(C, \Delta)$ [Muñoz and S., WG 2008]

- $\Delta = 1$: $M(C, 1) = 1$ for all $C$ (trivial).
- $\Delta = 2$: $M(C, 2) = 2$ for all $C$ (not difficult).
- $\Delta = 3$: Cubic graphs. First "interesting" case:
  - If $C \leq 3$, then $M(C, 3) = 3$.
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1 Traffic grooming

2 Statement of the problem

3 The parameter $M(C, \Delta)$

4 Previous work (Muñoz and S., WG 2008)

5 Our results
- Case $\Delta = 3$, $C = 4$
- Case $\Delta \geq 4$ even
- Case $\Delta \geq 5$ odd
- Improved lower bound when $\Delta \equiv C \pmod{2C}$

6 Conclusions

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# Case $\Delta = 3$, $C = 4$

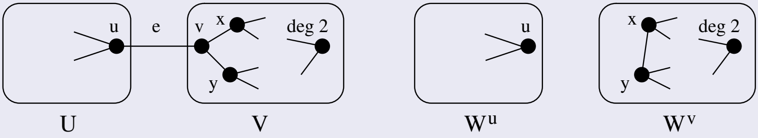
## Proposition

$M(4,3) = 2$.

## Idea of the proof.

(in fact, we prove a slightly stronger result)

- Let $G$ be a minimal counterexample.
- If has no bridges, then it can be "easily" proved.
- If $G$ has a bridge $e$, then the property is true for $U$ and $V$.




(a) (b)

- Finally, we merge "carefully" the partitions of $U$ and $V$ to obtain a

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# Case $\Delta \geq 4$ even

## Theorem

*Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.*

## Proof.

- The lower bound follows from [Muñoz and S., WG 2008].
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
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  - Each vertex $v$ appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2} \left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

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# Case $\Delta \geq 5$ odd

## Proposition

*Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C,\Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.*

## Sketch of proof.

- Since $\Delta$ is odd, $|V(G)|$ is even. Add a perfect matching $M$ to $G$ to obtain a $(\Delta+1)$-regular multigraph $G'$. Orient the edges of $G'$ in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta+1)/2$ out-edges $E_v^+$.
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# Case $\Delta \geq 5$ odd (II)

## Corollary

*Let* $\Delta \geq 5$ *be odd. If* $\Delta \pmod{2C} = 1$ *or* $\Delta \pmod{2C} \geq C + 1$*, then* $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

## Corollary (Value of $M(C, \Delta)$ for $C = 2$)

*For any* $\Delta \geq 5$ *odd,* $M(2, \Delta) = \left\lceil \frac{3\Delta}{4} \right\rceil$.

## Proposition

*Let* $\Delta \geq 5$ *be odd and let* $\mathcal{C}$ *be the class of* $\Delta$*-regular graphs than contain a perfect matching. Then* $M(C, \Delta, \mathcal{C}) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

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# Improved lower bound when $\Delta \equiv C \pmod{2C}$

**Theorem**

*Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.*

**Corollary (Value of $M(C,\Delta)$ for $C = 3$)**

*For any $\Delta \geq 5$ odd, $M(3,\Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.*

**Idea of the proof of the theorem.**

- We prove that if $\Delta = kC$ with $k$ odd, then $M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.
- Since both $\Delta$ and $k$ are odd, so is $C$, and therefore $\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a $\Delta$-regular graph $G$ with no $C$-edge-partition where each vertex is incident to at most $k \cdot \frac{C+1}{2}$ subgraphs.
- First, we construct a graph $H$ where all vertices have degree $\Delta$ except one which has degree $\Delta - 1$. Furthermore, we build $H$ so that it has girth strictly greater than $C$. Such a graph $H$ exists by [Chandran, SIAM J. Dicr. Math., 2003].

# Improved lower bound when $\Delta \equiv C \pmod{2C}$

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*Let* $\Delta \geq 5$ *be odd. If* $\Delta \equiv C \pmod{2C}$, *then* $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.

## Corollary (Value of $M(C, \Delta)$ for $C = 3$)

*For any* $\Delta \geq 5$ *odd,* $M(3, \Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

## Idea of the proof of the theorem.

- We prove that if $\Delta = kC$ with $k$ odd, then $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.
- Since both $\Delta$ and $k$ are odd, so is $C$, and therefore $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a $\Delta$-regular graph $G$ with no $C$-edge-partition where each vertex is incident to at most $k \cdot \frac{C+1}{2}$ subgraphs.
- First, we construct a graph $H$ where all vertices have degree $\Delta$ except one which has degree $\Delta - 1$. Furthermore, we build $H$ so that it has girth strictly greater than $C$. Such a graph $H$ exists by [Chandran, SIAM J. Dicr. Math., 2003].

# Improved lower bound when $\Delta \equiv C \pmod{2C}$

## Theorem

*Let* $\Delta \geq 5$ *be odd. If* $\Delta \equiv C \pmod{2C}$*, then* $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.

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# Improved lower bound when $\Delta \equiv C \pmod{2C}$ (II)

## Continuation of the proof.

- Make $\Delta$ copies of $H$ and add a cut-vertex $v$ joined to all vertices of degree $\Delta - 1$ to make our $\Delta$-regular graph $G$.
- Now suppose for the sake of contradiction that there is a $C$-edge-partition $\mathcal{B}$ of $G$ where each vertex is incident to at most $\mathrm{LB}(C, \Delta)$ subgraphs.
- Since the girth of $G$ is greater than $C$, all the subgraphs in $\mathcal{B}$ are trees.
- Since $LB(C, \Delta) < \Delta$, $v$ must have degree at least 2 in some subgraph $T' \in \mathcal{B}$.
- Since $|E(T')| \leq C$, the tree $T'$ contains at most $\lfloor \frac{C-2}{2} \rfloor = \frac{C-3}{2}$ edges of a copy $H'$ of $H$ intersecting $T'$.
- Now we only work in $H'$. Let $\alpha = |E(T' \cap H')| \leq \frac{C-3}{2}$.
- Let $\mathcal{B}' = \{B \cap H'\}_{B \in (\mathcal{B} - \{T'\})}$, with the empty subgraphs removed. That is, $\mathcal{B}'$ contains the subgraphs in $\mathcal{B}$ that partition the edges in $H'$ that are not in $T'$.
- Let $n = |V(H')|$, which is odd as in $H'$ there is one vertex of degree $\Delta - 1$ and all the others have degree $\Delta$.

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# Improved lower bound when $\Delta = C \pmod{2C}$ (II)

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# Improved lower bound when $\Delta \equiv C \pmod{2C}$ (III)

## Continuation of the proof (II).

- Therefore, the total number of edges of the trees in $\mathcal{B}'$ is

$$\sum_{T \in \mathcal{B}'} |E(T)| = |E(H')| - \alpha = \frac{n\Delta - 1}{2} - \alpha = \frac{nkC - 1}{2} - \alpha. \quad (1)$$

- As $\alpha \leq \frac{C-3}{2}$, from (1) we get

$$\sum_{T \in \mathcal{B}'} |E(T)| \geq \frac{nkC - 1}{2} - \frac{C - 3}{2} = \left(\frac{nk - 1}{2}\right) \cdot C + 1. \quad (2)$$

- As each tree in $\mathcal{B}'$ has at most $C$ edges, from (2) we get that

$$|\mathcal{B}'| \geq \left\lceil \frac{nk - 1}{2} + \frac{1}{C} \right\rceil = \frac{nk - 1}{2} + \left\lceil \frac{1}{C} \right\rceil = \frac{nk - 1}{2} + 1. \quad (3)$$

- Clearly, $\sum_{T \in \mathcal{B}'} |V(T)| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'|$, and $|V(T' \cap H')| = \alpha + 1$.

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# Improved lower bound when $\Delta = C \pmod{2C}$ (IV)

## Continuation of the proof (III).

- Therefore, using (1) and (3), we get that the total number of occurrences of the vertices in $H'$ in some tree of $\mathcal{B}$ is

$$\begin{aligned}
\sum_{v \in V(H')} |\{T \in \mathcal{B} : v \in T\}| &= \sum_{T \in \mathcal{B}'} |V(T)| + |V(T' \cap H')| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'| + \alpha + 1 \\
&= \frac{nkC - 1}{2} - \alpha + |\mathcal{B}'| + \alpha + 1 \geq \frac{nkC - 1}{2} + \frac{nk - 1}{2} + 1 + \\
&= nk \cdot \frac{C + 1}{2} + 1 = n \cdot \mathrm{LB}(C, \Delta) + 1 \,,
\end{aligned}$$

- which implies that at least one vertex of $H'$ appears in at least $\mathrm{LB}(C, \Delta) + 1$ subgraphs, which is a contradiction to $\mathcal{B}$ being a $C$-edge-partition of $G$ in which each vertex appears in at most $\mathrm{LB}(C, \Delta)$ subgraphs.

□

# Improved lower bound when $\Delta \equiv C \pmod{2C}$ (IV)

## Continuation of the proof (III).

- Therefore, using (1) and (3), we get that the total number of occurrences of the vertices in $H'$ in some tree of $\mathcal{B}$ is

$$\begin{aligned}
\sum_{v \in V(H')} |\{T \in \mathcal{B} : v \in T\}| &= \sum_{T \in \mathcal{B}'} |V(T)| + |V(T' \cap H')| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'| + \alpha + 1 \\
&= \frac{nkC - 1}{2} - \alpha + |\mathcal{B}'| + \alpha + 1 \geq \frac{nkC - 1}{2} + \frac{nk - 1}{2} + 1 + 1 \\
&= nk \cdot \frac{C+1}{2} + 1 = n \cdot \mathrm{LB}(C, \Delta) + 1 ,
\end{aligned}$$

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□

# Next section is...

1. Traffic grooming
2. Statement of the problem
3. The parameter $M(C, \Delta)$
4. Previous work (Muñoz and S., WG 2008)
5. Our results
6. **Conclusions**

# Summary of results: values of $M(C, \Delta)$

| $C \| \Delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... | $\Delta$ even | $\Delta$ odd |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... | $\Delta$ | $\Delta$ |
| 2 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | ... | $\frac{3\Delta}{4}$ | $\frac{3\Delta}{4}$ |
| 3 | 1 | 2 | 3 (2) | 3 | 4 | 5 (4) | 5 | ... | $\frac{2\Delta}{3}$ | $\frac{2\Delta+1}{3} \left(\frac{2\Delta}{3}\right)$ |
| 4 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | ... | $\frac{5\Delta}{8}$ | $\geq \frac{5\Delta}{8}$ $(=)$ |
| 5 | 1 | 2 | 2 | 3 | 4 (3) | 4 | 5 | ... | $\frac{3\Delta}{5}$ | $\geq \frac{3\Delta}{5}$ $(=)$ |
| 6 | 1 | 2 | 2 | 3 | $\geq 3$ $(=)$ | 4 | 5 | ... | $\frac{7\Delta}{12}$ | $\geq \frac{7\Delta}{12}$ $(=)$ |
| 7 | 1 | 2 | 2 | 3 | $\geq 3$ $(=)$ | 4 | 5 (4) | ... | $\frac{4\Delta}{7}$ | $\geq \frac{4\Delta}{7}$ $(=)$ |
| 8 | 1 | 2 | 2 | 3 | $\geq 3$ $(=)$ | 4 | $\geq 4$ $(=)$ | ... | $\frac{9\Delta}{16}$ | $\geq \frac{9\Delta}{16}$ $(=)$ |
| 9 | 1 | 2 | 2 | 3 | $\geq 3$ $(=)$ | 4 | $\geq 4$ $(=)$ | ... | $\frac{5\Delta}{9}$ | $\geq \frac{5\Delta}{9}$ $(=)$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| $C$ | 1 | 2 | 2 | 3 | $\geq 3$ $(=)$ | 4 | $\geq 4$ $(=)$ | ... | $\frac{C+1}{C}\frac{\Delta}{2}$ | $\geq \frac{C+1}{C}\frac{\Delta}{2}$ $(=)$ |

Table: Known values of $M(C, \Delta)$. The red cases remain open. The (blue) cases in brackets only hold if the graph has a perfect matching. The symbol "$(=)$" means that the corresponding lower bound is attained.

# Conclusions and further research

- We have studied a new model of **traffic grooming** that allows the network to support **dynamic** traffic without reconfiguring the electronic equipment at the nodes.
- We established the value of $M(C,\Delta)$ for "almost all" values of $C$ and $\Delta$, leaving **open** only the case where:
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  - Determine $M(C,\Delta)$ for the remaining cases:

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Gràcies!