Kernelization of MAXIMUM MINIMAL VERTEX COVER

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Séminaire AlGCo
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Outline of the talk

Next section is...

For a minimization problem, it is natural to consider its "maximum minimal" version: worst-case of a greedy heuristic.

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- MAXIMUM MINIMAL DOMINATING SET: UPPER DOMINATION.
- MAXIMUM MINIMAL HITTING SET.
- MAXIMUM MINIMAL FEEDBACK VERTEX SET.

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In this talk:

Maximum Minimal Vertex Cover (MMVC)
Input: A graph G and an integer k.
Question: Does G contain a minimal vertex cover of size at least k?

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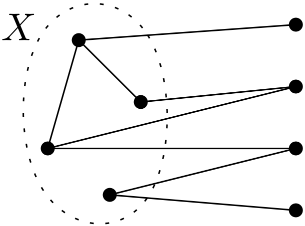
Dual problem of MMVC: Minimum Independent Dominating Set.



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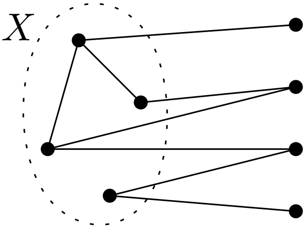

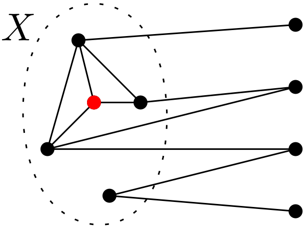

A set $X \subseteq V(G)$ is a minimal vertex cover of G $\Leftrightarrow$
X is a vertex cover of G and, for every vertex $v \in X$, $N(v) \not\subseteq X$.

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- Inapproximability of MMVC in subexponential time. [Bonnet, Paschos. 2018] [Bonnet, Lampis, Paschos. 2018]

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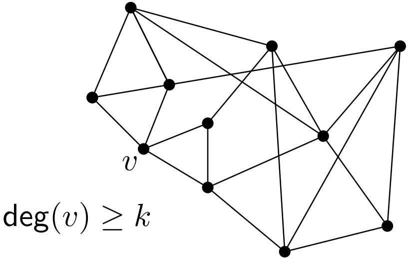

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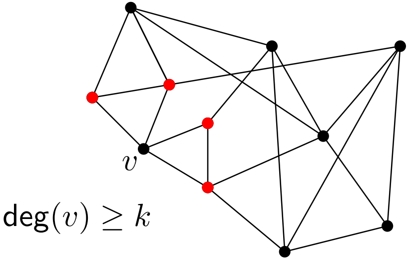

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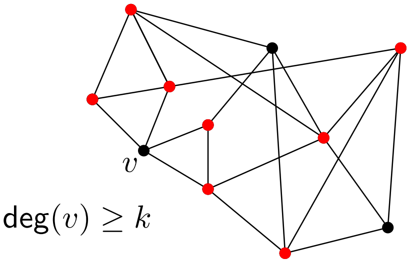

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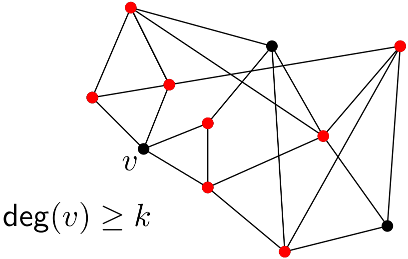

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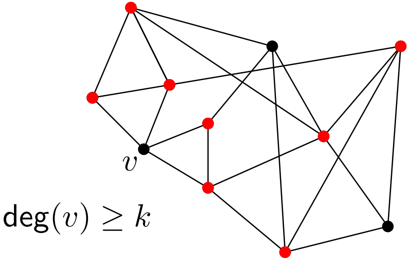

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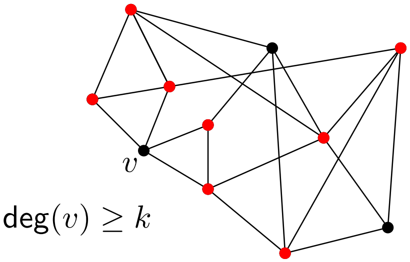

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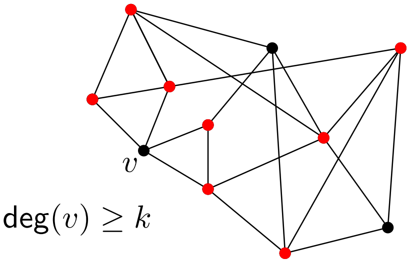

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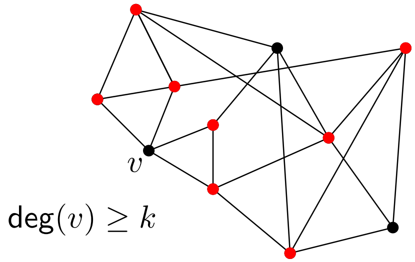

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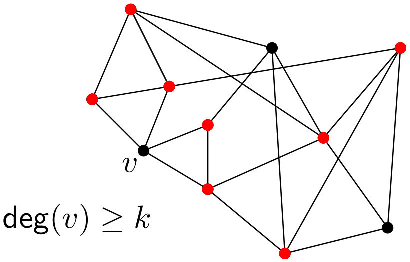

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By removing isolated vertices, it follows that $|V(G)| \leq |X| \cdot k$.

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Thus, we trivially have a kernel with $|V(G)| < k^2$.

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Strategy to obtain a linear kernel: [Fernau. 2005]

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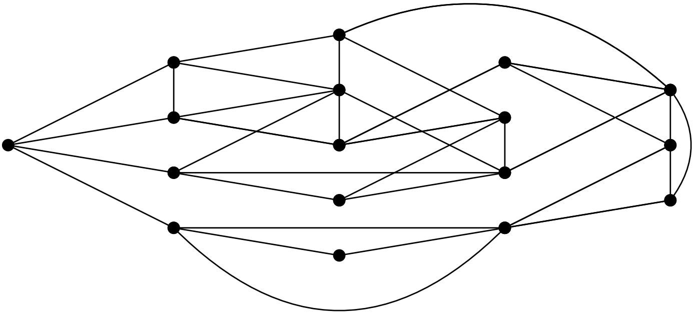

Given a graph G and a parameter k.

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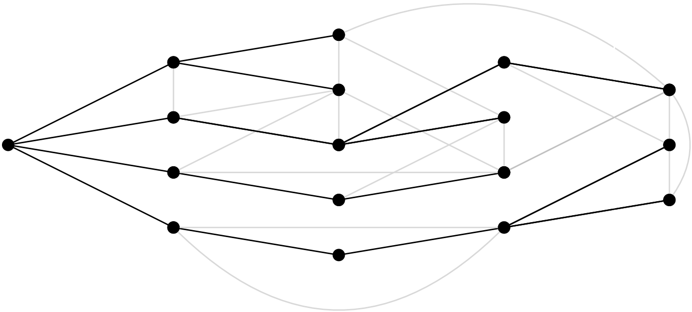

Consider an arbitrary spanning tree T and root it at a vertex r.

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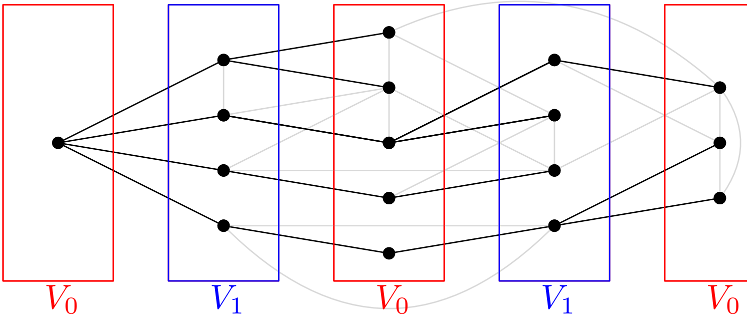

Partition $V(G)$ into V_0 and V_1 according to the distance from r.

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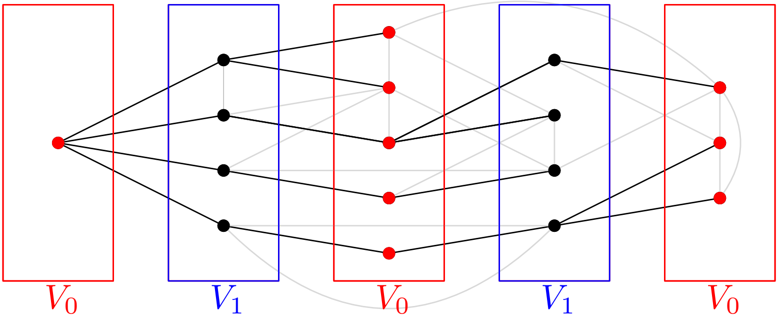

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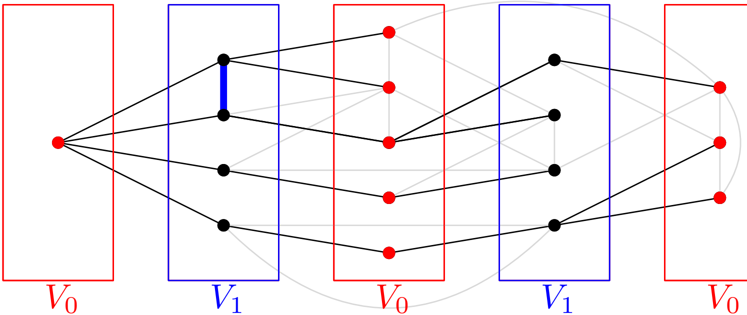

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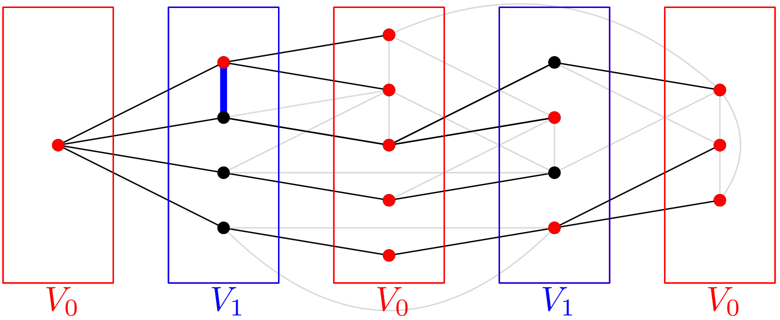

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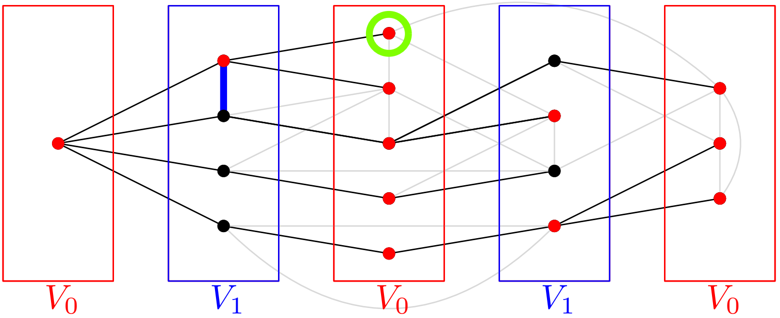

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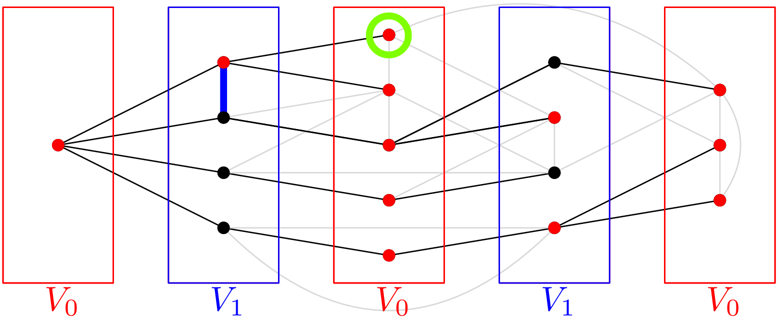

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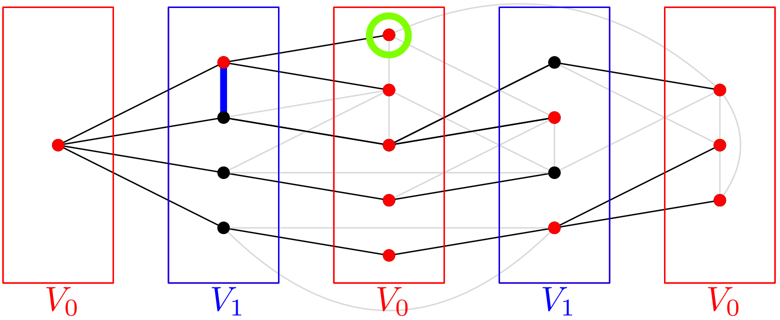

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Deciding whether $S \subseteq V(G)$ can the extended to a minimal vertex cover of G is NP-complete. [Casel, Fernau, Ghadikolaei, Monnot, Sikora. 2019]

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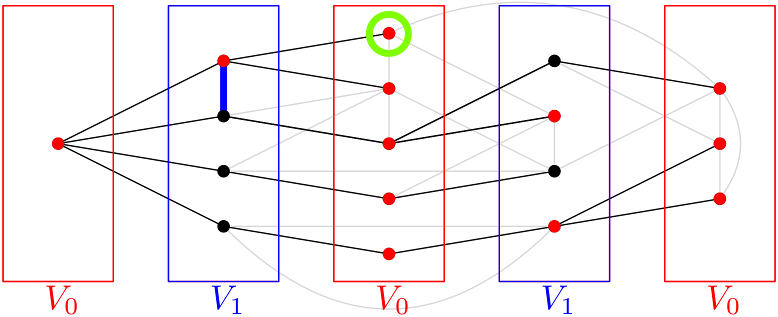

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The existence of a kernel with $o(k^2)$ vertices has been asked by [Boria, Della Croce, Paschos. 2015]

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Our result rules out the existence of polynomial kernels for MMVC parameterized by treewidth as well.

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We introduce a new framework to provide kernelization lower bounds.

We consider a general vertex-maximization problem Π,
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(The definitions can be adapted to vertex-minimization problems as well.)

A kernel for Π with parameter k is a polynomial-time algorithm that, given an instance (G, k), produces an instance (G', k') with $|V(G')| \leq s(k)$ for some function $s : \mathbb{N} \to \mathbb{N}$, called the size of the kernel, s.t.

1. (G', k') Yes-instance $\Rightarrow$ (G, k) Yes-instance.
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We also allow a lop-kernel to answer 'YES' directly.

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Example of a rule that is not a lop-rule for MMVC:

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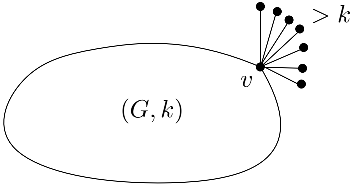

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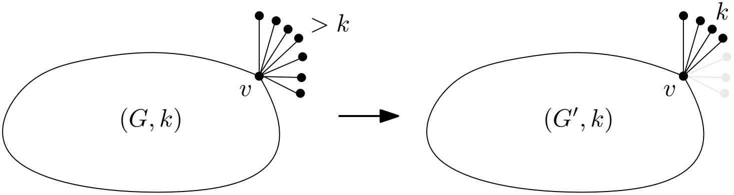

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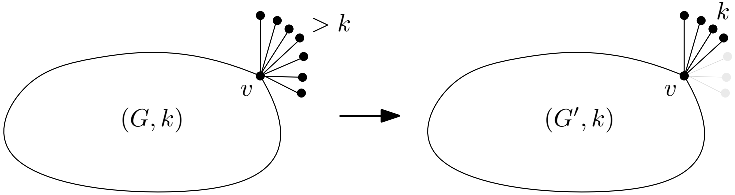

Want: mmvc(G) $\geq$ k

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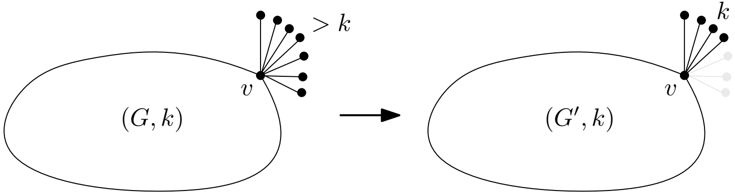

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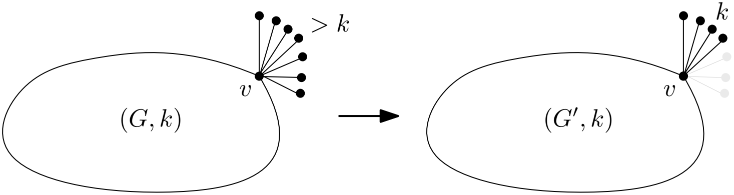

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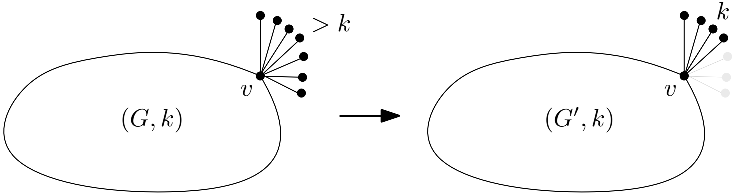

Want: $\text{mmvc}(G) \geq k \Rightarrow \text{mmvc}(G') \geq \text{mmvc}(G) - (k - k') = \text{mmvc}(G)$.

But $\text{mmvc}(G) > \text{mmvc}(G')$ may happen.

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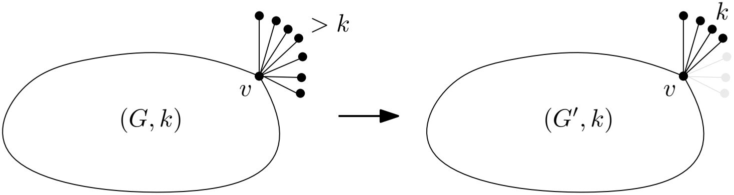

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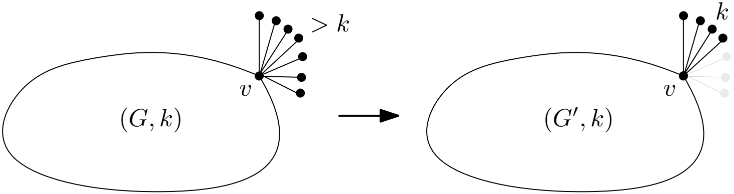

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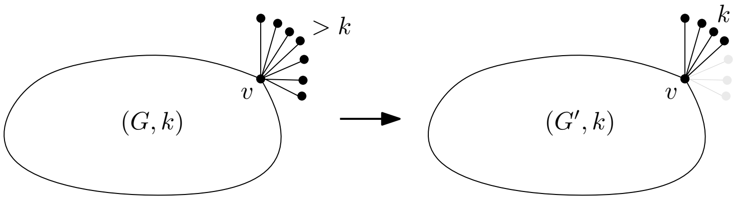

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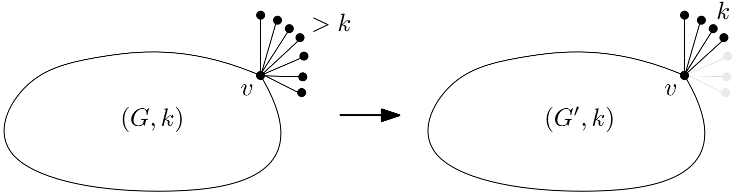

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So far, we don't know of any reduction rule that is not a lop-rule!

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(Similar statement for vertex-minimization problems.)

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Weak points needs strong inapproximability result, only vertex problems.

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Thus, the trivial quadratic kernel is "essentially" optimal.

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Next section is...

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We present a PPT from the Monotone Sat problem parameterized by the number of variables, which is known not to admit a polynomial kernel.

[Fortnow, Santhanam. 2011]

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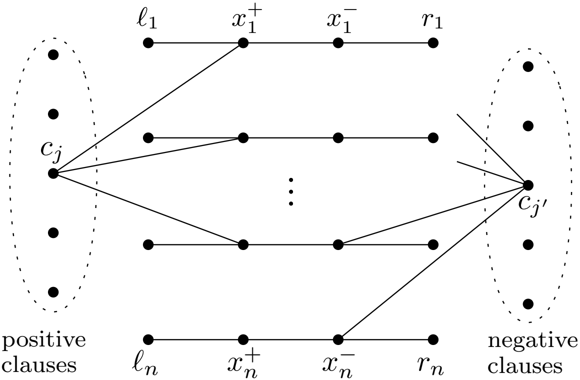

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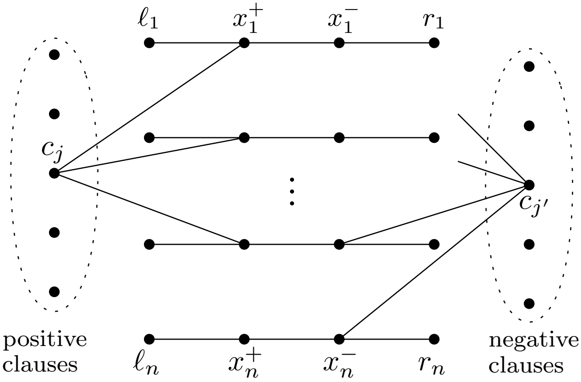

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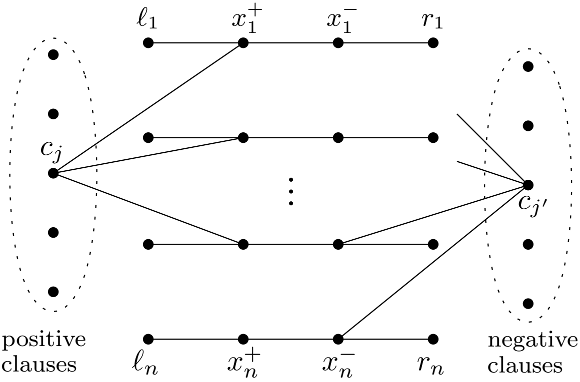

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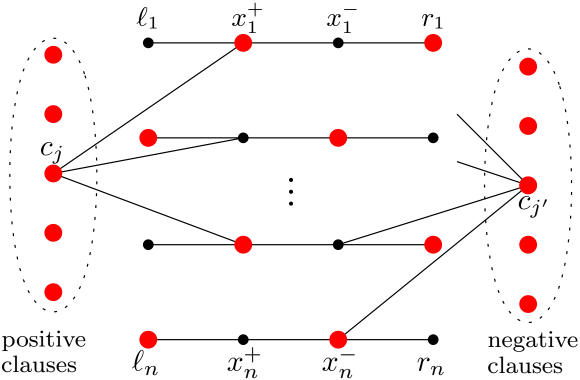

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Sketch of proof

Let Π be a vertex-maximization problem parameterized by solution size k.

Suppose that Π admits a lop-kernel of size $\mathcal{O}(k^c)$ for some constant $c \geq 1$.

Then, given an instance (G, k), in poly time either we conclude that

- $\text{opt}_\Pi(G) \geq k$, or
- we obtain an equivalent instance (G', k') such that

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It holds with $\varepsilon' := \varepsilon^2 \cdot \frac{(1-r)^2}{r}$.

Subquadratic kernels on particular graph classes

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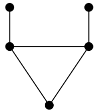
The Maximum Minimal Vertex Cover *problem parameterized by* k *restricted to bull-free graphs admits a kernel with* $\mathcal{O}(k^{7/4})$ *vertices.*



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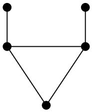

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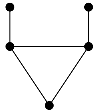

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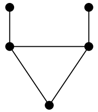

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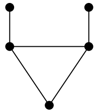

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For all the known cases, such a clique or independent set of size n^{δ} can be found in polynomial time.

Two useful lemmas

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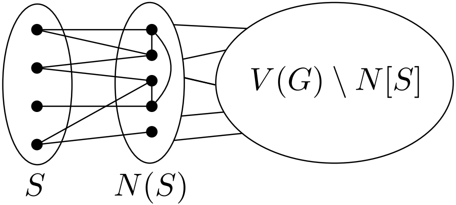

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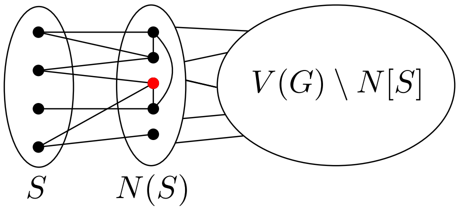

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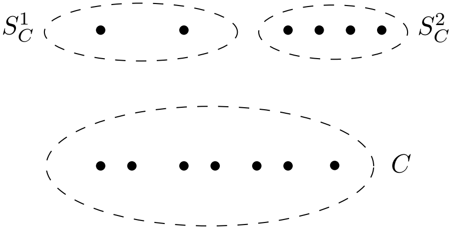

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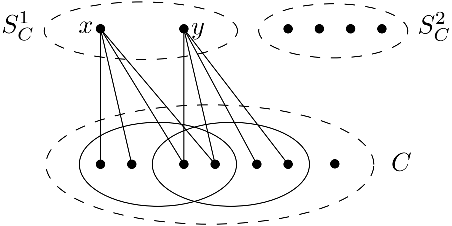

Partition $N_S(C) = S_C^1 \uplus S_C^2$ so that S_C^1 is a maximal subset of $N_S(C)$ s.t. the neighborhoods of its vertices pairwise do not cover all the clique C.

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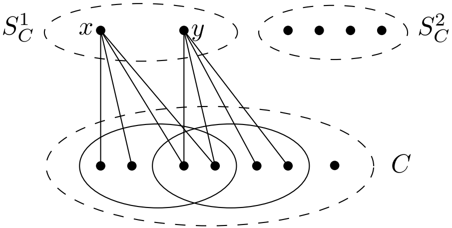

Claim: The vertices in S_C^1 can be ordered $x_1, \ldots, x_p$ so that $N_C(x_i) \subseteq N_C(x_j)$ whenever $i \leq j$.

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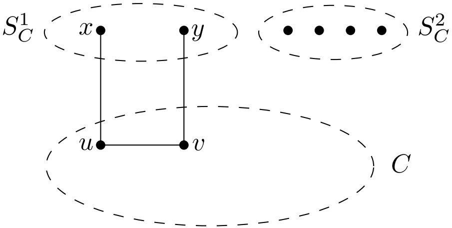

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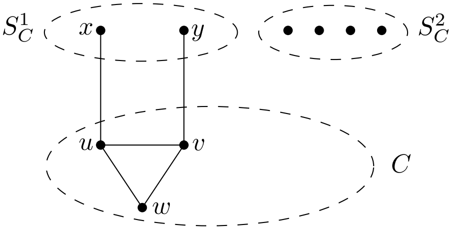

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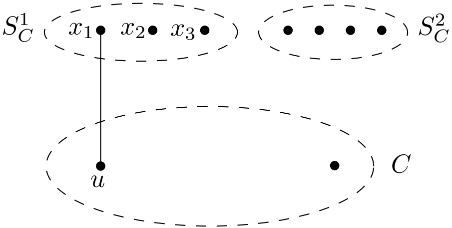

By the Claim, there is a vertex $u \in \bigcap_{x \in S_C^1} N_C(x)$.

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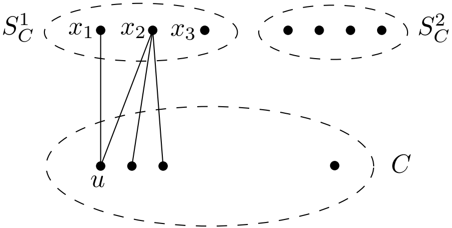

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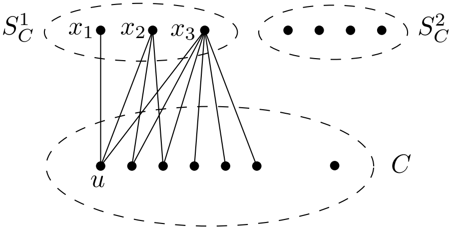

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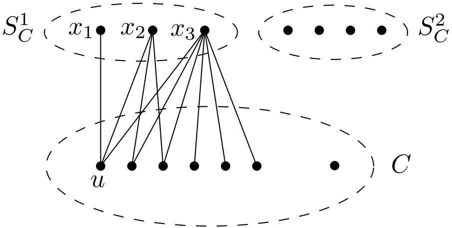

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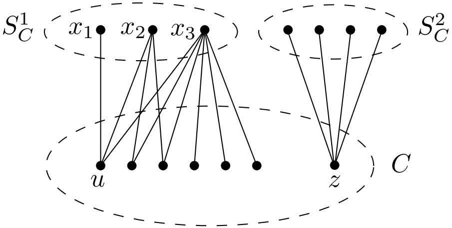

There exists a vertex $z \in C \setminus \bigcup_{y \in S_C^1} N_C(y)$.

It follows that $z \in \bigcap_{x \in S_C^2} N_C(x)$, and since $\deg(z) \leq k-1$, $|S_C^2| \leq k-1$.

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  $\mathrm{mmvc}(G) \geq \lfloor n^{1/2} \rfloor$. [Boria, Della Croce, Paschos. 2015]

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  $\mathrm{mmvc}(G) \geq \lfloor n^{1/2} \rfloor$. [Boria, Della Croce, Paschos. 2015]

  This immediately yields a quadratic kernel for MMVC.

Further research

lop-kernel of size $\mathcal{O}(k^{\frac{1}{1-r}-\varepsilon}) \Rightarrow \mathcal{O}(n^{r-\varepsilon'})$-approximation algorithm

- Can the "lop" assumption be removed?
- Are there natural reduction rules that are not lop-rules?
- Apply our framework to vertex-minimization problems.

Subquadratic kernels for MMVC on H-free graphs using the EH-property

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  If so, it would immediately yield a subquadratic kernel.

Gràcies!