Kernelization of Maximum Minimal Vertex Cover

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March 30th, 2021

Outline of the talk

1. Introduction
2. Our results
3. Some proofs
4. Further research

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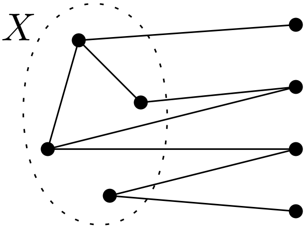

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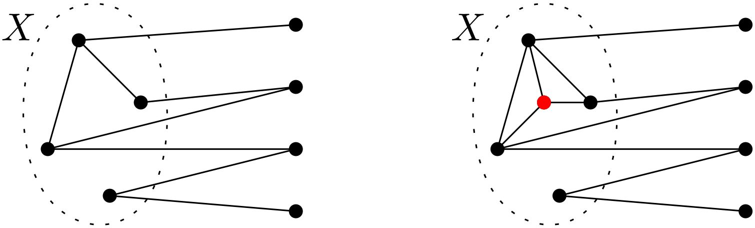

A set $X \subseteq V(G)$ is a minimal vertex cover of G $\Leftrightarrow$ X is a vertex cover of G and, for every vertex $v \in X$, $N(v) \not\subseteq X$.

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- Inapproximability of MMVC in subexponential time.
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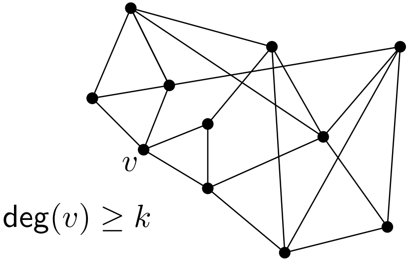

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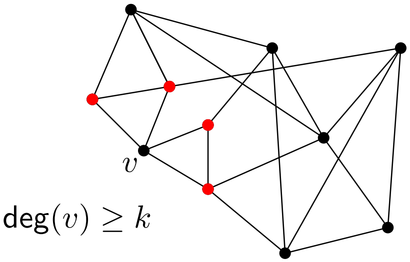

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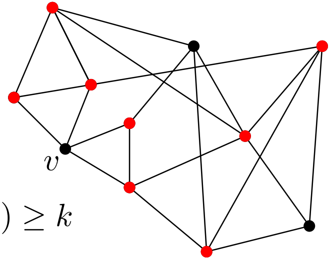

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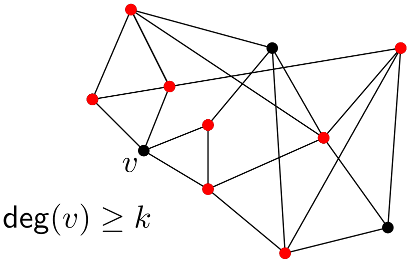

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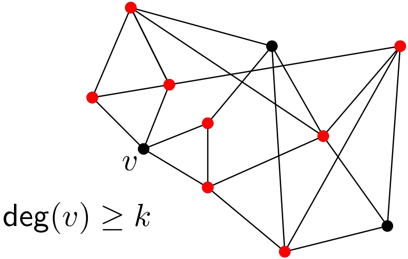

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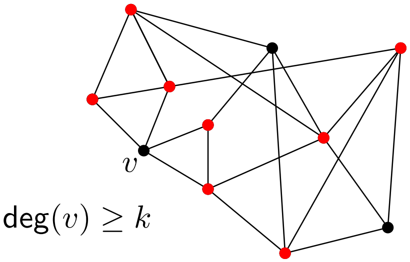

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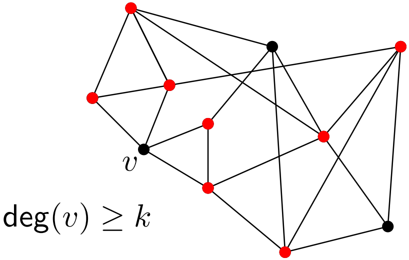

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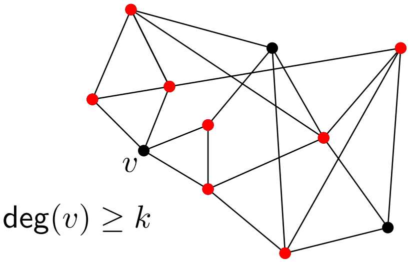

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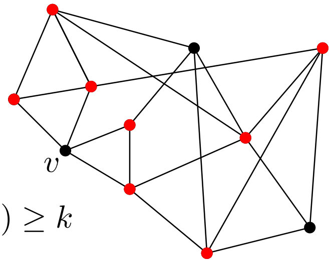

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Thus, we trivially have a kernel with $|V(G)| < k^2$.

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Strategy to obtain a linear kernel: [Fernau. 2005]

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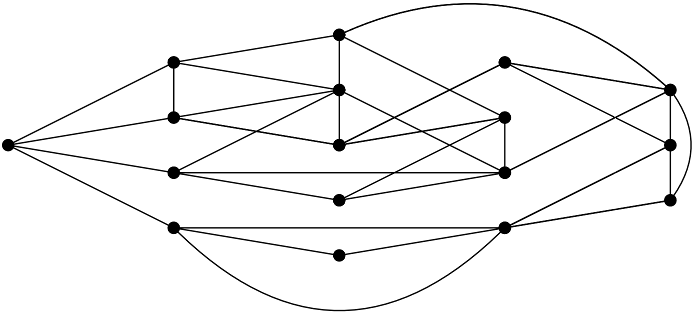

Given a graph G and a parameter k.

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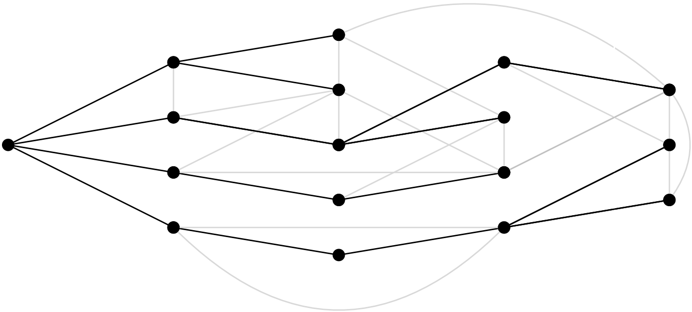

Consider an arbitrary spanning tree T and root it at a vertex r.

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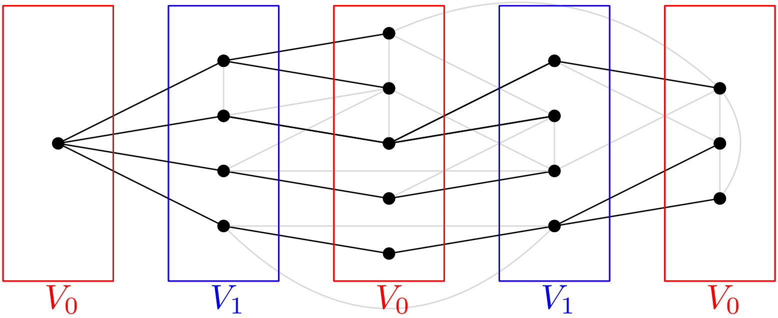

Partition $V(G)$ into V_0 and V_1 according to the distance from r.

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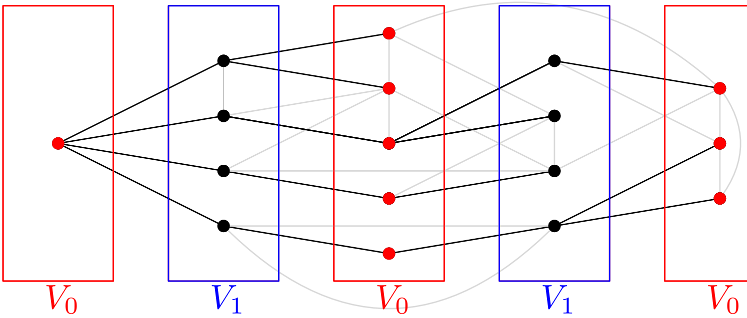

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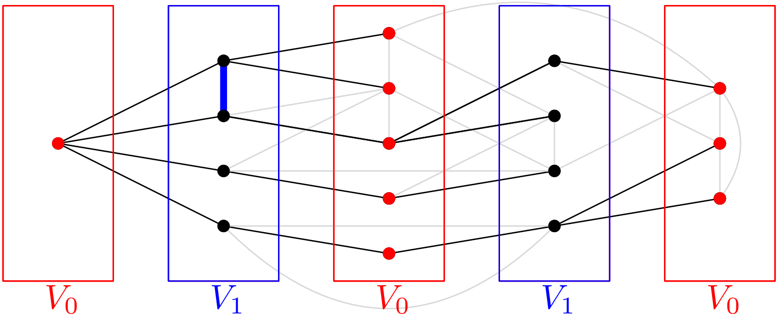

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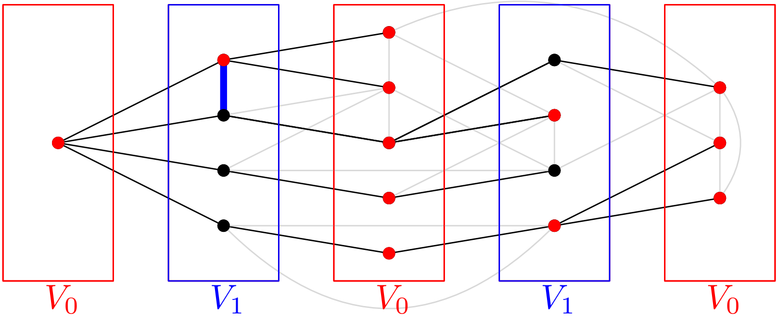

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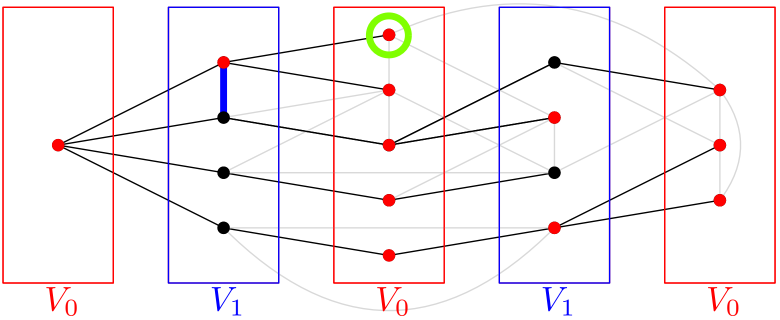

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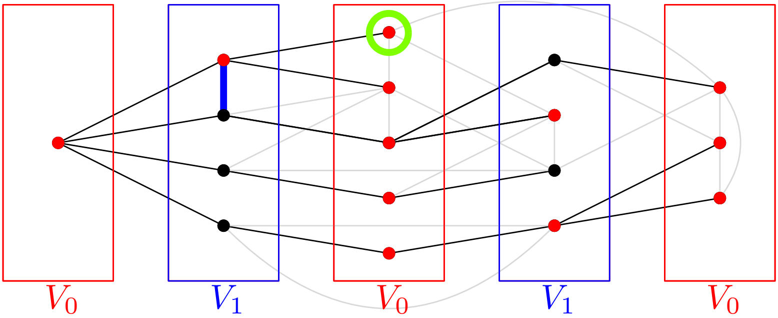

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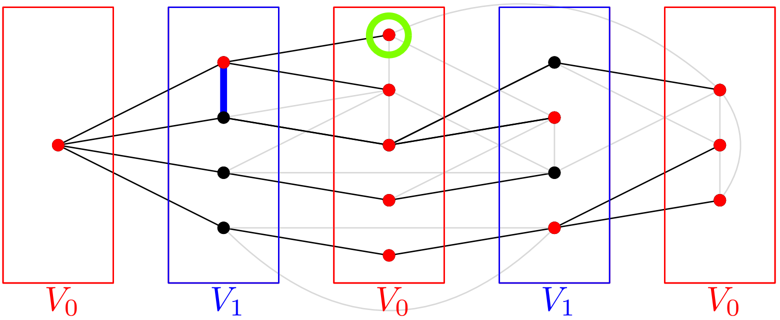

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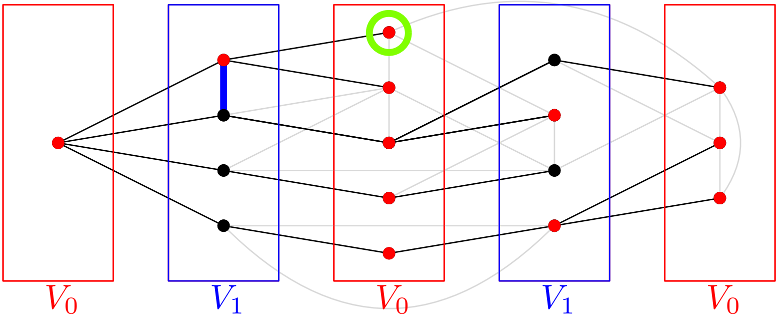

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The existence of a kernel with $o(k^2)$ vertices has been asked by [Boria, Della Croce, Paschos. 2015]

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Our result rules out the existence of polynomial kernels for MMVC parameterized by treewidth as well.

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(The definitions can be adapted to vertex-minimization problems as well.)

A $\boxed{\text{kernel}}$ for Π with parameter k is a polynomial-time algorithm that, given an instance (G, k), produces an instance (G', k') with $|V(G')| \leq s(k)$ for some function $s : \mathbb{N} \to \mathbb{N}$, called the size of the kernel, s.t.

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We also allow a lop-kernel to answer 'YES' directly.

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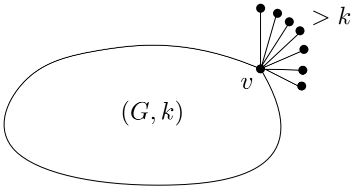

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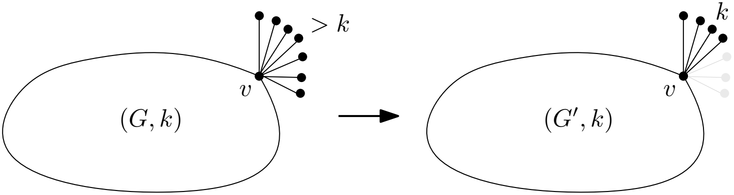

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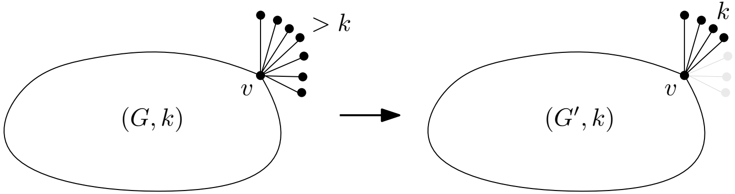

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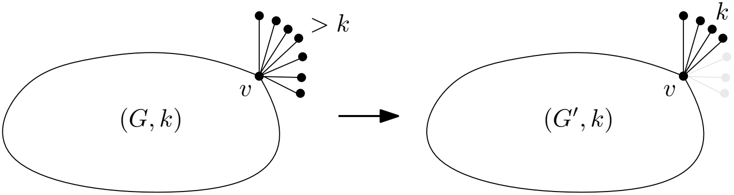

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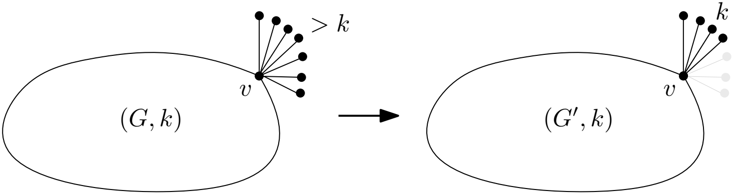

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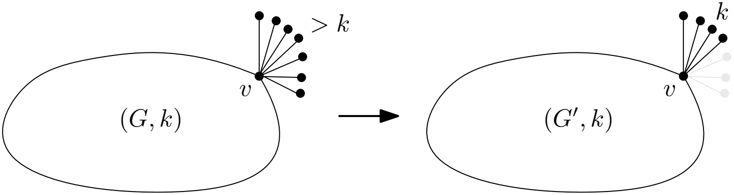

Want: $\quad \mathsf{mmvc}(G) \geq k \Rightarrow \mathsf{mmvc}(G') \geq \mathsf{mmvc}(G) - (k - k') = \mathsf{mmvc}(G)$.

But $\mathsf{mmvc}(G) > \mathsf{mmvc}(G')$ may happen.

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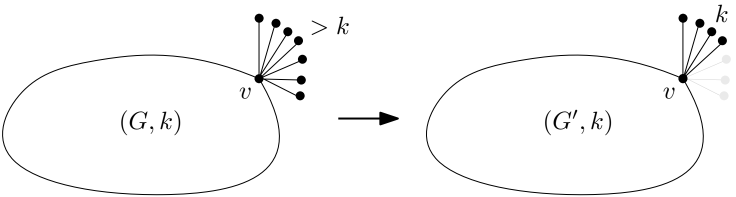

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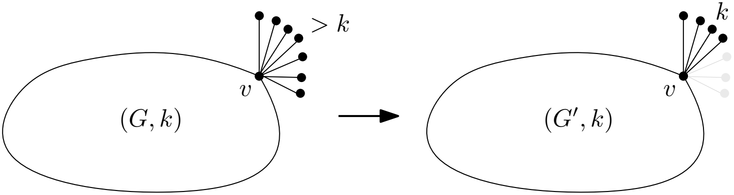

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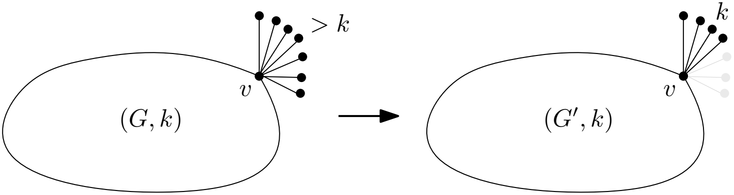

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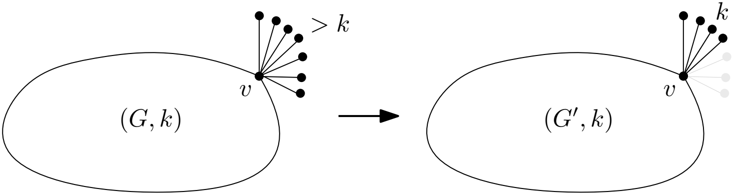

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So far, we don't know of any reduction rule that is not a lop-rule!

A general result

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(Similar statement for vertex-minimization problems.)

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Strong points: immediate application, weak hypothesis ($\mathrm{P} \neq \mathrm{NP}$).

Weak points: needs strong inapproximability result, only vertex problems.

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Thus, the trivial quadratic kernel is "essentially" optimal.

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Next section is...

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We present a PPT from the Monotone Sat problem parameterized by the number of variables, which is known not to admit a polynomial kernel.

[Fortnow, Santhanam. 2011]

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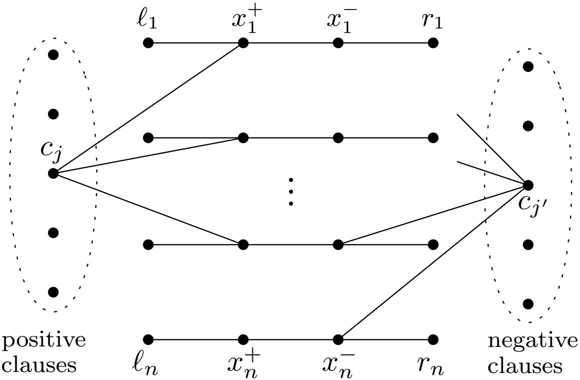

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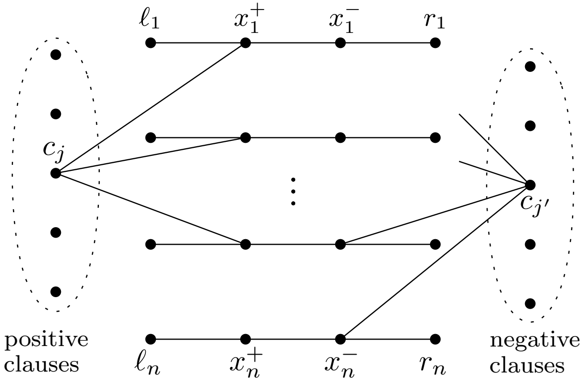

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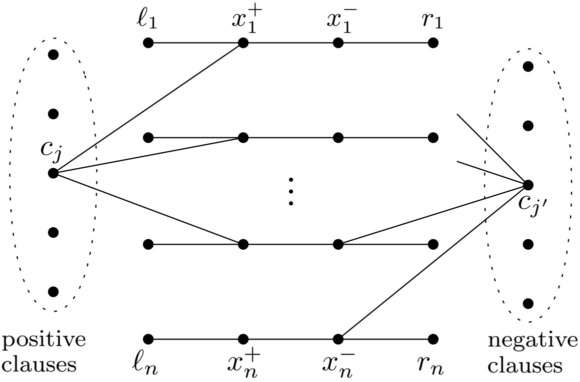

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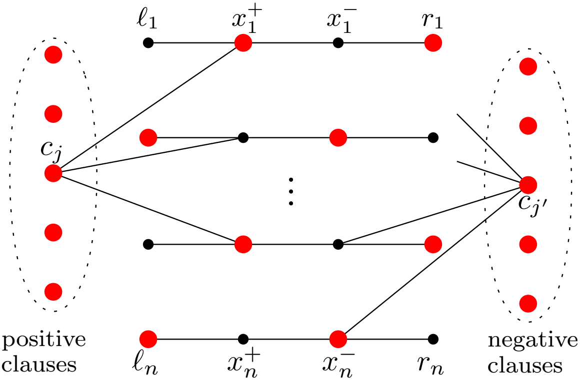

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It holds with $\varepsilon' := \varepsilon^2 \cdot \frac{(1-r)^2}{r}$.

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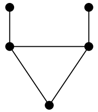

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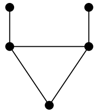

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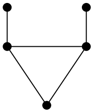

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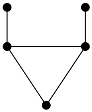

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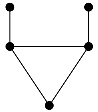

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For all the known cases, such a clique or independent set of size n^{δ} can be found in polynomial time.

Two useful lemmas

Lemma

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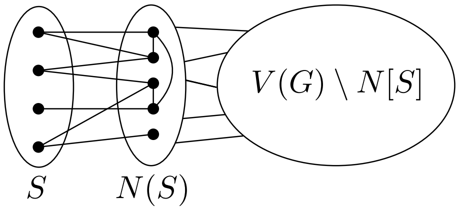

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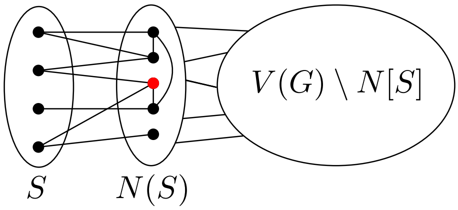

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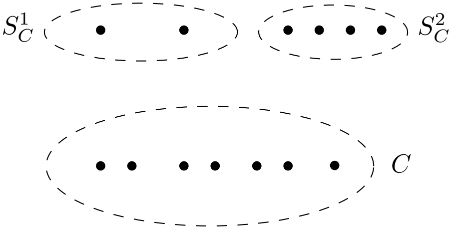

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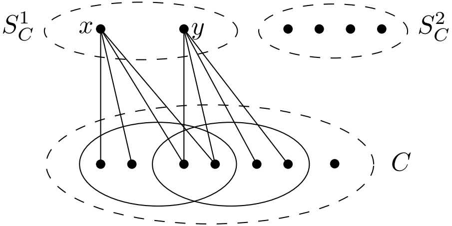

Partition $N_S(C) = S_C^1 \uplus S_C^2$ so that S_C^1 is a maximal subset of $N_S(C)$ s.t. the neighborhoods of its vertices pairwise do not cover all the clique C.

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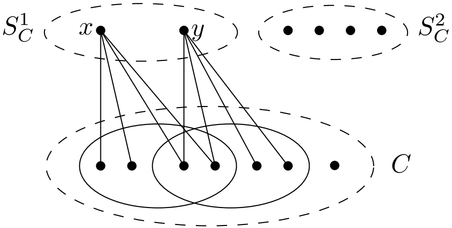

Claim: The vertices in S_C^1 can be ordered $x_1, \ldots, x_p$ so that $N_C(x_i) \subseteq N_C(x_j)$ whenever $i \leq j$.

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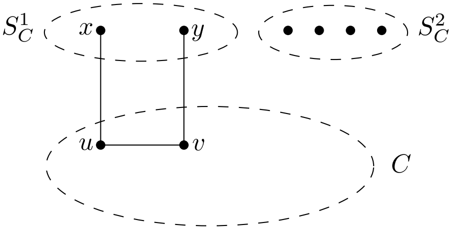

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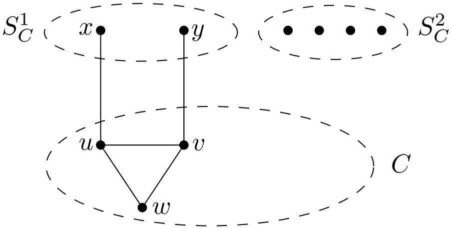

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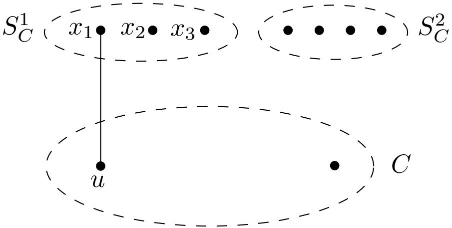

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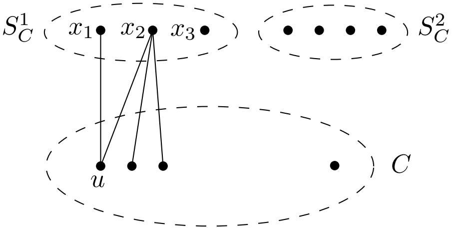

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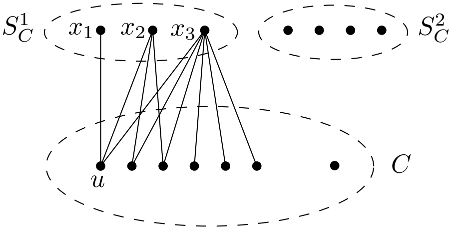

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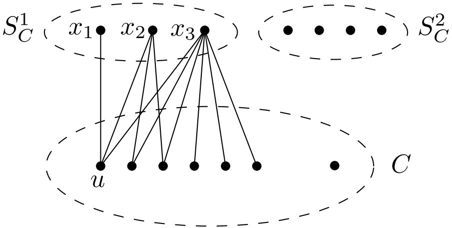

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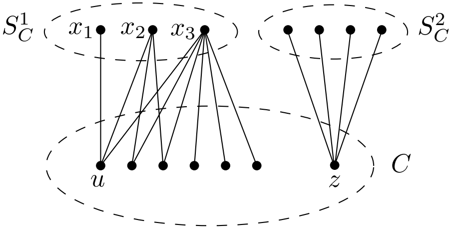

There exists a vertex $z \in C \setminus \bigcup_{y \in S_C^1} N_C(y)$.

It follows that $z \in \bigcap_{x \in S_C^2} N_C(x)$, and since $\deg(z) \leq k-1$, $|S_C^2| \leq k - 1$.

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- Other graphs H satisfying the E-H property: C_4, the diamond, P_5, C_5.

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- Other graphs H satisfying the E-H property: C_4, the diamond, P_5, C_5.
- The complexity of MMVC on P_5-free graphs is open.

Further research

lop-kernel of size $\mathcal{O}(k^{\frac{1}{1-r}-\varepsilon}) \Rightarrow \mathcal{O}(n^{r-\varepsilon'})$-approximation algorithm

- Can the "lop" assumption be removed?
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Subquadratic kernels for $\textsc{MMVC}$ on H-free graphs using the EH-property

- Other graphs H satisfying the E-H property: C_4, the diamond, P_5, C_5.
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- If G is a graph on n vertices without isolated vertices, then
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  If so, it would immediately yield a subquadratic kernel.

Gràcies!