Introducing lop-kernels: the case of Maximum Minimal Vertex Cover

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GRAA: Graphes en Rhône-Alpes et Auvergne
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Outline of the talk

1. Introduction
2. Our results
3. Some proofs
4. Further research

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- Maximum Minimal Dominating Set: Upper Domination.
- Maximum Minimal Hitting Set.
- Maximum Minimal Feedback Vertex Set.

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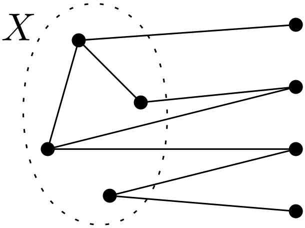

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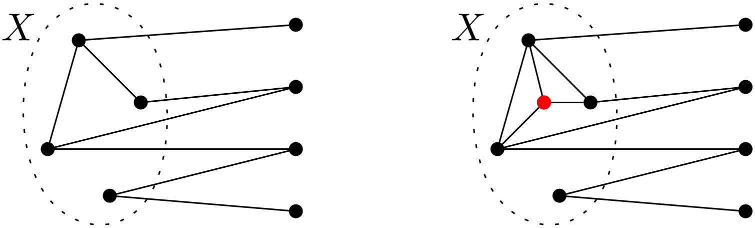

A set $X \subseteq V(G)$ is a minimal vertex cover of G $\Leftrightarrow$
X is a vertex cover of G and, for every vertex $v \in X$, $N(v) \not\subseteq X$.

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- Inapproximability of MMVC in subexponential time. [Bonnet, Paschos. 2018] [Bonnet, Lampis, Paschos. 2018]

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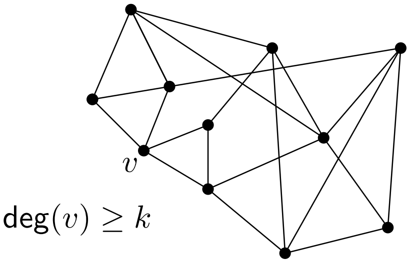

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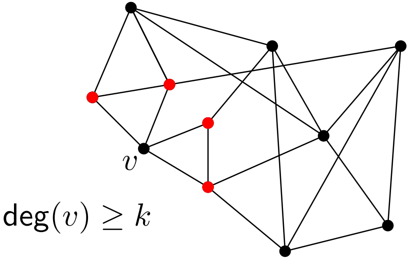

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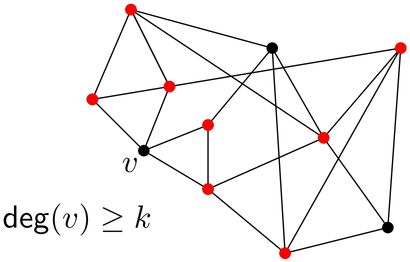

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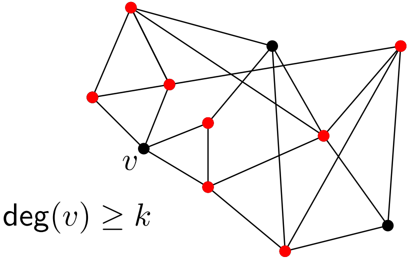

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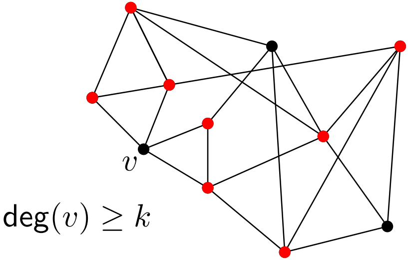

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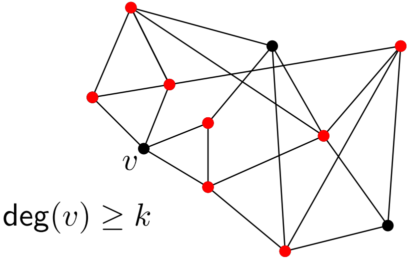

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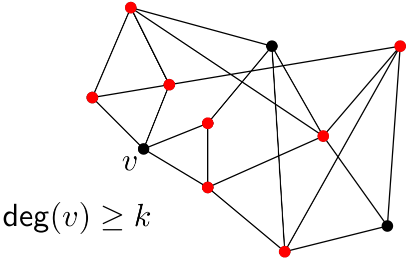

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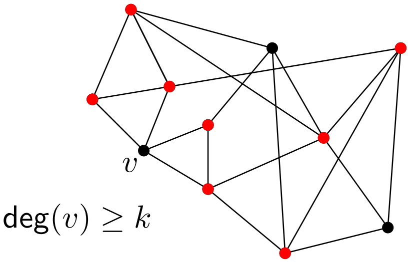

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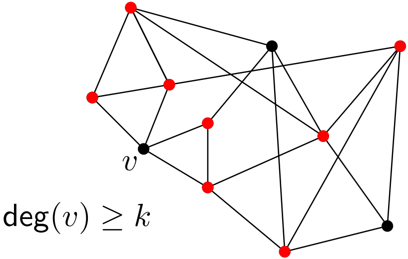

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Thus, we trivially have a kernel with $|V(G)| < k^2$.

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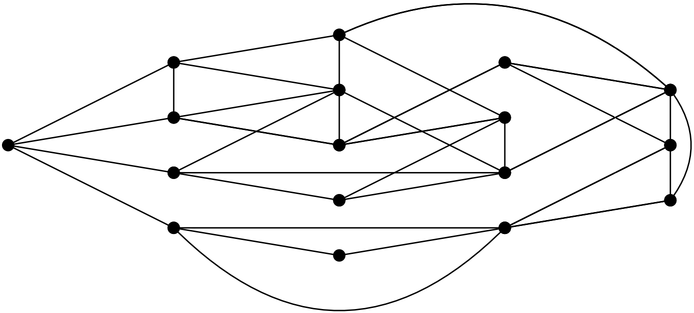

Given a graph G and a parameter k.

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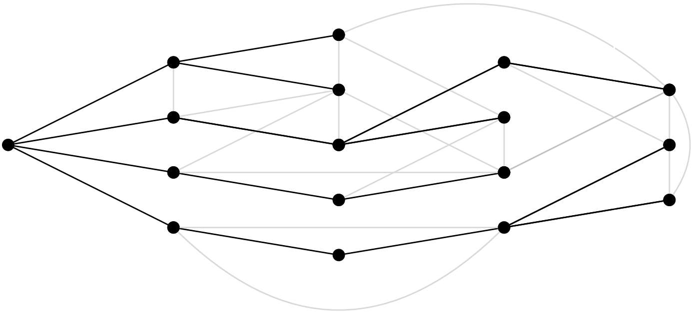

Consider an arbitrary spanning tree T and root it at a vertex r.

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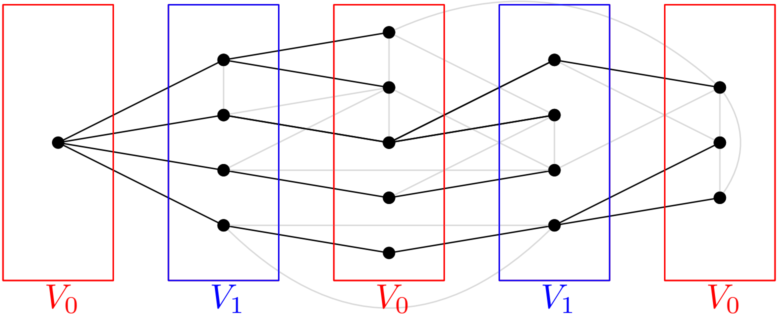

Partition $V(G)$ into V_0 and V_1 according to the distance from r.

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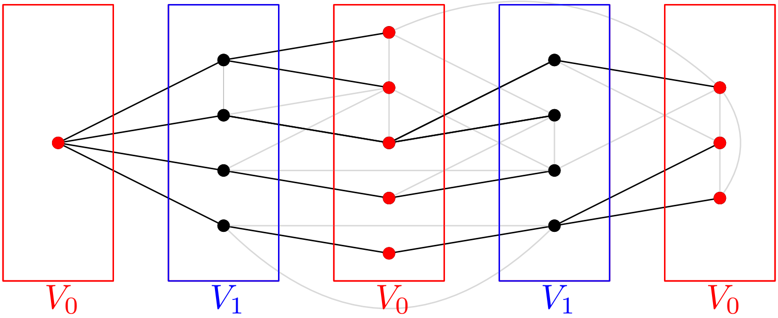

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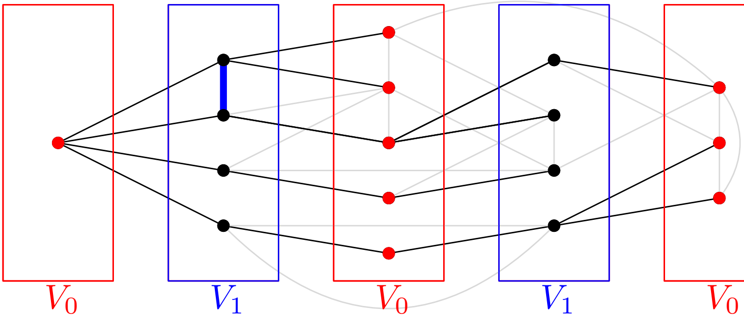

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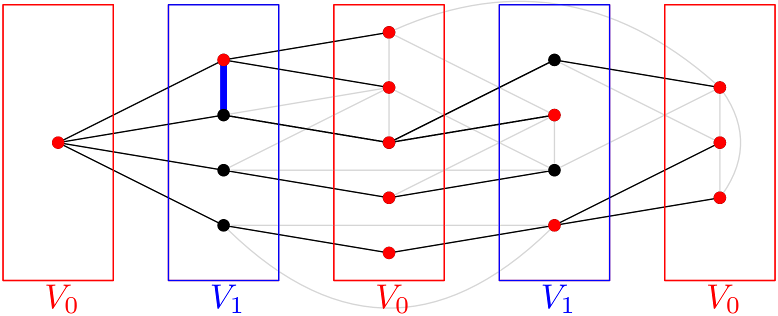

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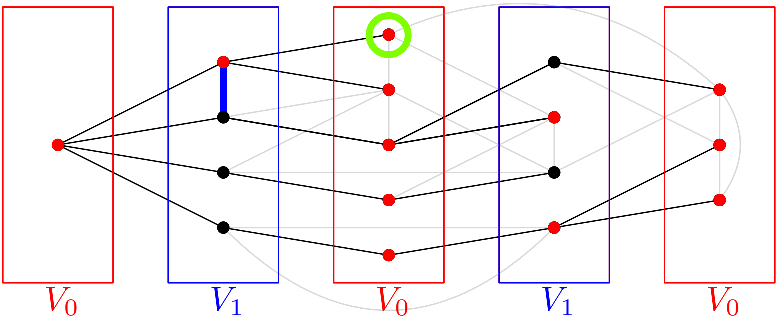

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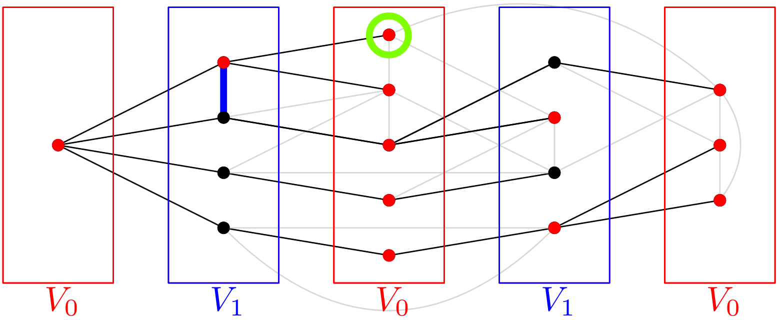

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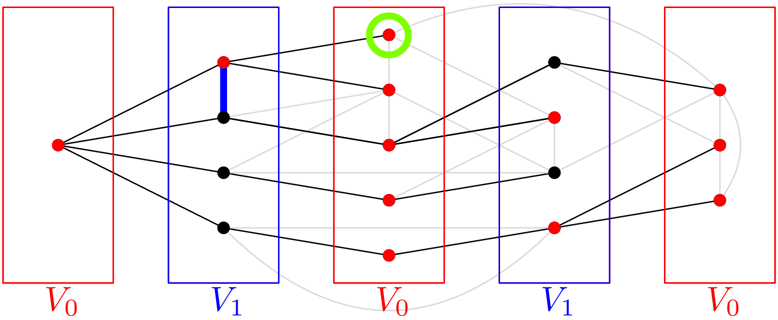

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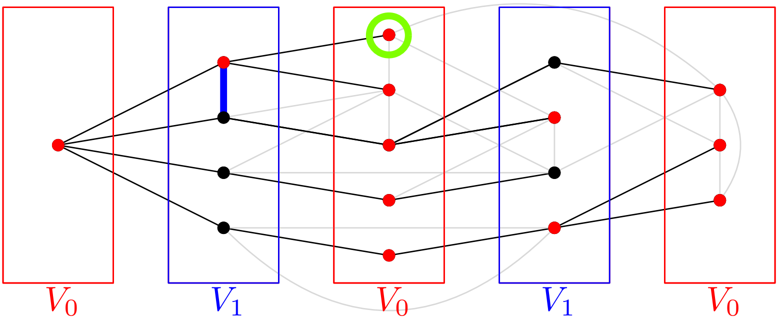

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The existence of a kernel with $o(k^2)$ vertices has been asked by [Boria, Della Croce, Paschos. 2015]

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We introduce a new framework to provide kernelization lower bounds.

We consider a general vertex-maximization problem Π,
parameterized by the solution size k.

(The definitions can be adapted to vertex-minimization problems as well.)

A **kernel** for Π with parameter k is a polynomial-time algorithm that, given an instance (G, k), produces an instance (G', k') with $|V(G')| \leq s(k)$ for some function $s : \mathbb{N} \to \mathbb{N}$, called the size of the kernel, s.t.

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We also allow a lop-kernel to answer 'YES' (or 'NO') directly.

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Example of a rule that is not a lop-rule for MMVC:

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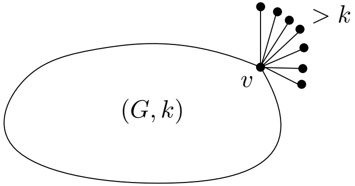

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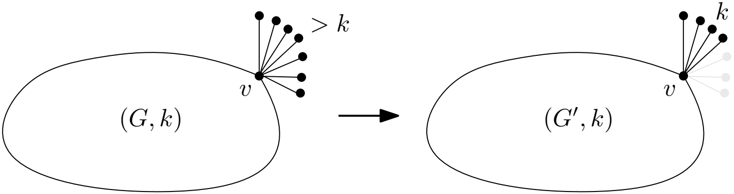

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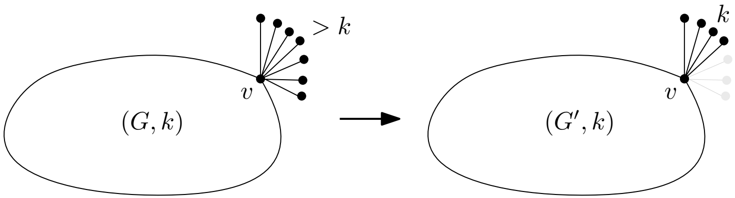

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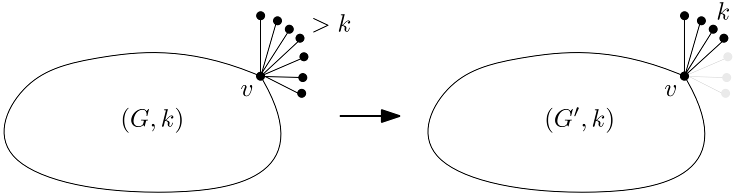

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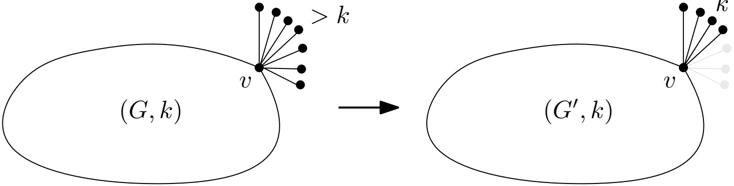

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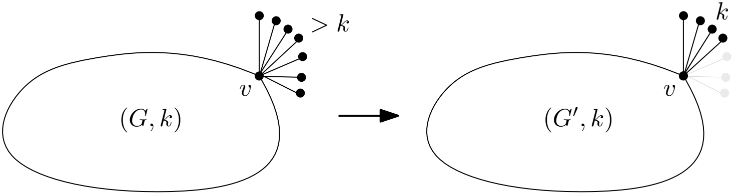

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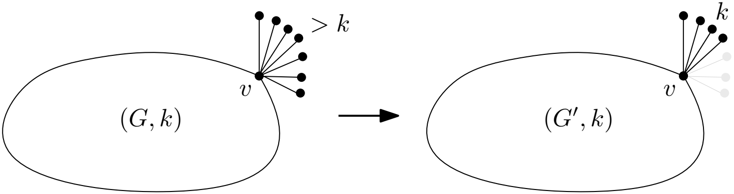

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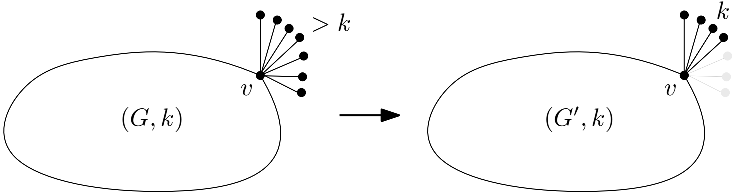

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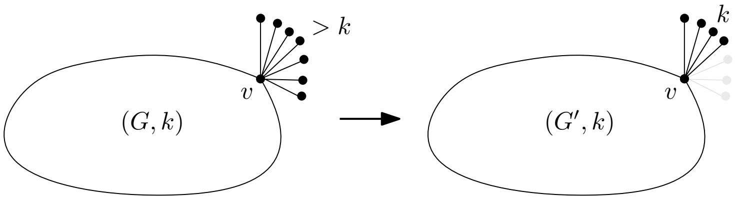

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Known examples of lop-rules:

- Classical reduction rules for VERTEX COVER:
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- Rules based on protrusion replacement, matroids, ...

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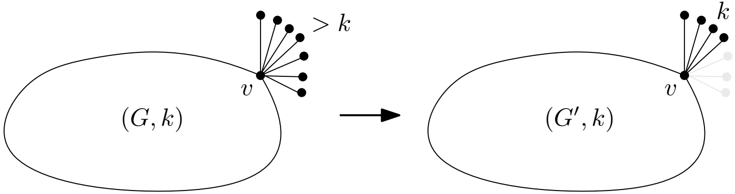
Example of a rule that is not a lop-rule for MMVC:



Want: $\text{mmvc}(G) \geq k \Rightarrow \text{mmvc}(G') \geq \text{mmvc}(G) - (k - k') = \text{mmvc}(G)$.

But $\text{mmvc}(G) > \text{mmvc}(G')$ may happen.

Anyway, we can just answer 'YES', so no problem!

Known examples of lop-rules:

- Classical reduction rules for VERTEX COVER:
  high-degree, crown decomposition, Nemhauser-Trotter.
- Rules based on protrusion replacement, matroids, ...

We have not seen any non-lop-rule for a vertex-maximization problem!

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(Similar statement for vertex-minimization problems.)

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- Rule out polynomial kernels, assuming $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$.

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Thus, the trivial quadratic kernel is "essentially" optimal.

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By just plugging $r = \frac{2}{3}$ in our general result we obtain:

Corollary

If $\mathsf{P} \neq \mathsf{NP}$, Maximum Minimal Feedback Vertex Set *parameterized by the solution size does not admit a lop-kernel with* $\mathcal{O}(k^{3-\varepsilon})$ *vertices.*

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This complements the FPT algorithms for MMVC with this parameter.

[Boria, Della Croce, Paschos. 2015]

[Zehavi. 2017]

Our result rules out the existence of polynomial kernels for MMVC parameterized by treewidth as well.

Next section is...

1. Introduction
2. Our results
3. Some proofs
4. Further research

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From this, it is not difficult to see that we can obtain a polynomial-time approximation algorithm for Π with the desired ratio:

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It holds with $\varepsilon' := \varepsilon^2 \cdot \frac{(1-r)^2}{r}$.

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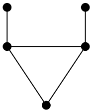

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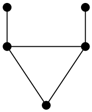

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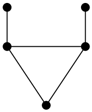

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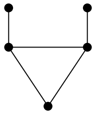

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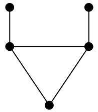

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For all the known cases, such a clique or independent set of size n^{δ} can be found in polynomial time.

Two useful lemmas

Lemma

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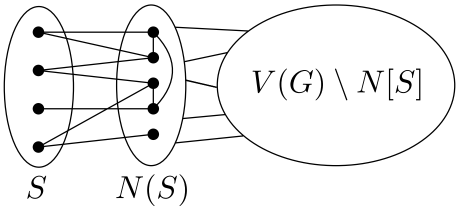
Let G be a graph and let $S \subseteq V(G)$ be an independent set. Then there exists a minimal vertex cover of G containing $N(S)$.



Two useful lemmas

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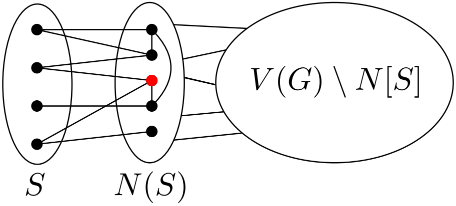
Let H be a graph satisfying the constructive Erdős-Hajnal property with constant $\delta > 0$, and let G be an H-free graph. Then $V(G)$ can be partitioned in polynomial time into a collection of cliques $\mathcal{C}$ and a collection of independent sets $\mathcal{I}$ such that $|\mathcal{C}| + |\mathcal{I}| = \mathcal{O}(|V(G)|^{1-\delta})$.

Lemma

Let G be a graph and let $S \subseteq V(G)$ be an independent set. Then there exists a minimal vertex cover of G containing $N(S)$.



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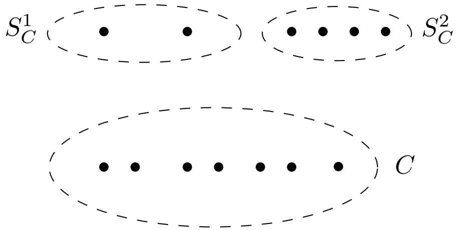

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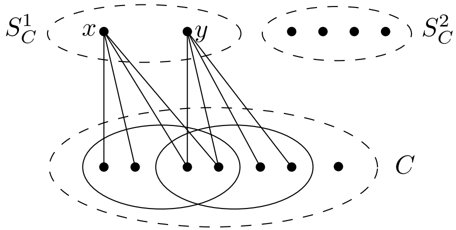

Partition $N_S(C) = S_C^1 \uplus S_C^2$ so that S_C^1 is a maximal subset of $N_S(C)$ s.t. the neighborhoods of its vertices pairwise do not cover all the clique C.

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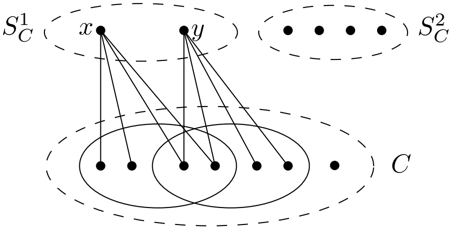

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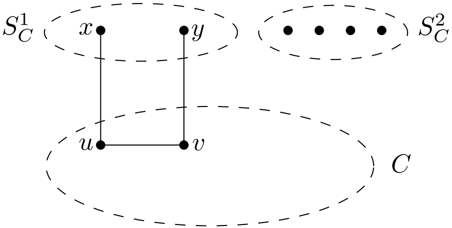

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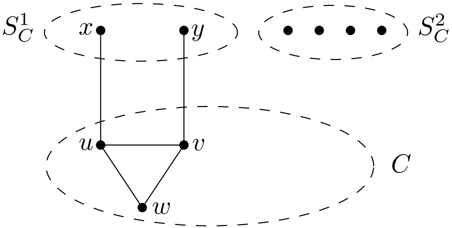

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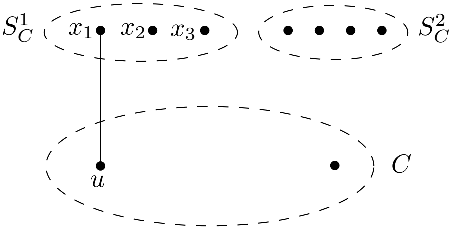

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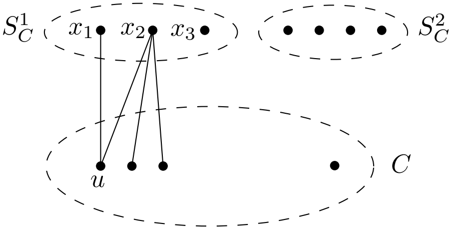

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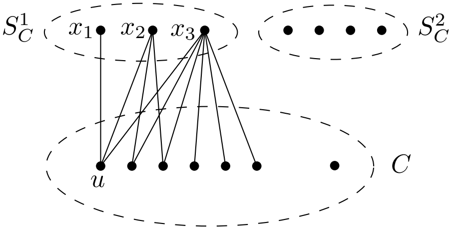

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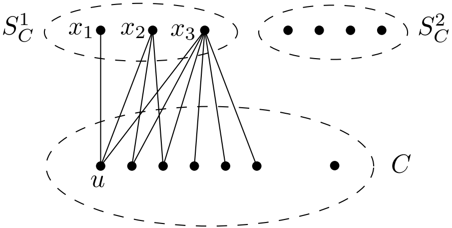

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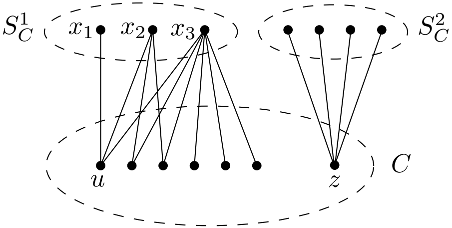

There exists a vertex $z \in C \setminus \bigcup_{y \in S_C^1} N_C(y)$.

It follows that $z \in \bigcap_{x \in S_C^2} N_C(x)$, and since $\deg(z) \leq k-1$, $|S_C^2| \leq k-1$.

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We present a PPT from the MONOTONE SAT problem parameterized by the number of variables, which is known not to admit a polynomial kernel.

[Fortnow, Santhanam, 2011]

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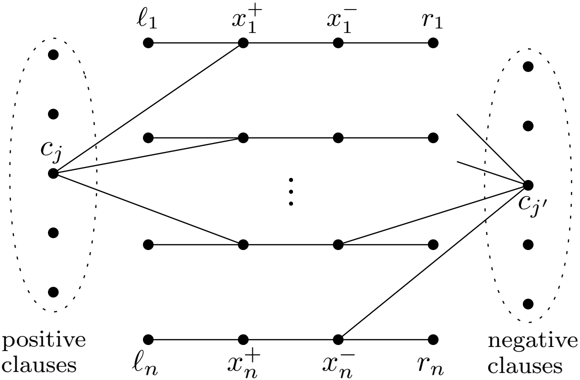

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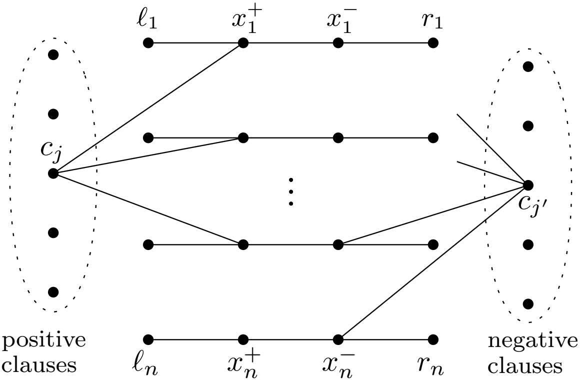

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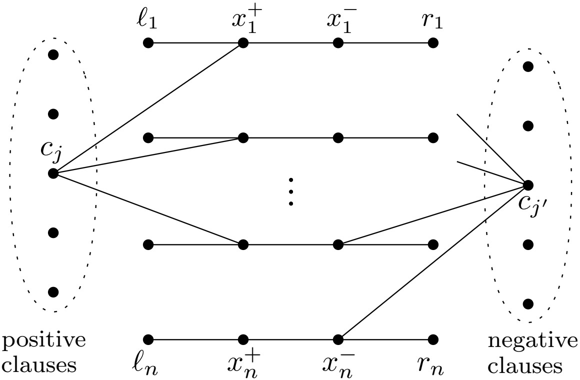

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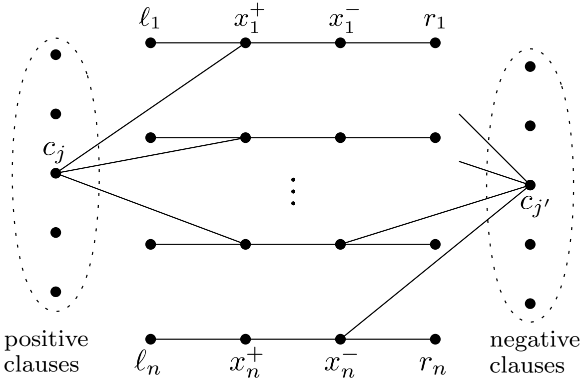

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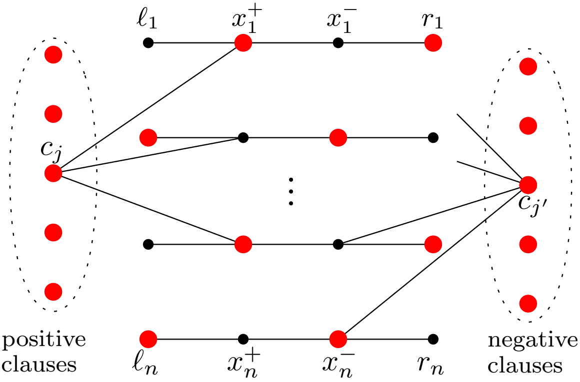

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- Other graphs H satisfying the E-H property: C_4, the diamond, P_5, C_5.
- The complexity of MMVC on P_5-free graphs is open.
- If G is a graph on n vertices without isolated vertices, then $\mathrm{mmvc}(G) \geq \lfloor n^{1/2} \rfloor$. [Boria, Della Croce, Paschos. 2015]

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  If so, it would immediately yield a subquadratic kernel.

Gràcies!