Single-exponential algorithms and linear kernels via protrusion decompositions

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Outline of the talk

1. Preliminaries

2. Protrusion decompositions
   - Definitions
   - A simple algorithm to compute them

3. Single-exponential algorithm for Planar-$\mathcal{F}$-Deletion
   - Motivation and our result
   - Sketch of proof
   - Further research

4. Linear kernels on graphs without topological minors
   - Motivation and our result
   - Idea of proof
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$$f(k) \cdot n^{O(1)}, \quad \text{for some function } f.$$

  Examples: k-VERTEX COVER, k-LONGEST PATH.

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  Examples: k-Vertex Cover, k-Longest Path.

- A single-exponential parameterized algorithm is an FPT algo s.t.

$$f(k) = 2^{O(k)}.$$

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- FPT algorithms based on the structural decomposition result of H-minor-free graphs. [Graph Minors theory by Robertson and Seymour]
- Linear-time algorithms based on modular decompositions.

Next section is...

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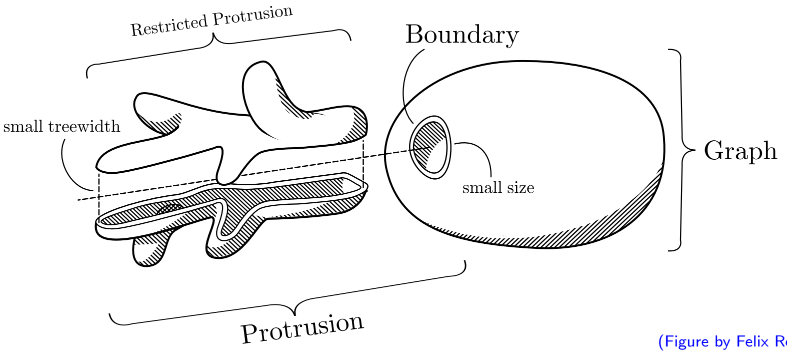
Protrusions

[Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh, Thilikos '09]

- Given a graph G, a set $W \subseteq V(G)$ is a t-protrusion of G if

$$|\partial_G(W)| \leqslant t \quad \text{and} \quad \mathrm{tw}(G[W]) \leqslant t$$



(Figure by Felix Reidl)

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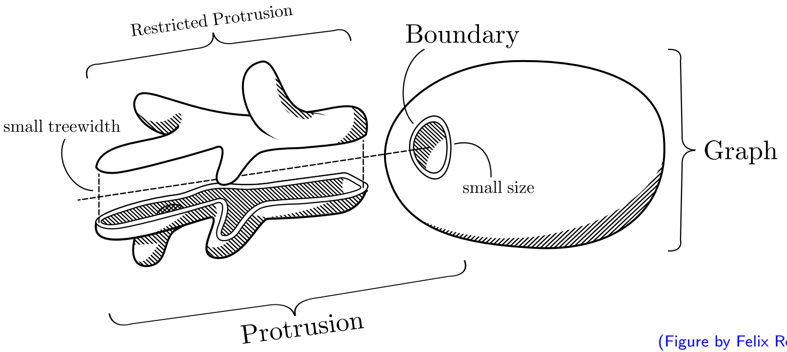


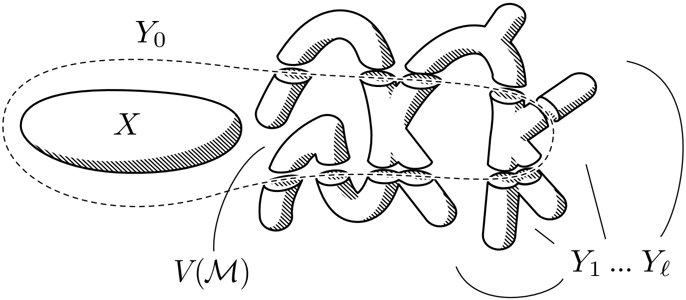
(Figure by Felix Reidl)

- The vertex set $W' = W \setminus \partial_G(W)$ is the restricted protrusion of W.
- We call $\partial_G(W)$ the boundary and $|W|$ the size of W.

Protrusion decompositions

An (α, t)-protrusion decomposition of a graph G is a partition $\mathcal{P} = Y_0 \uplus Y_1 \uplus \cdots \uplus Y_\ell$ of $V(G)$ such that:

- for every $1 \leqslant i \leqslant \ell$, $N(Y_i) \subseteq Y_0$;
- for every $1 \leqslant i \leqslant \ell$, $Y_i \cup N_{Y_0}(Y_i)$ is a t-protrusion of G;
- $\max\{\ell, |Y_0|\} \leqslant \alpha$.



The set Y_0 is called the separating part of $\mathcal{P}$.

(Figure by Felix Reidl)

Next subsection is...

Main (informal) ideas of our algorithm

- Protrusion decompositions have already been used in the literature.

[Bodlaender, Fomin, Lokshtanov, Saurabh, Thilikos '09-12]

Main (informal) ideas of our algorithm

- Here we present a new algorithm to compute protrusion decompositions for graphs G that come equipped with a set

$$\boxed{X \subseteq V(G) \text{ s.t. } \mathrm{tw}(G - X) \leqslant t}$$

for some constant $t > 0$.

The set X is called a t-treewidth-modulator.

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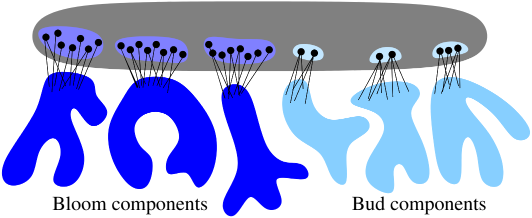
- Given tree-decompositions of the conn. comp. of $G - X$ with $\geqslant r$ neighbors in X, we identify a set of bags $\mathcal{M}$ in a bottom-up manner.



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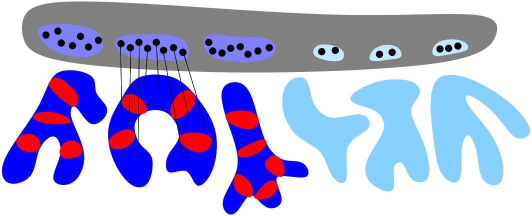

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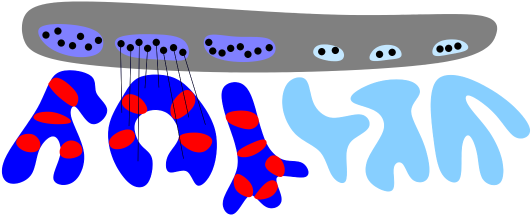

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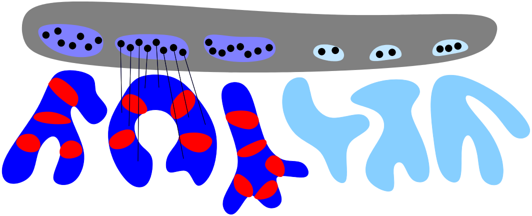

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- Some marked bags will be mapped bijectively into pairwise vertex-disjoint connected subgraphs of $G - X$, each of which has $\geqslant r$ neighbors in X.
- Finally, to guarantee that the conn. comp. of $G - (X \cup V(\mathcal{M}))$ form protrusions with small boundary, the set $\mathcal{M}$ is closed under taking LCA.

Description of the bag marking algorithm

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    if B is the LCA of two marked bags of $\mathcal{M}$:
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Return $Y_0 = X \cup V(\mathcal{M})$.

Some properties of the bag marking algorithm

Lemma

The bag marking algorithm can be implemented to run in $O(n)$ time, where the hidden constant depends only on t and r.

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Given a graph G and a subset $S \subseteq V(G)$, a cluster of $G - S$ is a maximal collection of connected components of $G - S$ with the same neighborhood in S.

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Proposition

- *Let* r, t *be two positive integers,*
- *let* G *be a graph and* $X \subseteq V(G)$ *such that* $\mathrm{tw}(G - X) \leqslant t - 1$,
- *let* $Y_0 \subseteq V(G)$ *be the output of the algorithm with input* (G, X, r), *and*
- *let* $Y_1, \ldots, Y_\ell$ *be the set of clusters of* $G - Y_0$.

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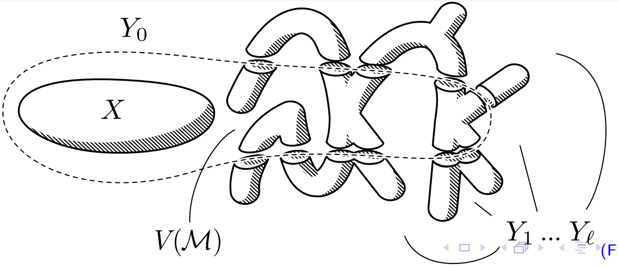
Then $\mathcal{P} := Y_0 \uplus Y_1 \uplus \cdots \uplus Y_\ell$ is a $(\max\{\ell, |Y_0|\}, 2t + r)$-protrusion decomp. of G.



(Figure by Felix Reidl)

Next section is...

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Some particular cases:

1. $\mathcal{F} = \{K_2\}$: $\equiv$ Vertex Cover
   $\equiv$ Treewidth-zero Vertex Deletion
2. $\mathcal{F} = \{K_3\}$: $\equiv$ Feedback Vertex Set
   $\equiv$ Treewidth-one Vertex Deletion
3. $\mathcal{F} = \{K_4\}$: $\equiv$ Treewidth-two Vertex Deletion

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- $\mathcal{F} = \{\theta_c\}$ $O^*(c^k)$ [Joret, Paul, S., Saurabh, Thomassé '11]
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- $2^{O(k \log k)} \cdot n^2$ -time algorithm. [Fomin, Lokshtanov, Misra, Saurabh '11]

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- $2^{O(k \log k)} \cdot n^2$ -time algorithm. [Fomin, Lokshtanov, Misra, Saurabh '11]
- $2^{O(k)} \cdot n \log^2 n$ -time algorithm for PLANAR-CONNECTED-$\mathcal{F}$-DELETION. [Fomin, Lokshtanov, Misra, Saurabh '12]

Our result

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The PLANAR-$\mathcal{F}$-DELETION *problem can be solved in time* $2^{O(k)} \cdot n^2$.

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- This result unifies a number of algorithms in the literature.
- No hope for a $2^{o(k)} \cdot n^{O(1)}$-time algorithm (under ETH). [Chen *et al.* '05]

  That is, the function $2^{O(k)}$ in our theorem is best possible.

Next subsection is...

First step: use iterative compression

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Input: A graph G, a non-negative integer k, and a set $X \subseteq V(G)$ with $|X| = k$ s.t. $G - X$ is $\mathcal{F}$-minor-free.

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Question: Does G have a set $\boxed{\tilde{X} \subseteq V(G) \setminus X}$ such that $|\tilde{X}| < k$ and $G - X$ is H-minor-free for every $H \in \mathcal{F}$?

We call $\tilde{X}$ an alternative solution.

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Lemma (well-kwown)

If DISJOINT PLANAR-$\mathcal{F}$-DELETION *can be solved in time* $O^*(c^k)$ *for some* $c \in \mathbb{N}^+$, *then* PLANAR-$\mathcal{F}$-DELETION *can be solved in* $O^*((c+1)^k)$.

Working hypothesis: an alternative solution $\tilde{X}$ does exist in $G - X$.

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Observation:

If (G, X, k) is a YES-instance of DISJOINT PLANAR-$\mathcal{F}$-DELETION, then

- $G[X]$ is $\mathcal{F}$-minor-free
- $G[V \setminus X]$ is $\mathcal{F}$-minor-free

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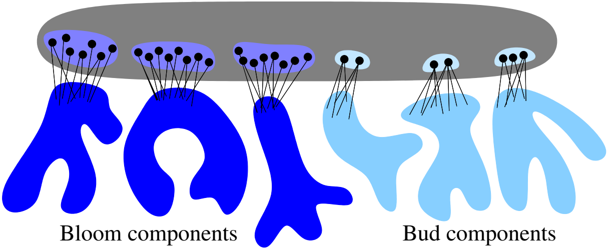
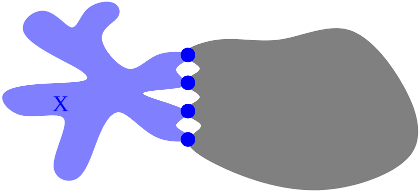


Linear protrusion decompositions

⋆ Recall that a β-protrusion in a graph G is a subset $Y \subseteq V(G)$ such that

$$|\partial(Y)| \leqslant \beta \text{ and } \mathrm{tw}(G[Y]) \leqslant \beta$$



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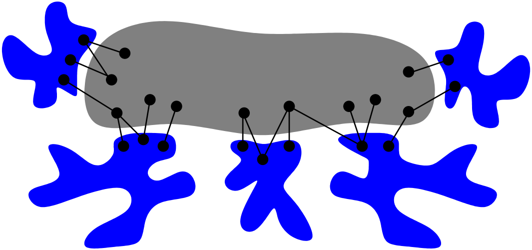

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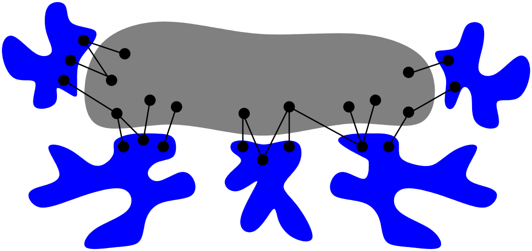

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- ⋆ $\mathcal{P}$ is linear with respect to a parameter k whenever $\alpha = O(k)$.

Algorithm to solve Disjoint Planar-$\mathcal{F}$-Deletion

⋆ We will use our algorithm to compute protrusion decompositions.

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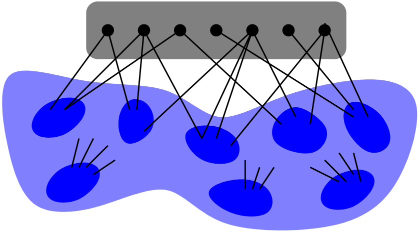
⋆ Both steps can be done in single-exponential time.

First step: analysis of the bag marking algorithm

Lemma (edge simulation to chop bloom components)

If $C_1, \ldots, C_\ell$ is a collection of connected pairwise vertex-disjoint subgraphs of $G - X$ such that $|N_X(C_i)| \geqslant r$ for $1 \leqslant i \leqslant \ell$, then $\boxed{\ell \leqslant (1 + \alpha_r) \cdot k}$.



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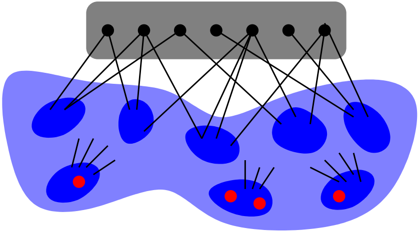

Proposition (Thomason '01)

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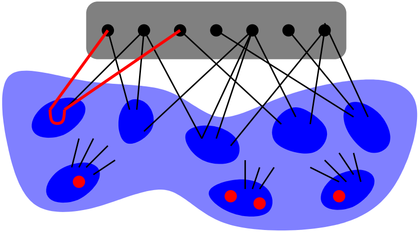

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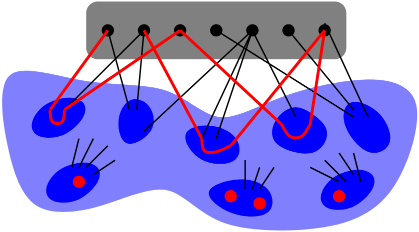

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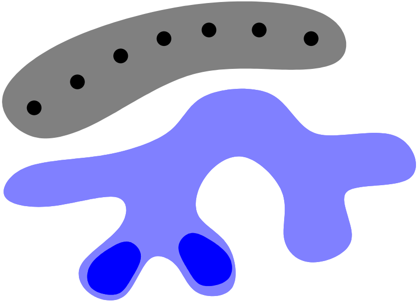

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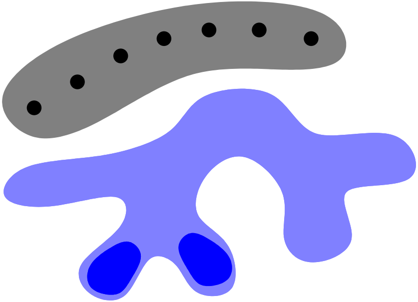

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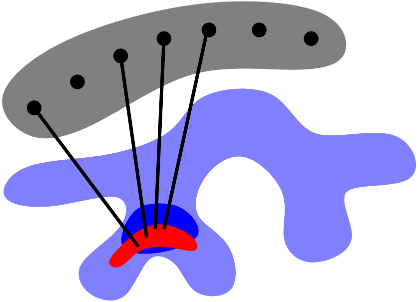

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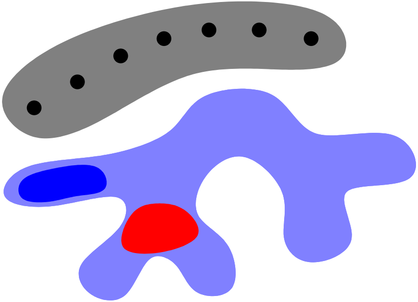

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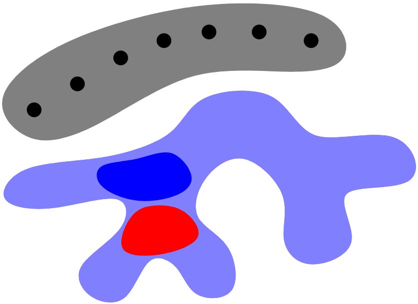

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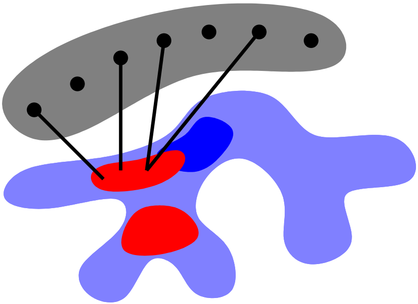

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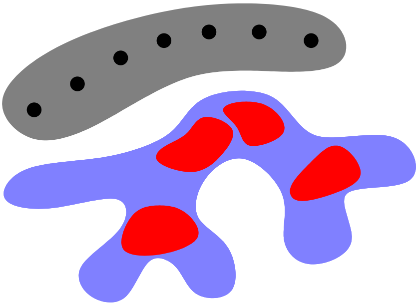

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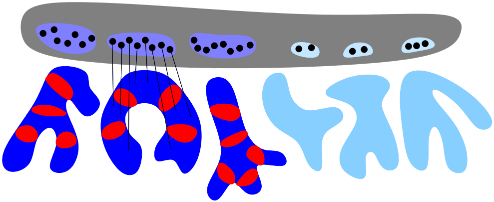

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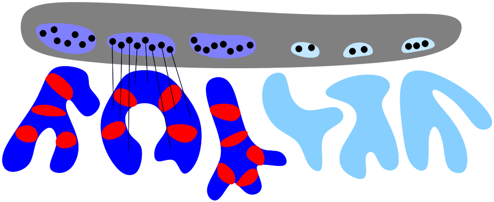

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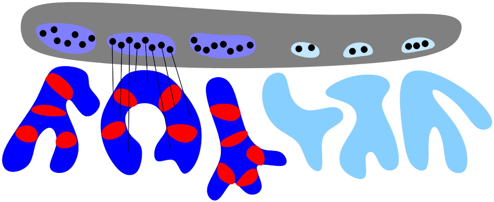

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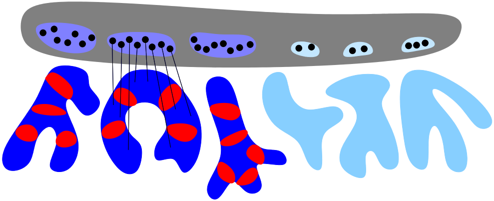

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- $|\mathcal{M}| \leqslant (1 + \alpha_r) \cdot k$ (by the "edge simulation" Lemma)

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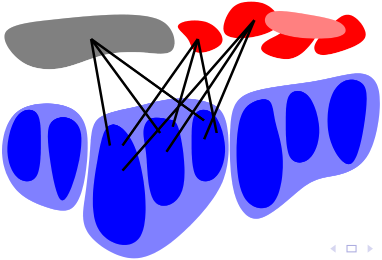
Let $G_I := G - I$. Recall that a cluster of $G_I - Y_0$ is a maximal set of connected components of $G_I - Y_0$ with the same neighborhood in Y_0.



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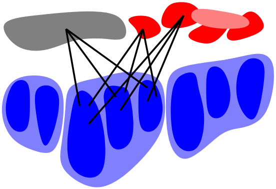

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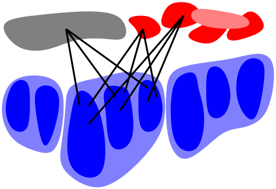

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Proposition (Fomin, Oum, Thilikos '10)

There exists a constant $\mu < 11.355$ such that for all $r > 2$, every n-vertex graph with no K_r-minor has at most $\mu_r \cdot n = 2^{\mu \cdot r \log\log r} \cdot n$ cliques.

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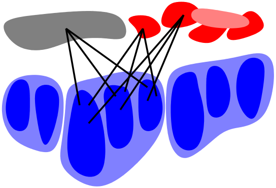

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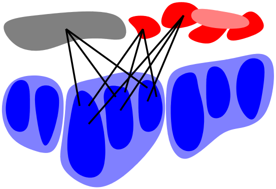
$\star$ At most $\ell' = k - |I|$ clusters $C_1, \ldots, C_{\ell'}$ intersect the alternative solution $\tilde{X}$.

Linear protrusion decomposition (2)



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We have that $G' = G_I - \cup_{i=1}^{\ell'} C_i$ is $\mathcal{F}$-minor-free.

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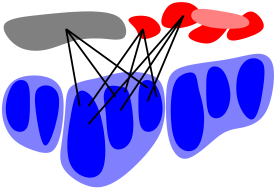

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⋆ Using edge simulation we construct a minor of G' on vertices of Y_0.

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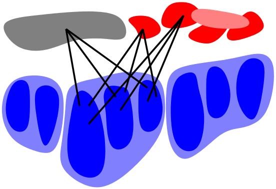

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⋆ As before, the number of clusters used so far is at most $\alpha_r \cdot k$.

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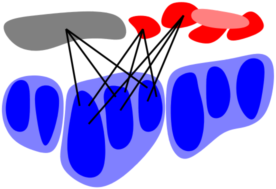

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⋆ When we cannot add more edges, all neighborhoods of clusters are cliques!

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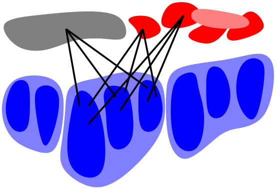

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⋆ Now we use the Proposition: the number of remaining clusters is $\mu_r \cdot k$.

Back to the road map of the algorithm

Therefore, the partition $\mathcal{P} = Y_0 \uplus C_1 \uplus \cdots \uplus C_\ell$ is a

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Recall the two main steps of our algorithm:

1 Guess the intersection $I = \tilde{X} \cap Y_0$ of the alt. solution $\tilde{X}$ with Y_0 s.t.:

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- with $X \subseteq Y_0$ and $\tilde{X} \setminus I \subseteq V(G) \setminus Y_0$.

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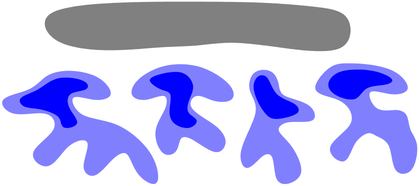
2 Finally, compute $\tilde{X} \setminus I$, given a linear protrusion decomposition.

Based on the finite index of MSO-definable properties (automaton theory)

Solving the problem when given a linear protrusion decomposition



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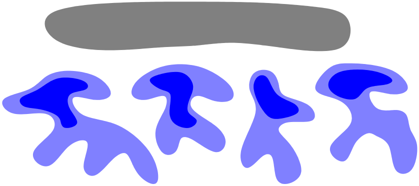

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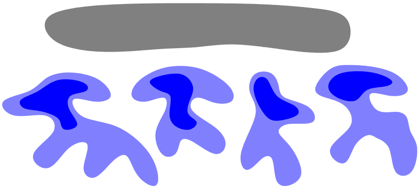

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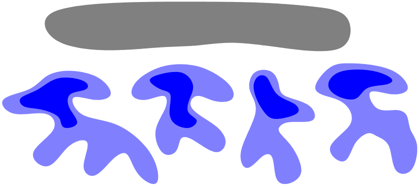

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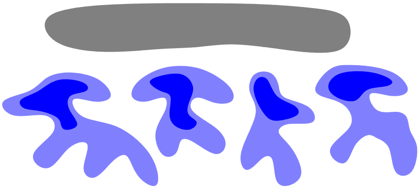

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- ⋆ We use a decomposability property of the solution: there exists a solution which is formed by the union of one representative per restricted protrusion.
- ⋆ To make the algorithm constructive and uniform on the family $\mathcal{F}$, we use classic arguments from tree automaton theory (like method of test sets).

Next subsection is...

1. Preliminaries
2. Protrusion decompositions
   - Definitions
   - A simple algorithm to compute them
3. Single-exponential algorithm for Planar-$\mathcal{F}$-Deletion
   - Motivation and our result
   - Sketch of proof
   - **Further research**
4. Linear kernels on graphs without topological minors
   - Motivation and our result
   - Idea of proof
   - Further research

Conclusions and further research

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- ★ We could forbid the family of graphs $\mathcal{F}$ according to another containment relation, like topological minor.

Next section is...

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Kernels

- A kernel for a parameterized problem Π is an algorithm that given (x, k) outputs, in time polynomial in $|x| + k$, an instance (x', k') s.t.:
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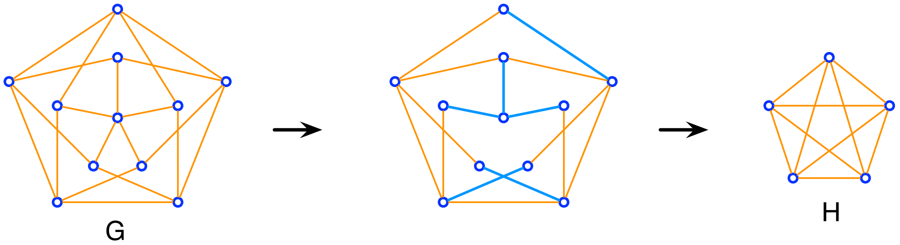
- Question: which FPT problems admit linear or polynomial kernels?

Minors and topological minors



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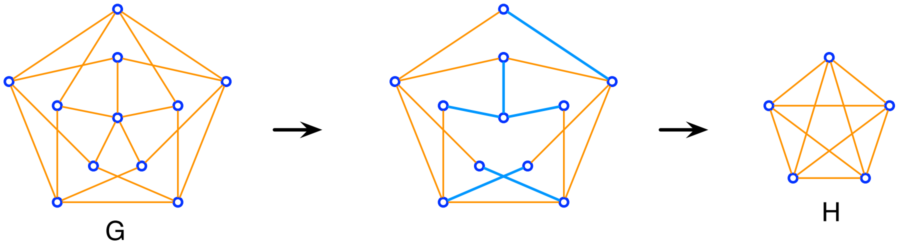

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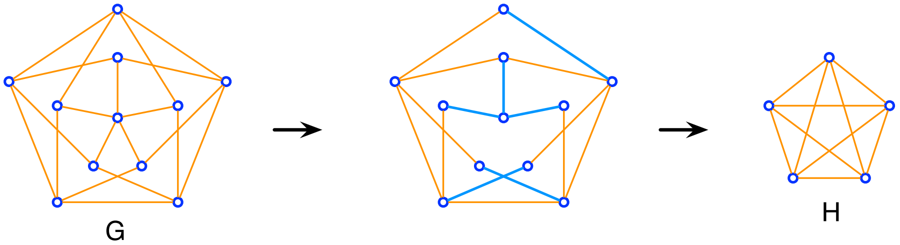

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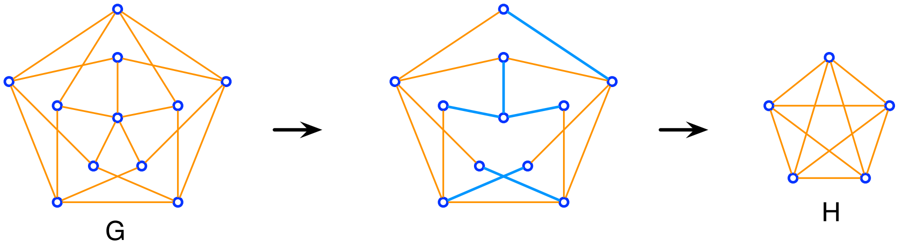

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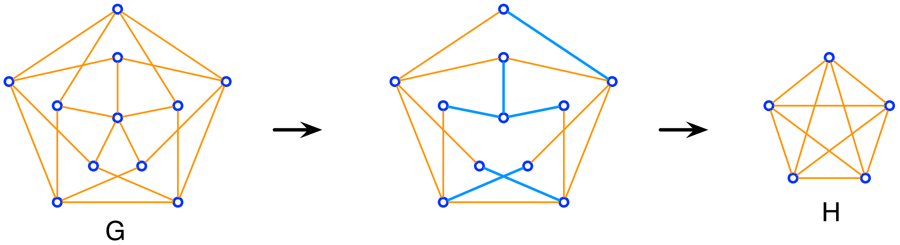

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- Fixed H: H-minor-free graphs $\subseteq$ H-topological-minor-free graphs.

Linear kernels on sparse graphs – an overview

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Our result

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Fix a graph H. Let Π be a parameterized graph problem on the class of H-topological-minor-free graphs that is treewidth-bounding and has finite integer index (FII). Then Π admits a linear kernel.

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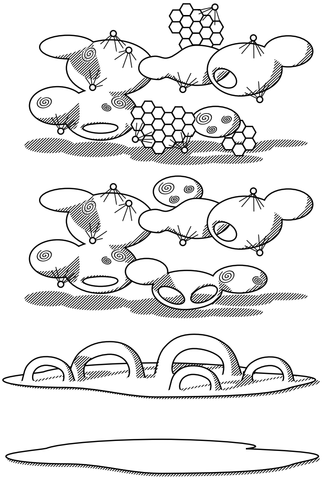
Problems affected by our result:

Treewidth-t Vertex Deletion, Chordal Vertex Deletion, Interval Vertex Deletion, Edge Dominating Set, Feedback Vertex Set, Connected Vertex Cover, ...

Linear kernels on sparse graphs – the conditions

H-topological-minor-free		treewidth-bounding
$\bigcup$		
H-minor-free		bidimensional, separation property
$\bigcup$		
bounded genus		quasi-compact
$\bigcup$		
planar		“distance-property”

(Figure by Felix Reidl)

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- Thus, our results imply the linear kernels of [Fomin, Lokshtanov, Saurabh, Thilikos '10]

Next subsection is...

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- We say that $G_1 \equiv_{\Pi,t} G_2$ if there exists a constant $\Delta_{\Pi,t}(G_1, G_2)$ such that for all t-boundaried graphs H and for all k:
  1. $G_1 \oplus H \in \mathcal{G}$ iff $G_2 \oplus H \in \mathcal{G}$;
  2. $(G_1 \oplus H, k) \in \Pi$ iff $(G_2 \oplus H, k + \Delta_{\Pi,t}(G_1, G_2)) \in \Pi$.

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Disconnected PLANAR-$\mathcal{F}$-DELETION has not FII

- We prove: if $\mathcal{F}$ is a family of graphs containing some disconnected graph H, then PLANAR-$\mathcal{F}$-DELETION has not FII (in general).

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- Then, if we take $1 \leqslant n < m$,

$$\begin{aligned} \pi(G_n \oplus H_{n-1}) - \pi(G_m \oplus H_{n-1}) &= (n-1) - (n-1) = 0, \\ \pi(G_n \oplus H_m) - \pi(G_m \oplus H_m) &= n - m < 0. \end{aligned}$$

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- Thus, $G_n, G_m \notin$ same equiv. class of $\sim_{\Pi,1}$ whenever $1 \leqslant n < m$.

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$\forall$ *fixed* t, $\exists$ *finite set* $\mathcal{R}_t$ *of* t*-boundaried graphs s.t. for each* t*-boundaried graph* $G \in \mathcal{G}_t$ $\exists$ $G' \in \mathcal{R}_t$ *s.t.* $G \equiv_{\Pi,t} G'$ *and* $\Delta_{\Pi,t}(G, G') \geqslant 0$.

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If one is given a t-protrusion $X \subseteq V(G)$ s.t. $\rho'_\Pi(t) < |X|$, then one can, in time $O(|X|)$, find a $2t$-protrusion W s.t. $\rho'_\Pi(t) < |W| \leqslant 2 \cdot \rho'_\Pi(t)$.

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Lemma (Replacing protrusions of constant size)

For $t \in \mathbb{N}$, suppose that the set $\mathcal{R}_t$ of representatives of $\equiv_{\Pi,t}$ is given. If W is a t-protrusion of size at most a fixed constant c, then one can decide in constant time which $G' \in \mathcal{R}_t$ satisfies $G' \equiv_{\Pi,t} G[W]$.

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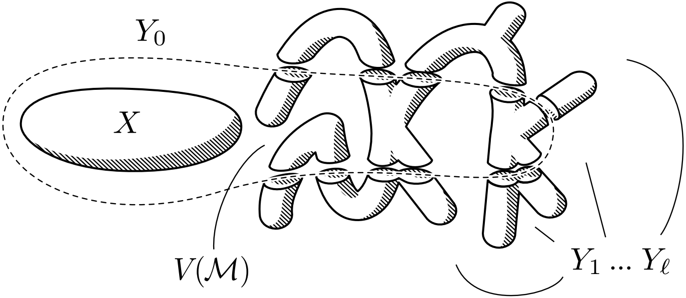
It runs in polynomial time ... given the sets of representatives!

Protrusion decompositions (in case someone forgot!)

An (α, t)-protrusion decomposition of a graph G is a partition $\mathcal{P} = Y_0 \uplus Y_1 \uplus \cdots \uplus Y_\ell$ of $V(G)$ such that:

- for every $1 \leqslant i \leqslant \ell$, $N(Y_i) \subseteq Y_0$;
- for every $1 \leqslant i \leqslant \ell$, $Y_i \cup N_{Y_0}(Y_i)$ is a t-protrusion of G;
- $\max\{\ell, |Y_0|\} \leqslant \alpha$.



(Figure by Felix Reidl)

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   Let Π be a parameterized graph problem that has FII and is t-treewidth-bounding, both on the class of H-topological-minor-free graphs. Then any reduced YES-instance (G, k) has a protrusion decomposition $V(G) = Y_0 \uplus Y_1 \uplus \cdots \uplus Y_\ell$ s.t.:

   1. $|Y_0| = O(k)$;
   2. $|Y_i| \leqslant \rho'_\Pi(2t + \omega_{\mathcal{H}})$ for $1 \leqslant i \leqslant \ell$; and
   3. $\ell = O(k)$.

Next subsection is...

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  2. Graphs of bounded expansion (contains H-topological-minor-free)?

     Obtaining a kernel for Treewidth-t Vertex Deletion on graphs of bounded expansion is as hard as on general graphs.

     Best known kernel: $k^{O(t)}$. [Fomin, Lokshtanov, Misra, Saurabh '12]

- Constructing the kernels? Finding the sets of representatives!!

- Explicit constants? Lower bounds on their size?

Gràcies!