Optimal Permutation Routing on Hexagonal Networks

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COST 293

Outline

- Introduction
  - Statement of the problem
  - Topologies
  - Preliminaries
  - Example
- Algorithm
  - Description
  - Correctness
  - Optimality
- Conclusions

Permutation routing

- The **permutation routing** problem is a **packet routing** problem.
- Each processor is the **origin of at most one packet** and the **destination of no more than one packet**.
- The goal is to **minimize the number of time steps** required to route all packets to their respective destinations.

Statement of the problem

- **Input:**
  - a directed graph $G = (V, E)$ (the *host* graph),
  - a subset $S \subseteq V$ of nodes,
  - and a permutation $\pi : S \rightarrow S$.
    Each node $u \in S$ wants to send a packet to $\pi(u)$.
- **Output:** Find for each pair $(u, \pi(u))$, a path form u to $\pi(u)$ in G.
- **Constraints:**
  - At each step, a packet can move or stay at a node.
  - No arc can be crossed by two packets at the same step.
  - Cohabitation of multiple packets at the same node is allowed.
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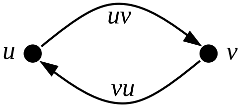
- We consider the **store-and-forward** and **Δ-port** model.
- **Full duplex link**: packets can be sent in the two directions of the link **simultaneously**.



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- If the network is **half-duplex** $\rightarrow$

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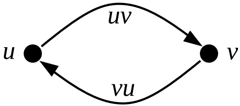

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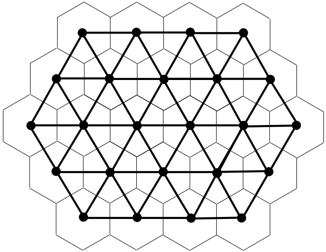
Network topologies

- There is an **ambiguity** in the notation in the literature:

$$\text{triangular grid} \leftrightarrow \text{hexagonal network},$$
$$\text{hexagonal grid} \leftrightarrow \text{honeycomb network}.$$

- Hexagonal network (△) and hexagonal tessellation (⬡):



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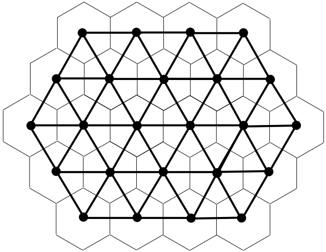

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Previous work

-The permutation routing problem has been studied in:

- Mobile Ad Hoc Networks
- Cube-Connected Cycle Networks
- Wireless and Radio Networks
- All-Optical Networks
- Reconfigurable Meshes...

-But, optimal algorithms:

- 2-circulant graphs, square grids.
- Hexagonal networks: **Two-terminal routing**
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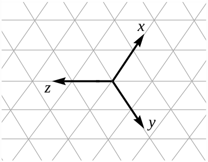
Notation and preliminary results

Nocetti, Stojmenović and Zhang [IEEE TPDS'02]:

Representation of the relative address of the nodes on a generating system ***i***, ***j***, ***k*** on the directions of the three axis x, y, z.



- This address **is not unique**, but we have that, being (a, b, c) and (a', b', c') the addresses of two $D - S$ pairs,

$$(a, b, c) = (a', b', c') \quad \Leftrightarrow \quad \exists \text{ an integer } d \text{ such that}$$

$$a' = a + d,$$
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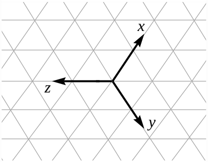

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Notation and preliminary results (2)

- A relative address $D - S = (a, b, c)$ is of the ***shortest path form*** if
  - there is a path C from S to D, C=$a\mathbf{i}$+$b\mathbf{j}$+$c\mathbf{k}$,
  - and C has the shortest length over all paths going from S to D.

Theorem (*NSZ'02*)

*An address (a, b, c) is of the **shortest path form** if and only if*

i) ***at least one component is zero*** *(that is, $abc = 0$),*

ii) *and **any two components do not have the same sign** (that is, $ab \leq 0,\ ac \leq 0,$ and $bc \leq 0$).*

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Corollary (*MSZ'02*)

*Any address has a **unique** shortest path form.*

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If $D - S = (a, b, c)$, then the shortest path form is one of those:

$$(0, b-a, c-a),$$
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and thus:

$$|D - S| = \min(|b-a| + |c-a|, |a-b| + |c-b|, |a-c| + |b-c|).$$

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- Given a packet p and its relative address (a, b, c) *in the shortest path form*,

$$\ell_p := |a| + |b| + |c|,$$

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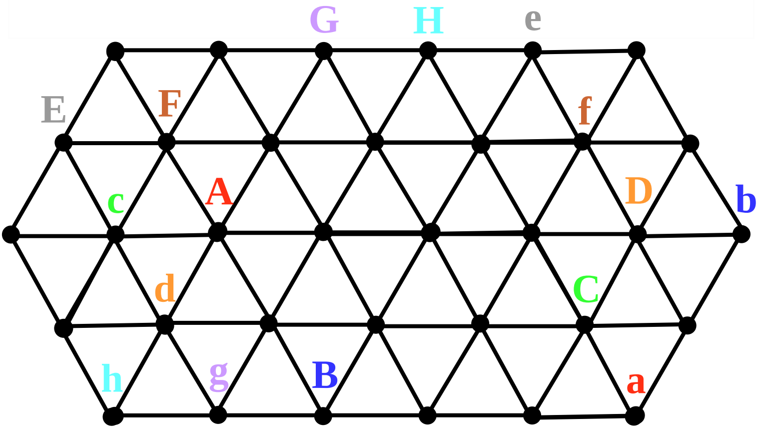
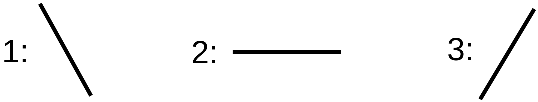
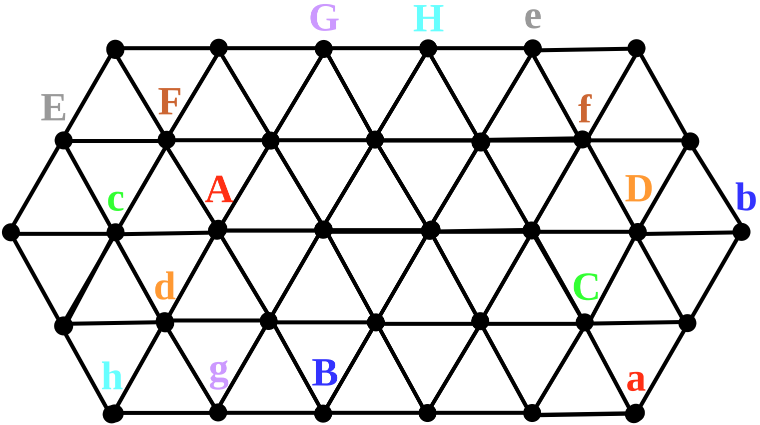
Example of an instance



A non-optimal intuitive algorithm





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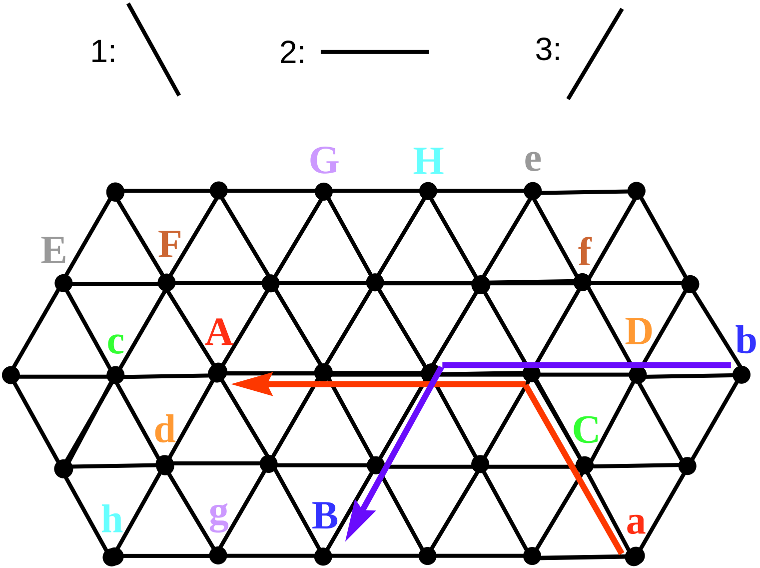

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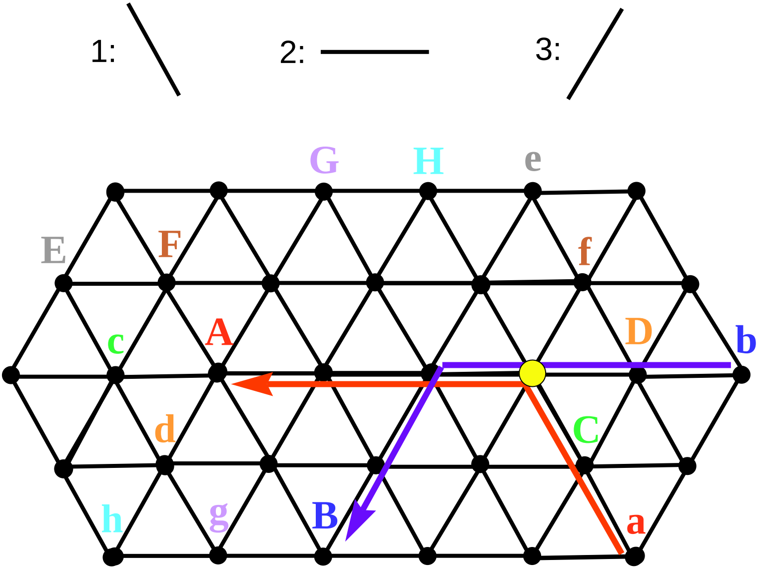

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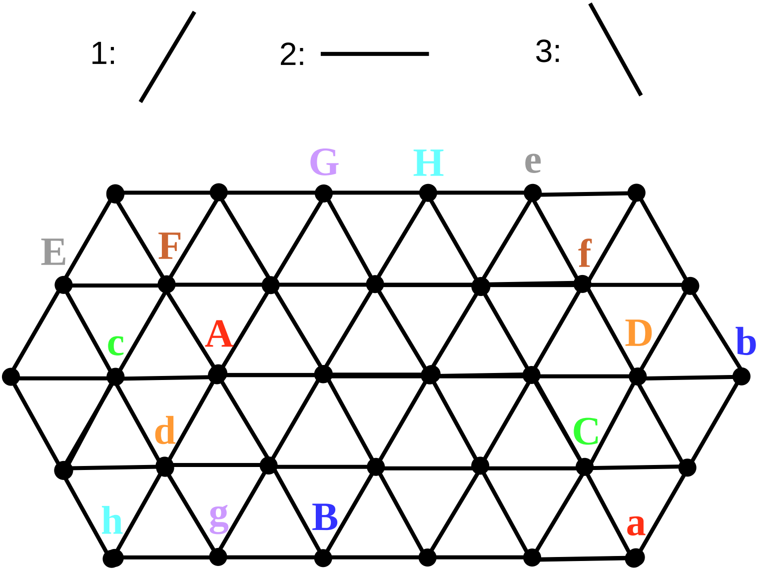

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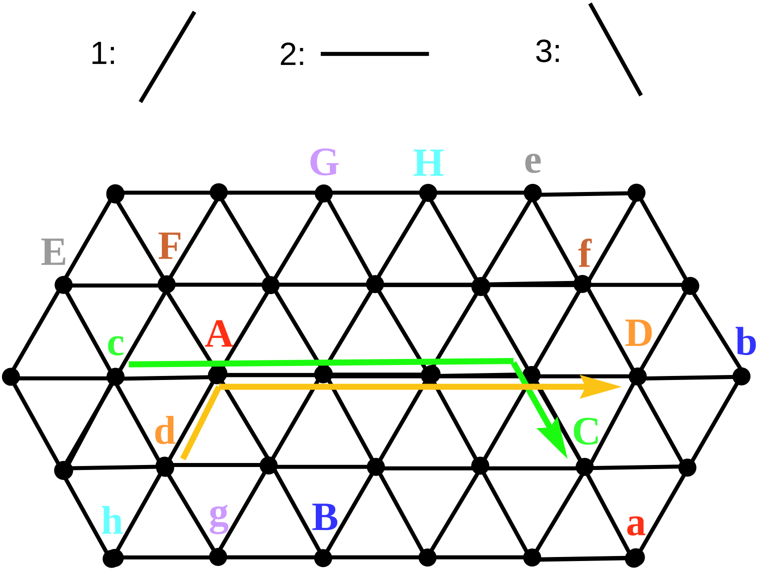

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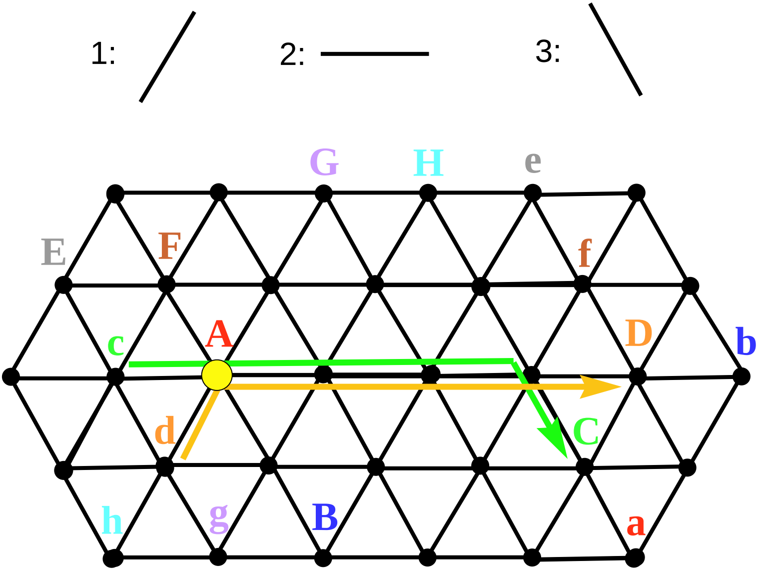

Description of Algorithm $\mathcal{A}$

At each node u of the network:

- ***Preprocessing*:** Initially, if there is a packet at u, compute the relative address $D - S$ of the message in the shortest path form, and add this information to the message.
- ***Reception phase*:** At each step, when a packet is received at u, its relative address is updated.
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  - **a)** If there are packets with **negative components**, send them **immediately** along the direction of this component.
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- For instance:
  - if the packet address is of the type $(-, 0, +) \rightarrow$
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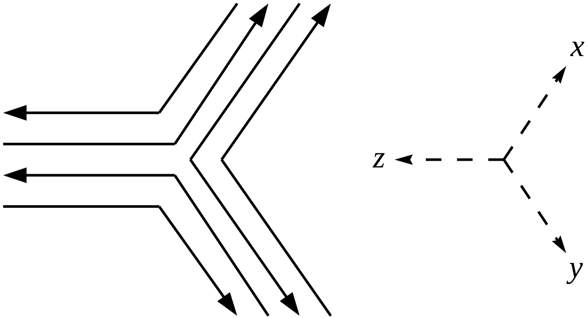
Routing the packets (2)

In this figure all the routing rules are summarized:



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  Packets can only **wait**, possibly, during their **last direction**.

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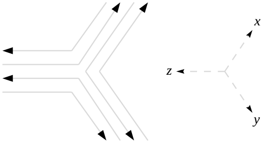

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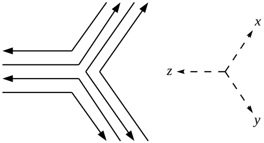

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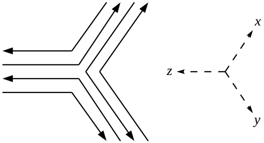

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- All the packets in in **b)** are moving along their **last direction**
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- Using this algorithm, at each step **all the packets with maximum remaining distance move**

  $\rightarrow$ every step the maximum remaining distance over all packets decreases by one

  $\rightarrow$ the total running time is at most ℓ_{max}, **meeting the lower bound**.

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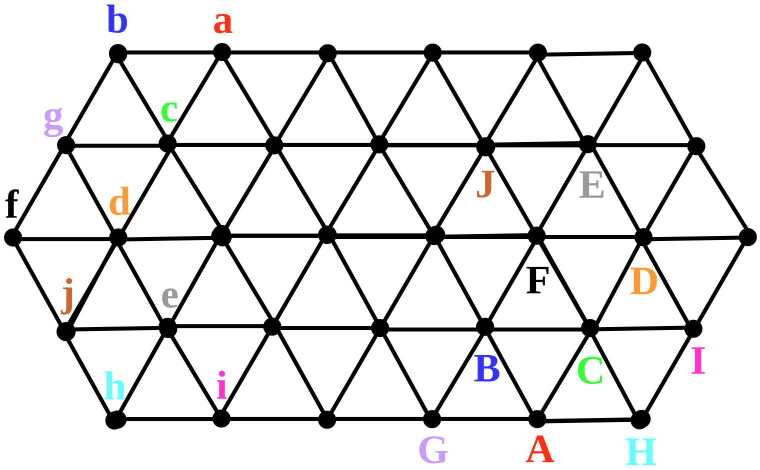

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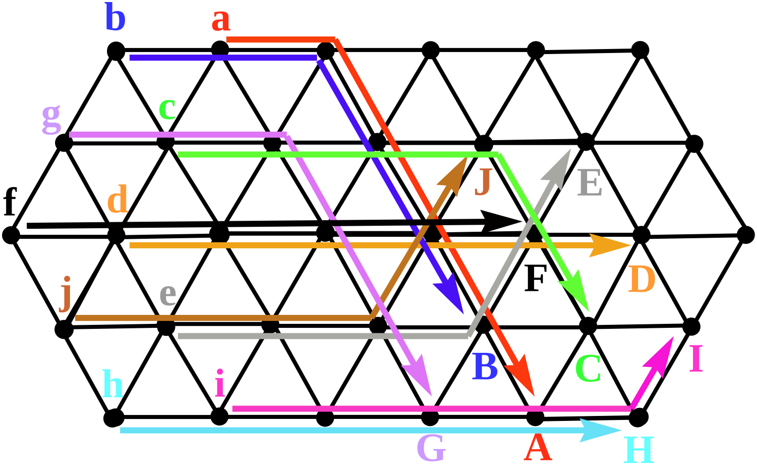

Summary

- We have solved the permutation routing problem for full-duplex triangular grids
- 2 factor approximation algorithm for half-duplex triangular grids
- 2 factor approximation algorithm for full-duplex hexagonal grids

Further research

- $(\ell - k)$-routing
- Hexagonal grid
- Half-duplex case for triangular/hexagonal grids
- Permutation routing on 3-circulant graphs
- Conceive algorithms under *average case* analysis

Thanks!