Maximum Degree-Bounded Connected Subgraph:

Hardness and Approximation

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Outline of the talk

- Definition of the problem
- Example
- State of the art + our results
- Approximation algorithm
- Preliminaries
- Hardness results
- Conclusions and further research

Definition of the problem

- **MAXIMUM DEGREE-BOUNDED CONNECTED SUBGRAPH (MDBCS$_d$)**:

  Input:
  - an undirected graph $G = (V, E)$,
  - an integer $d \geq 2$, and
  - a weight function $\omega : E \to \mathbb{R}^+$.

  Output:
  a subset of edges $E' \subseteq E$ of **maximum weight**, s.t. $G' = (V, E')$
  - is **connected**, and
  - has **maximum degree** $\leq d$.

- It is one of the classical **NP**-complete problems of *[Garey and Johnson, Computers and Intractability, 1979]*.
- If the output subgraph is not required to be connected, the problem is in **P** for any d (using matching techniques).
- For *fixed* $d = 2$ it is the well known LONGEST PATH (OR CYCLE)

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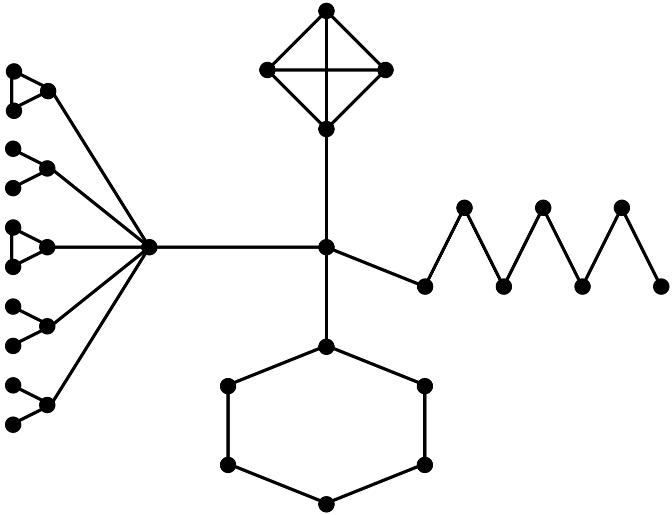
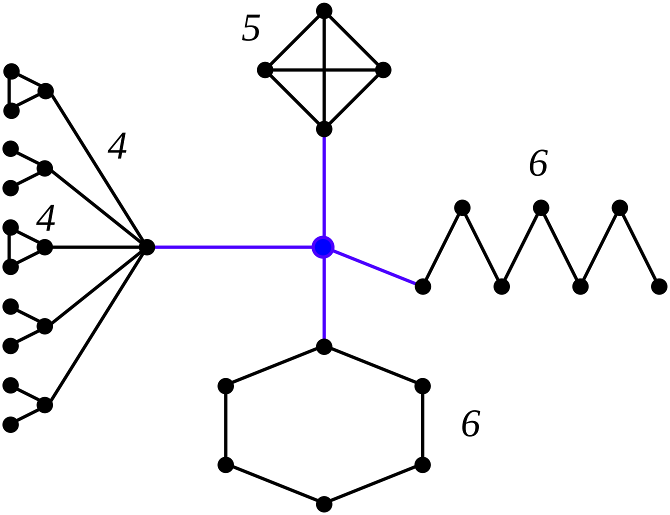
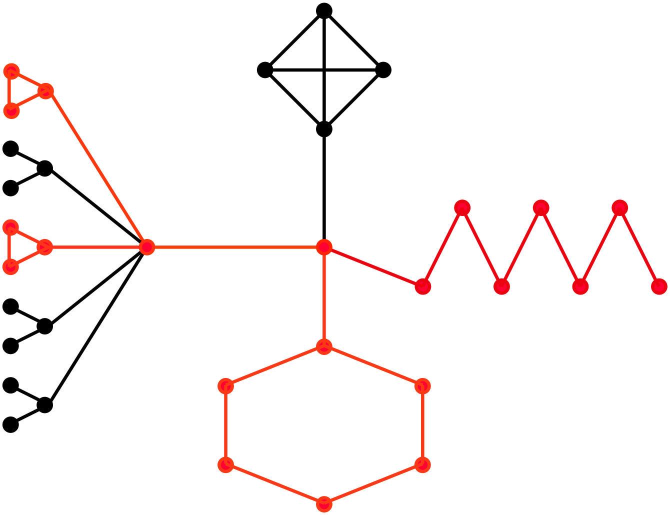
Example with $d = 3$, $\omega(e) = 1$ for all $e \in E(G)$



Example with $d = 3$ (II)



Example with $d = 3$ (III)



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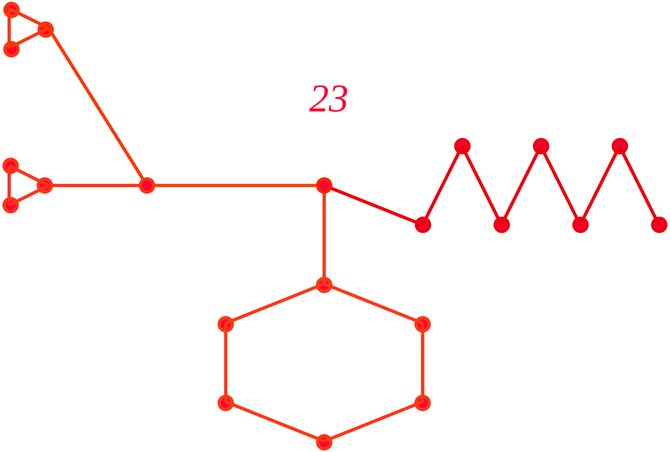

State of the art

To the best of our knowledge, there were no results in the literature except for the case $d = 2$, a.k.a. the **LONGEST PATH** problem:

- **Approximation algorithms**:
  $\mathcal{O}(n/\log n)$-approximation, using the **color-coding** method.
  N. Alon, R. Yuster and U. Zwick, STOC'94.

- **Hardness results**:
  It does not accept *any* constant-factor approximation.
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Our results

- **Approximation algorithms** ($n = |V(G)|$, $m = |E(G)|$):
  - $\min\{\frac{n}{2}, \frac{m}{d}\}$-approximation algorithm for **weighted** graphs.
  - $\min\{\frac{m}{\log n}, \frac{nd}{2 \log n}\}$-approximation algorithm for **unweighted** graphs.
  - when G **accepts a low-degree spanning tree**, in terms of d, then MDBCS$_d$ can be approximated within a **small constant factor**.
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Approximation algorithm for weighted graphs

Input: undirected graph $G = (V, E)$, a weight function $\omega : E \to \mathbb{R}^+$, and an integer $d \geq 2$. Let $n = |V|$, $m = |E|$, and $\rho = \min\{n/2, m/d\}$.

F: set of d heaviest edges in G, with weight $\omega(F)$.
W: set of endpoints of those edges. Let $H = (W, F)$.

Description of the algorithm: Two cases according to $H = (W, F)$:

(1) If $H = (W, F)$ is connected, the algorithm returns H.
Claim: this yields a ρ-approximation.

(2) If $H = (W, F)$ consists of a collection $\mathcal{F}$ of k connected components, we *glue* them in $k - 1$ phases. In each phase:

- For every two components $C, C' \in \mathcal{F}$, compute $d(C, C') = \min\{dist(u, u', G) \mid u \in C, u' \in C'\}$.
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Analysis of the algorithm

(a) Running time: clearly polynomial.

(b) Correctness:

- The output subgraph is connected.
- **Claim**: after i phases, $\Delta(H) \leq d - k + i + 1$.
  The proof is done by induction. When $i = k - 1$ we get $\Delta(H) \leq d$.

(c) Approximation ratio: follows from case (1).

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Preliminaries: hardness of approximation

- **Class APX (Approximable):**

  an NP-complete optimization problem is in APX if it can be approximated within a constant factor.

  ***Example**:* VERTEX COVER

- **Class PTAS (Polynomial-Time Approximation Scheme):**

  an NP-complete optimization problem is in PTAS if it can be approximated within a constant factor $1 + \varepsilon$, for *all* $\varepsilon > 0$ (the best one can hope for an NP-complete problem).

  ***Example**:* MAXIMUM KNAPSACK

Hardness result: idea of the proof

(1) First we prove that $\mathrm{MDBCS}_d \notin \mathrm{PTAS}$:
Reduction from $\mathrm{TSP}(1,2)$.

(2) Then we prove that $\mathrm{MDBCS}_d \notin \mathrm{APX}$:

- Let $\alpha > 1$ be the hardness factor of MDBCS_d given by (1).
- We use a technique called **error amplification**:
  - ⋆ We build a sequence of families of graphs $\mathcal{G}^k$, such that MDBCS_d is hard to approximate in $\mathcal{G}^k$ within a factor α^k, unless $\mathrm{P} = \mathrm{NP}$.
  - ⋆ This proves that the problem is not in APX.
    (for any constant C, $\exists\, k > 0$ such that $\alpha^k > C$).
- Let $G^1 = G$.
  We explain the construction of G^2: first take our graph G and...

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  - ★ We build a sequence of families of graphs $\mathcal{G}^k$, such that MDBCS_d is hard to approximate in $\mathcal{G}^k$ within a factor α^k, unless P $=$ NP.
  - ★ This proves that the problem is not in APX.

    (for any constant C, $\exists\ k > 0$ such that $\alpha^k > C$).

- Let $G^1 = G$.

  We explain the construction of G^2: first take our graph G and...

Hardness result: idea of the proof

(1) First we prove that $\mathrm{MDBCS}_d \notin$ PTAS:

Reduction from $\mathrm{TSP}(1,2)$.

(2) Then we prove that $\mathrm{MDBCS}_d \notin$ APX:

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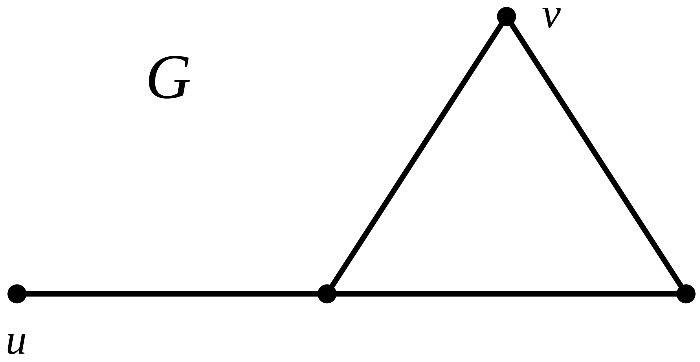
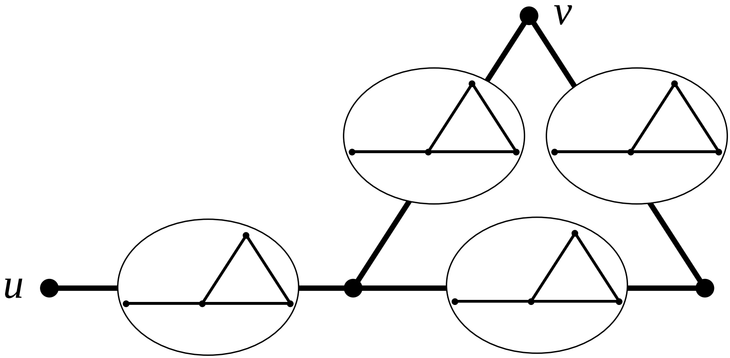
Error amplification to prove that $\text{MDBCS}_d \notin \text{APX}$

For each pair of vertices $\{u, v\} \in V^2$, $u \neq v$, we build the graph $G^2_{u,v}$ in the following way:



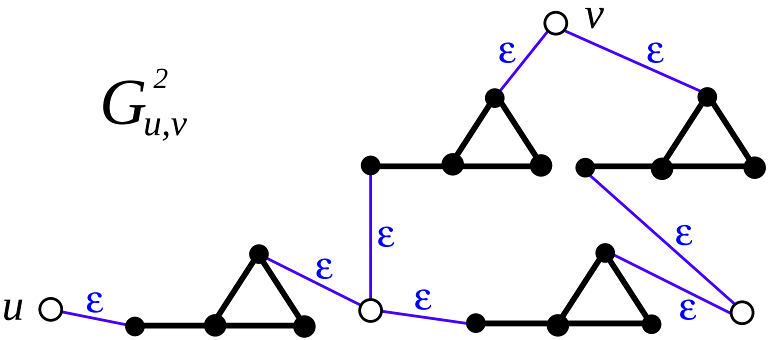
Error amplification to prove that $\mathrm{MDBCS}_d \notin \mathrm{APX}$ (II)

We replace each edge $e_i = (x, y) \in E(G)$ with a copy G_i of G, $i = 1, \ldots, m$:



Error amplification to prove that $\mathrm{MDBCS}_d \notin \mathrm{APX}$ (III)

The copy of the vertex $u \in V(G)$ in G_i is labeled u_i. For each $e_i = (x, y) \in E(G)$, we add the edges (x, u_i) and (y, v_i) with weight ε, $0 < \varepsilon << 1$.



Error amplification to prove that $\text{MDBCS}_d \notin \text{APX}$ (IV)

- Suppose we have an approx. algo $\mathcal{C}$ with ratio ρ. We define G^2 as the graph $G^2_{u,v}$ for which algorithm $\mathcal{C}$ gives the best solution.
- **Claim 1**: $OPT_2 \geq OPT_1^2 + 2\varepsilon \cdot OPT_1 \approx OPT_1^2$.
- **Claim 2**: Given any solution S_2 in G^2 with weight x, it is possible to find a solution S_1 in G with weight at least $\sqrt{x}$.

  To prove the claim, we distinguish two cases:
  - **Case a:** S_2 intersects at least $\sqrt{x}$ copies of G.
    Let S_1 be the subgraph of G induced by the edges corresponding to these copies of G in G^2.
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    Let S_1 be $S_2 \cap G_i$, with G_i being the copy of G in G^2 such that $|E(S_2 \cap G_i)|$ is maximized.

  In both cases S_1 is connected, has maximum degree at most d, and has at least $\sqrt{x}$ edges.
- Combining Claims 1 and 2: if there exists a ρ-approximation in G^2, then it is possible to find a solution for G with weight at least $\sqrt{\frac{OPT_2}{\rho}} \geq \frac{OPT_1}{\sqrt{\rho}} \Rightarrow$ we have a $\sqrt{\rho}$-approximation in G.

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Conclusions and further research

- We have proved that MDBCS_d, $d \geq 2$, is not in APX.

- We have provided approximation algorithms for any d:
    - $\min\{\frac{n}{2}, \frac{m}{d}\}$-approximation algorithm for **weighted** graphs.
    - $\min\{\frac{m}{\log n}, \frac{nd}{2 \log n}\}$-approximation algorithm for **unweighted** graphs.
    - We have also proved that when G **accepts a low-degree spanning tree**, in terms of d, then MDBCS_d can be approximated within a **small constant factor** in **unweighted** graphs.

- **Further Research:**
    - Close the huge complexity gap of MDBCS_d, $d \geq 2$.
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Moltes gràcies!

Força Barça!!

20h45: Manchester United - F.C. Barcelona