Compound Logics for Modification Problems

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Thanks Dimitrios for many of the slides!!

Plan of the talk

1. Motivation for algorithmic meta-theorems based on logic
2. Definition of the new logic(s) and our results
3. Necessity of the ingredients of the logic
4. Sketch of some ideas of the proofs
5. Further research

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Input:	A graph G and an integer k ("amount of modification").
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Want: algorithms in time $\boxed{f(\varphi) \cdot n^{\mathcal{O}(1)}}$, where $n = |V(G)|$.

Algorithmic Meta-Theorems (AMTs)

For some logic L *and some class* $\mathcal{C}$ *of combinatorial structures, every algorithmic problem* Π *that is expressible in* L*, there is an* efficient *algorithm solving* Π *for inputs that belong in* $\mathcal{C}$*.*

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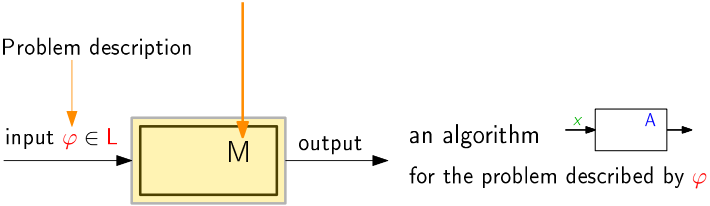

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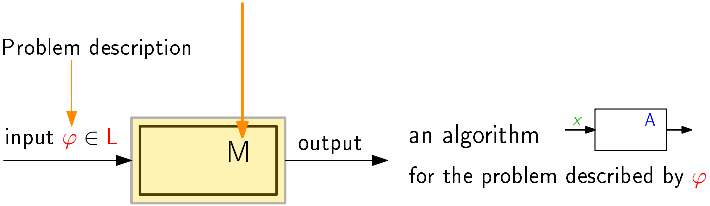
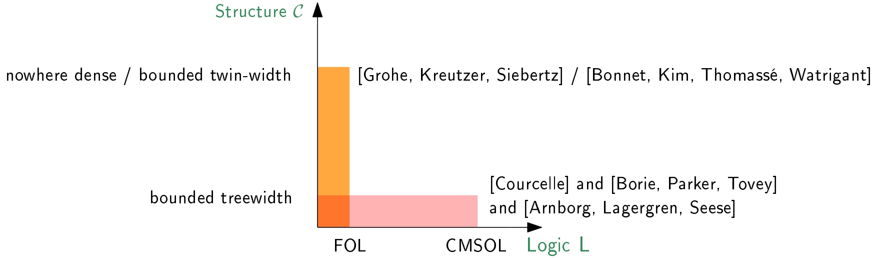


Two main logics for φ:

- FOL: First Order Logic
  - ▶ quantification on vertices or edges
- CMSOL: Counting Monadic Second Order Logic
  - ▶ quantification on sets of vertices or edges

Famous AMTs for model-checking in time FPT



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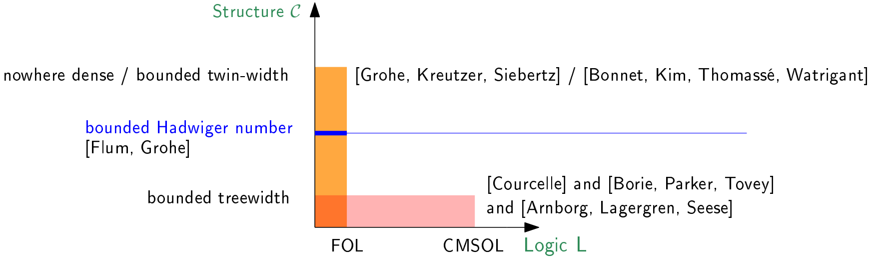

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Hadwiger number: **hw**$(G)=$ max clique-minor of the graph G

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VERTEX DELETION TO PLANARITY
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Modulator: $X = \{x_1, \ldots, x_k\}$

Target property: minor-exclusion of $\mathcal{H} = \{K_5, K_{3,3}\}$

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Topological minor exclusion:

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[Fomin, Lokshtanov, Panolan, Saurabh, Zehavi, STOC 2020]

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- Extensions to general minor-closed target classes $\mathcal{G}$?

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- More general modification operations do not seem to be captured...

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- $\mathcal{G}$: target graph class (example: planar + 3-regular).

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- Can we define successive target properties?
- Hierarchical clustering?
- Multi-level modification?
- Consider different modification scenarios?
- We may demand target conditions to be satisfied by the connected components (or even the blocks) of $G \setminus X$ (CMSOL-demand).
- MULTIWAY CUT or MULTICUT to some target property $\mathcal{G}$.
- We may demand vertex/edge removals with prescribed adjacencies.
- ...

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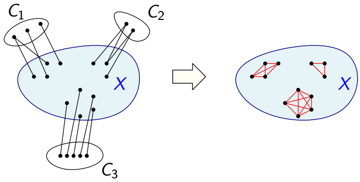

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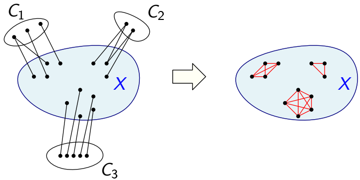

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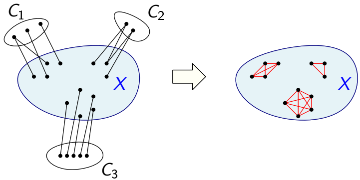

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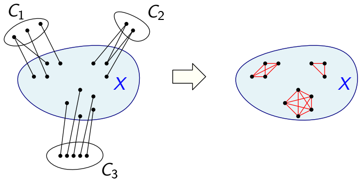

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- **p=bridge-depth**: $\mathcal{G}$-bridge-depth:

[Bougeret, Jansen, S., ICALP 2020]

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- or just $G[X] \models \beta_k$ for some $\beta_k \in \text{CMSOL}^{\text{tw}}$?

  - $\text{CMSOL}^{\text{tw}}[\mathtt{E}, \mathtt{X}]$ (on annotated graphs):
    every $\beta \in \text{CMSOL}[\mathtt{E}, \mathtt{X}]$ for which there exists some c_β such that the torsos of all the models of β have treewidth at most c_β.

Is there **one** meta-theorem that deals with **all** these cases?

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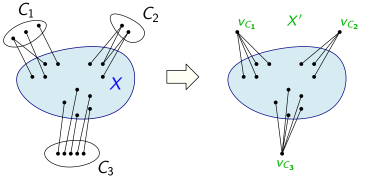

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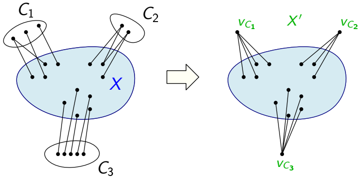

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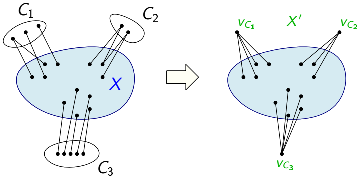

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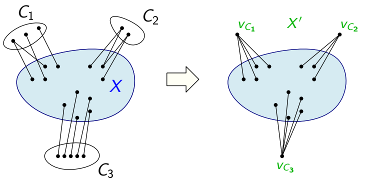

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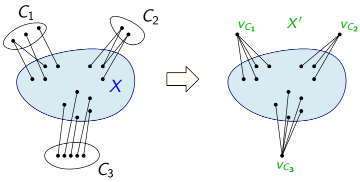

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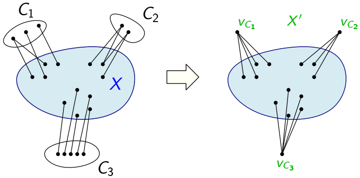

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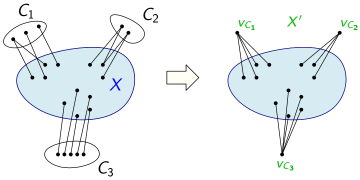

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Theorem (our result, in a less simple form)

For every $\beta \in \mathsf{CMSOL}^{\mathsf{tw}}$ and every $\gamma \in \Theta_0^{(\mathsf{c})}$, there is an algorithm deciding $\mathrm{Mod}(\beta \triangleright \gamma)$ in quadratic time.

- for $\varphi \in \mathsf{CMSOL}$, define $\varphi^{(\mathsf{c})}$: $G \models \varphi^{(\mathsf{c})}$ if $\forall C \in \mathsf{cc}(G), C \models \varphi$.
- for $\mathsf{L} \subseteq \mathsf{CMSOL}$, define $\mathsf{L}^{(\mathsf{c})} = \mathsf{L} \cup \{\varphi^{(\mathsf{c})} \mid \varphi \in \mathsf{L}\}$.

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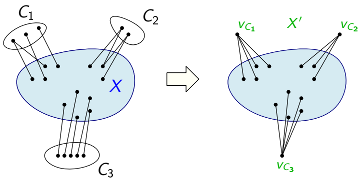

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Theorem (our result, in a simple form)

For every $\beta \in \mathsf{CMSOL}^{\mathsf{tw}}$ and every $\gamma \in \mathbf{MB}(\Theta_0^{(c)})$, there is an algorithm deciding $\mathrm{Mod}(\beta \triangleright \gamma)$ in quadratic time.

- **MB**(L): all monotone Boolean combinations of sentences in L.

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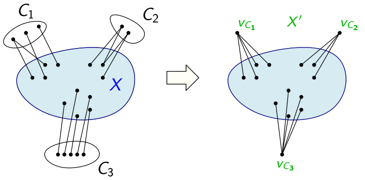

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*For every $\beta \in \text{CMSOL}^{\text{tw}}$ and every $\gamma \in$ **MB**$(\Theta_0^{(c)})$, there is an algorithm deciding $\text{Mod}(\beta \triangleright \gamma)$ in quadratic time.*

- This automatically implies algorithms in all aforementioned directions, beyond the applicability of the theorems of Courcelle and Grohe and Flum.

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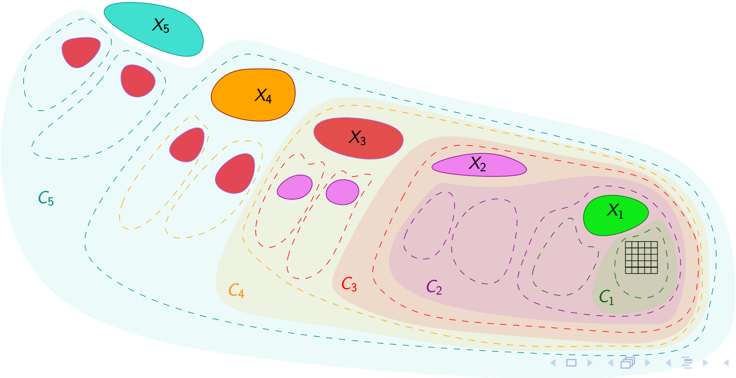

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Our results are **constructive**:

Theorem

There is a **Meta-Algorithm** M *that,*
with input a sentence $\theta \in \Theta$ and an upper bound c_θ on $\mathbf{hw}(\text{Mod}(\theta))$,
returns as output the algorithm $\mathbf{A}_\theta$.

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For every $\tilde{\theta} \in \tilde{\Theta}$, there is an algorithm deciding $\mathrm{Mod}(\tilde{\theta})$ in quadratic time on graphs of fixed Hadwiger number.

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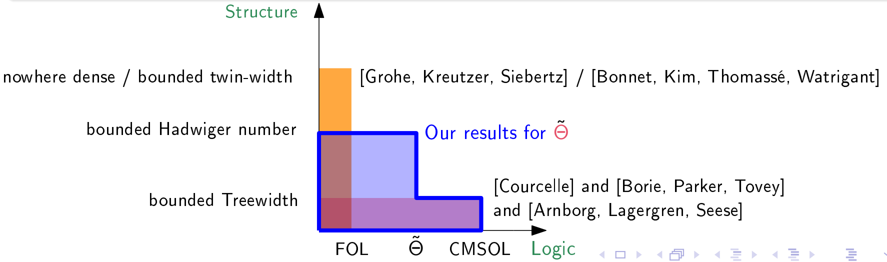

Generalization to extensions of FOL

First-Order Logic with Connectivity Operators

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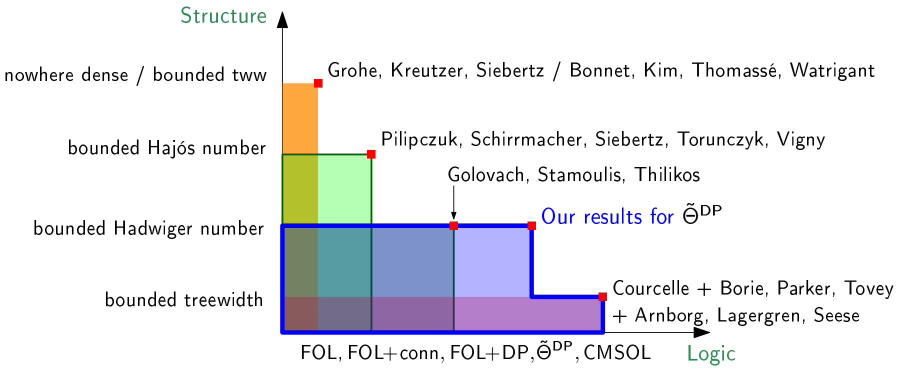
The current meta-algorithmic landscape



Missing: FOL + DP, FPT model-checking up to bounded Hajós number.

[Schirrmacher, Siebertz, Stamoulis, Thilikos, Vigny, arXiv 2023]

Necessity of the ingredients of our logic

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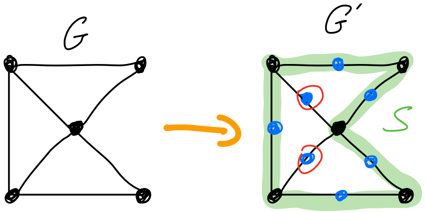

- G Hamiltonian $\Leftrightarrow$ G' has a vertex set S such that $G'[S]$ is a cycle and $G' \setminus S$ is edgeless.
- $\mathrm{tw}(G'[S]) = 2$ but $\mathrm{tw}(\mathsf{torso}(G',S)) = \mathrm{tw}(G)$ unbounded.

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3. Why the target sentence μ expresses proper minor-exclusion?

Expressing whether a graph G contains a clique on k vertices is FOL-expressible, while k-CLIQUE is W[1]-hard on general graphs
(again, consider a void modulator).

Basic ingredients and techniques of the proof(s)

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- The combinatorial/algorithmic results in

1. Ken-ichi Kawarabayashi, Robin Thomas, and Paul Wollan. A new proof of the flat wall theorem. *Journal of Combinatorial Theory, Series B*, 129:204–238, 2018.
2. Ignasi Sau, Giannos Stamoulis, and Dimitrios M. Thilikos. A more accurate view of the Flat Wall Theorem, 2021. arXiv:2102.06463.
3. Petr A. Golovach, Giannos Stamoulis, and Dimitrios M. Thilikos. Hitting topological minor models in planar graphs is fixed parameter tractable. In *Proc. of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms, (SODA)*, pages 931–950, 2020.
4. Julien Baste, Ignasi Sau, and Dimitrios M. Thilikos. A complexity dichotomy for hitting connected minors on bounded treewidth graphs: the chair and the banner draw the boundary. In *Proc. of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 951–970, 2020.
5. Ignasi Sau, Giannos Stamoulis, and Dimitrios M. Thilikos. k-apices of minor-closed graph classes. I. Bounding the obstructions. Transactions on Algorithms 2022.
6. Fedor V. Fomin, Petr A. Golovach, Giannos Stamoulis, and Dimitrios M. Thilikos. An algorithmic meta-theorem for graph modification to planarity and FOL. In *Proc. of the 28th Annual European Symposium on Algorithms (ESA)*, volume 173 of *LIPIcs*, pages 51:1–51:17, 2020.

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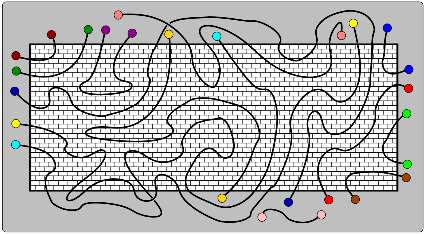
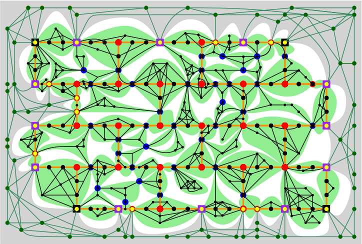
Irrelevant Vertex Technique (> 1200 citations and used in > 120 papers)

- Neil Robertson and Paul D. Seymour. Graph minors. XIII. The disjoint paths problem. *Journal of Comb. Theory, Ser. B*, 63(1):65–110, 1995.





Ultra-sketch of proof

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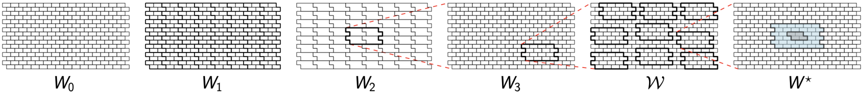

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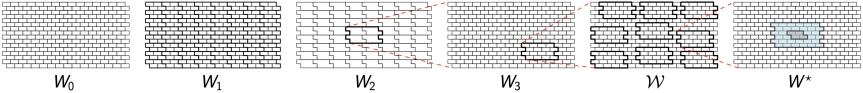
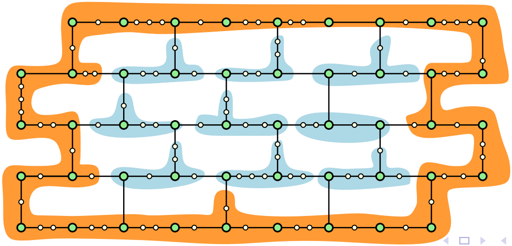


Crucial fact: the fact that the modulator sentence $\beta \in \mathrm{CMSOL}^{\mathrm{tw}}$ allows to prove that the removal of the modulator X does not destroy a flat wall too much.



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High-level sketch of proof

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Theorem (Flat Wall Theorem. Robertson and Seymour. 1995)

There exist recursive functions $f_1 : \mathbb{N}^2 \to \mathbb{N}$ *and* $f_2 : \mathbb{N} \to \mathbb{N}$, *such that for every graph* G *and every* $q, r \in \mathbb{N}$, *one of the following holds:*

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There are several different variants and optimizations of this theorem...

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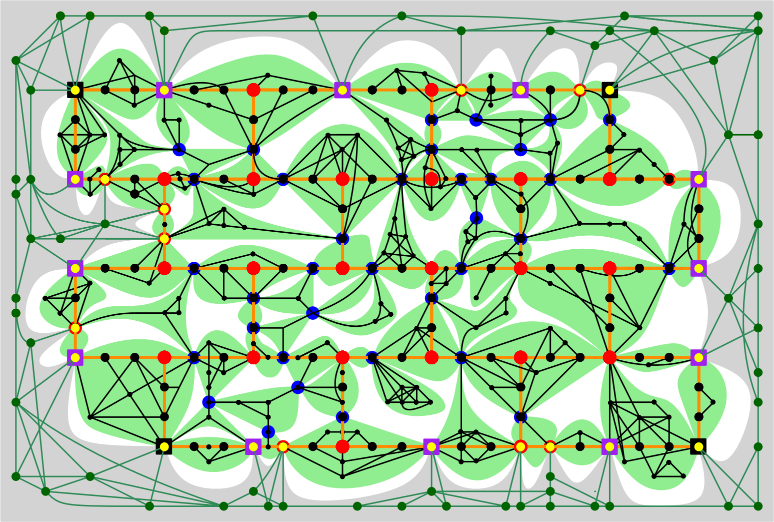
Important: possible to find one of the outputs in time $f(q, r) \cdot |V(G)|$.

How does a flat wall look like?



[Figures by Dimitrios M. Thilikos]

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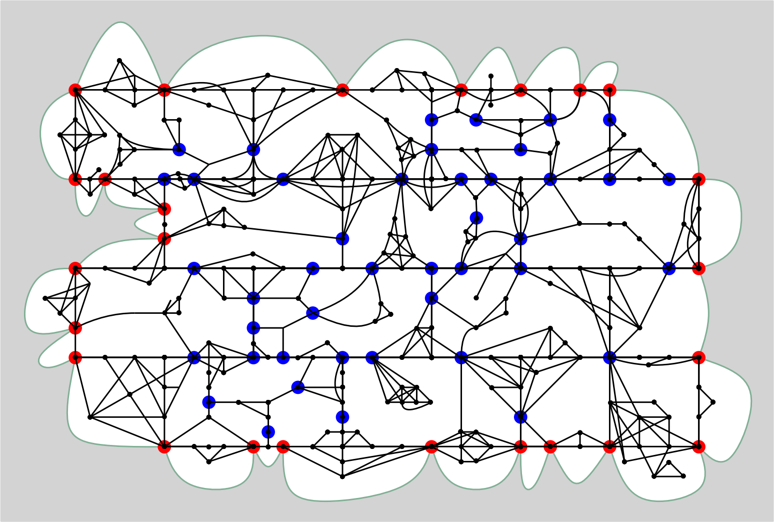

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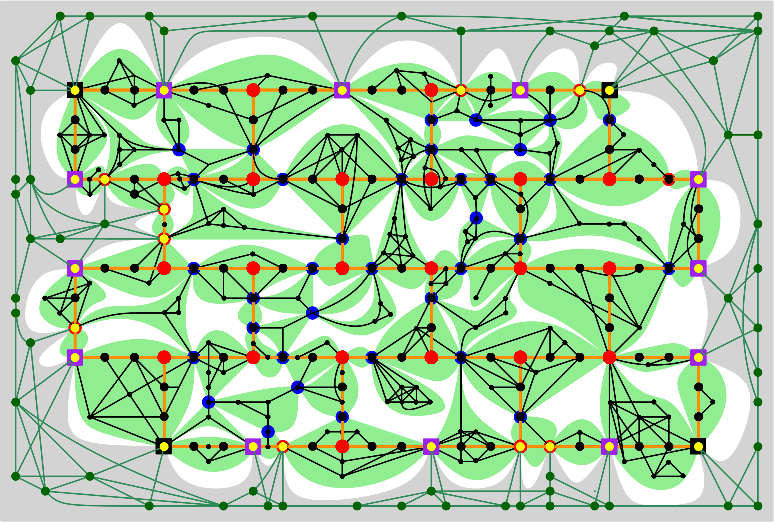

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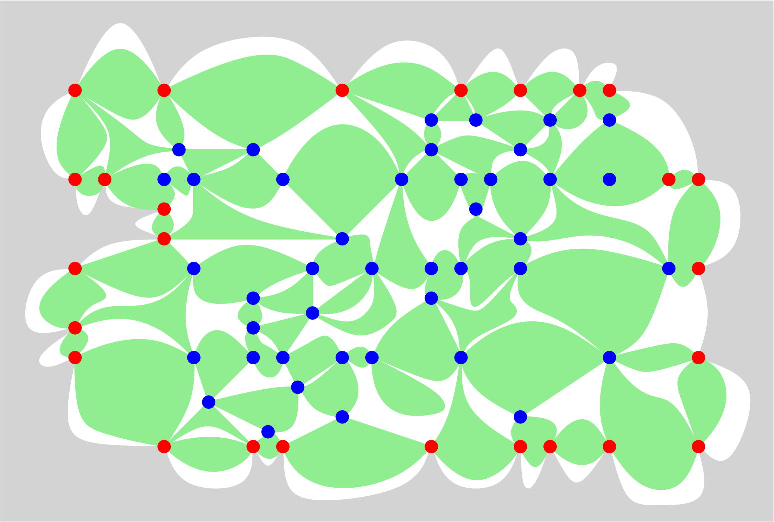

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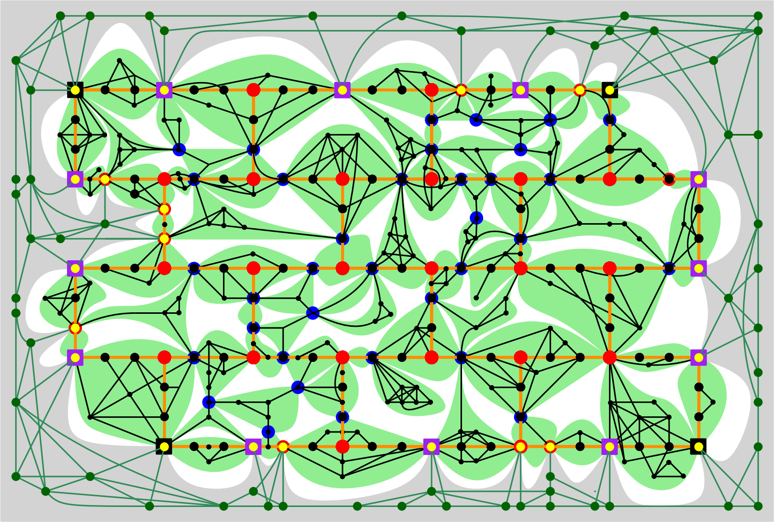

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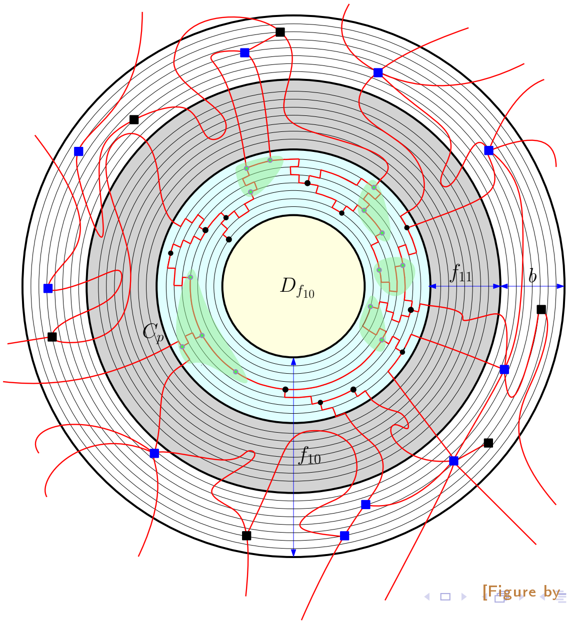
skip

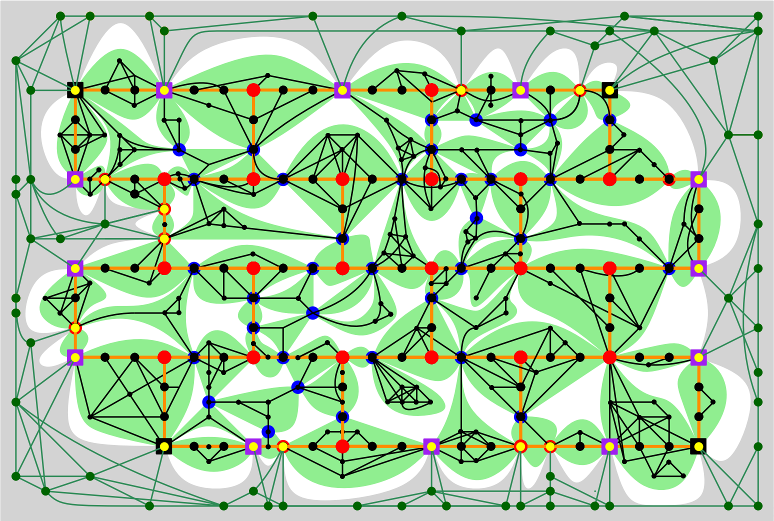
Rerouting inside a flat wall can be painful...



[Figure by Dimitrios M. Thilikos]

Crucial notion: homogeneity

In order to declare a vertex irrelevant for some problem, usually we need to consider a homogenous flat wall, which we proceed to define.



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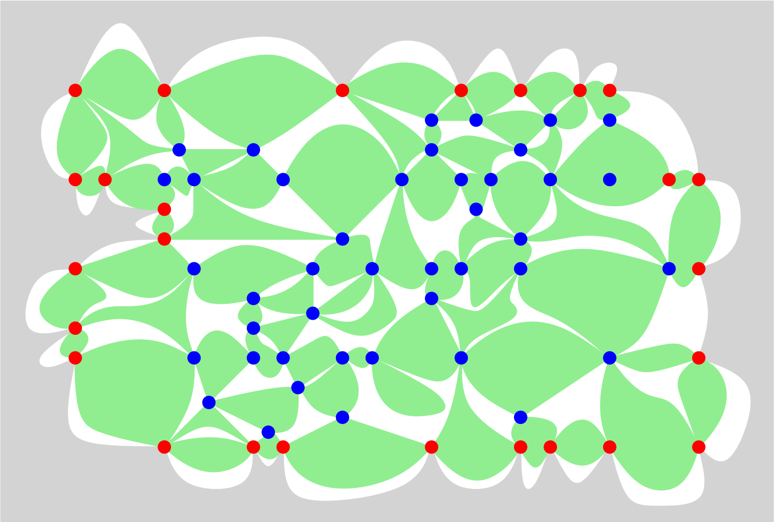
We consider a flap-coloring encoding the relevant information of our favorite problem inside each flap (similar to tables of DP).



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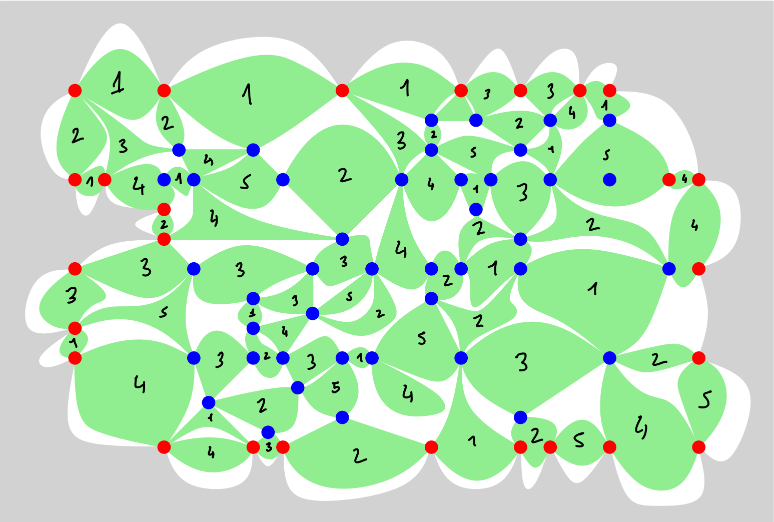
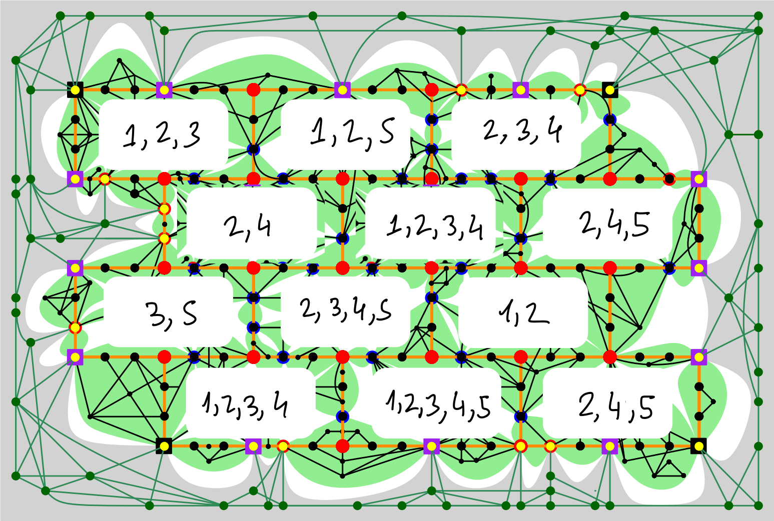


Crucial notion: homogeneity

For every brick of the wall, we define its palette as the colors appearing in the flaps it contains.



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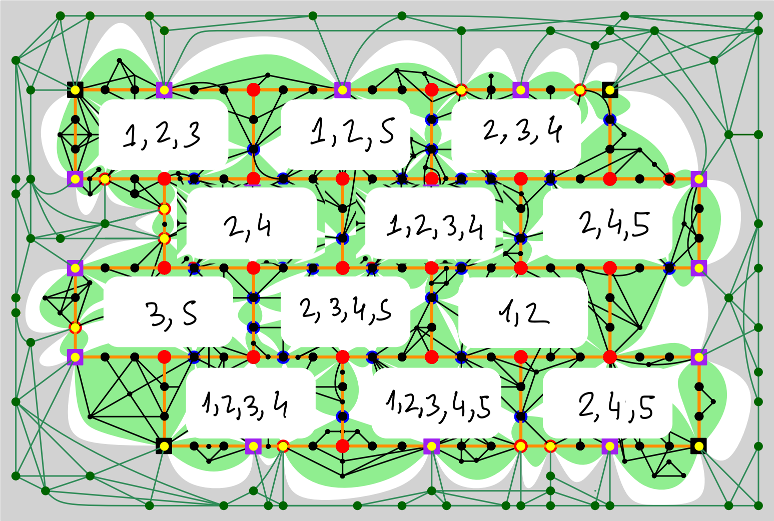
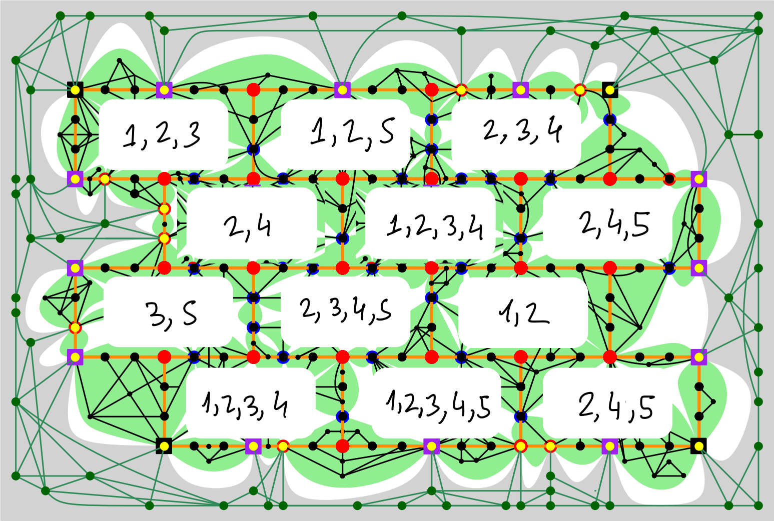
A flat wall is homogenous if every (internal) brick has the same palette.
Fact: every brick of a homogenous flat wall has the same "behavior".

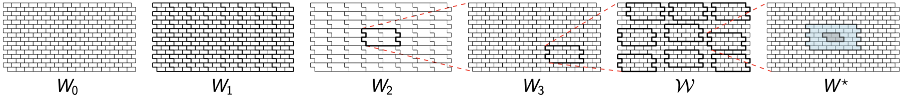


Crucial notion: homogeneity

Price of homogeneity to obtain a homogenous flat r-wall (zooming):
If we have c colors, we need to start with a flat r^c-wall. (why?)



Back to the proof: zooming and zooming inside a flat wall



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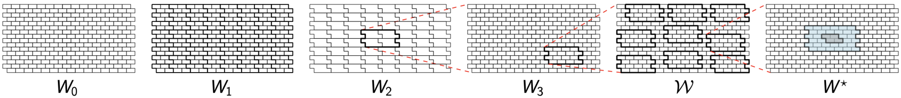

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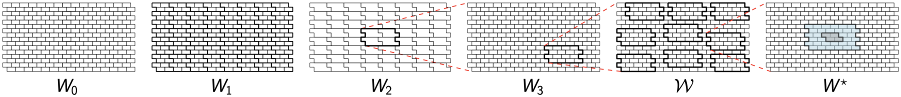

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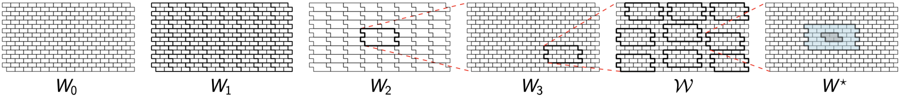

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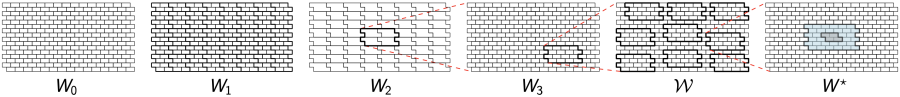

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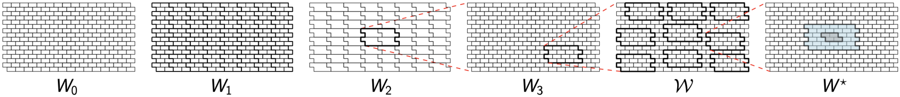

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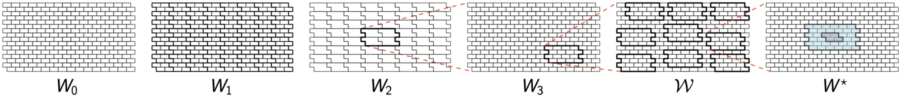
From now on, we can forget the minor-exclusion part of θ.

We keep on zooming...



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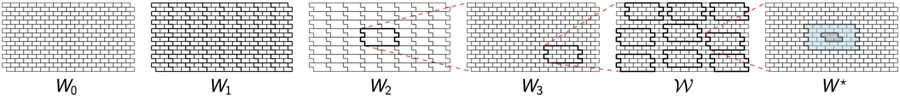

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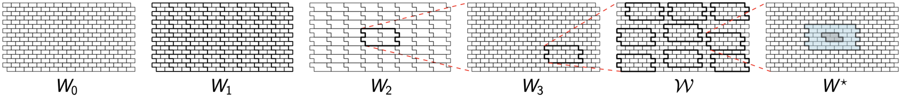

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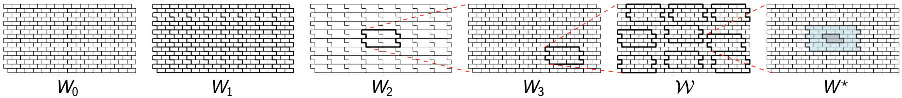

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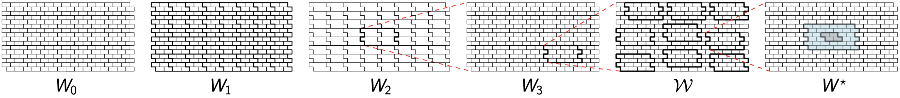

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Hardest part of the proof: prove that the central part of $W^\star$ is indeed irrelevant.

skip

Exploiting the bounded-treewidth property of β

Compound logic We define $\beta \triangleright \gamma$ so that

$$G \models \beta \triangleright \gamma \text{ if } \exists X \subseteq V(G) \text{ so that } \boxed{(\mathsf{stell}(G, X), X) \models \beta} \text{ and } \boxed{G \setminus X \models \gamma}.$$

$\gamma = \sigma \wedge \mu$, where $\sigma \in \mathsf{FOL}[\mathtt{E}]$ and μ expresses minor-exclusion.

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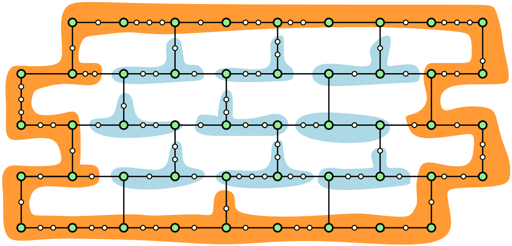

Defining the θ-characteristic of a wall: privileged component

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$$\theta = \theta^{\text{in}} \wedge \theta^{\text{out}}$$

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This splitting gives rise to the in-signature and out-signature of a wall.

skip

In-signature of a wall

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In-signature of a wall

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Approach inspired from the technique for modification to planarity + FOL.
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Main new difficulty: deal with the apices corresponding to the flat wall.

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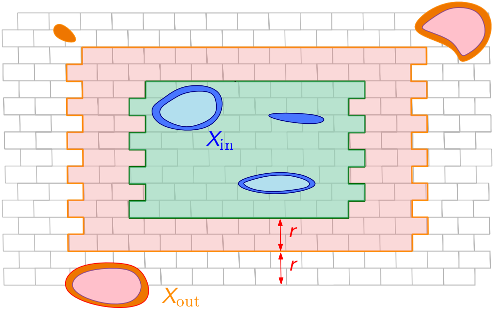
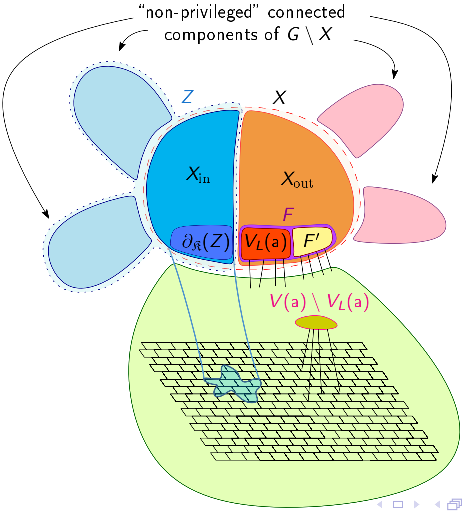


Out-signature of a wall

θ^{out}: conjunction of the modulator sentence β and the target sentence γ in the non-privileged components of $G \setminus X$.



Some final remarks

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- Limitations
  - are torsos really necessary?
  - which are the optimal combinatorial assumptions on FOL+CMSOL?
- Extensions
  - irrelevant friendliness (bipartiteness)
  - other modification operations (blocks, contractions, ...)
- Open problems
  - constants hidden in $\mathcal{O}_{|\theta|}(n^2)$
  - is the Θ-hierarchy proper?
  - Is quadratic time improvable?
  - Further than minor-exclusion?