# On the existence of polynomial kernels for structural parameterizations of hitting problems

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Based on joint work with Marin Bougeret and Bart M. P. Jansen
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# Outline of the talk

1. Introduction to structural parameterizations
2. Graph classes closed under minors
3. Graph classes closed under (induced) subgraphs
4. Some ideas of the techniques
   - Upper bounds
   - Lower bounds

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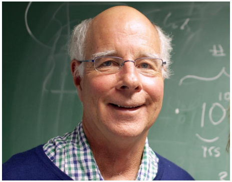
# Parameterized complexity in a nutshell

$\boxed{\text{Idea}}$ Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established and very active area.

# Parameterized problems

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- $k$-Vertex Cover: Does a graph $G$ contain a set $S \subseteq V(G)$, with $|S| \leq k$, containing at least an endpoint of every edge?
- $k$-Clique: Does a graph $G$ contain a set $S \subseteq V(G)$, with $|S| \geq k$, of pairwise adjacent vertices?
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These three problems are NP-hard, but are they $\boxed{\text{equally}}$ hard?

# They behave quite differently...

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Major goal in parameterized complexity:

$\boxed{\text{Which FPT problems admit polynomial kernels?}}$

# Structural parameterizations

**VERTEX COVER**
**Input**: A graph $G$ and an integer $k$.
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What about instances whose solution is large? For instance, a path?

Idea: consider parameters that can be smaller than the solution size.

The existence of a polynomial kernel for such a parameter would be a stronger result: better preprocessing guarantees.

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Examples:

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- feedback vertex set number: $\mathcal{C} =$ forests.

Very influential result:

**Theorem (Jansen and Bodlaender, 2011)**

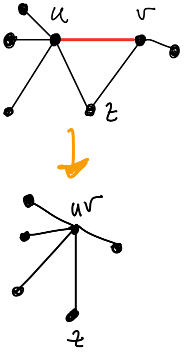
*Vertex Cover admits a polynomial kernel parameterized by the feedback vertex set number of the input graph.*

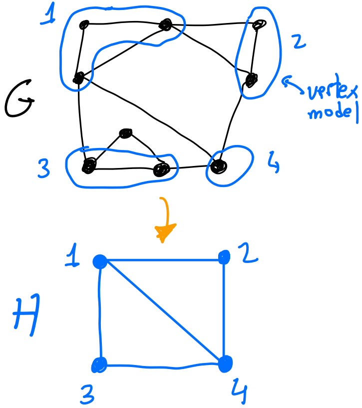
Note that, for every graph $G$, $\mathrm{fvs}(G) \leq \mathrm{vc}(G)$.

# Next section is...

# Graph minors

A graph $H$ is a minor of a graph $G$, denoted by $H \leqslant_m G$, if $H$ can be obtained by a subgraph of $G$ by contracting edges.



# Minor-closed graph classes

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Examples of minor-closed graph classes:

- Independent sets.
- Forests.
- Series-parallel graphs.
- Planar graphs.
- Graphs embeddable in a fixed surface.
- Linklessly embeddable graphs.
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- ...

# Characterizing a graph class by excluded minors

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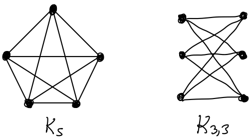
- If $\mathcal{C}$ = independent sets, then $\mathcal{C} = \text{exc}(K_2)$.
- If $\mathcal{C}$ = forests, then $\mathcal{C} = \text{exc}(K_3)$.
- If $\mathcal{C}$ = series-parallel graphs, then $\mathcal{C} = \text{exc}(K_4)$.
- If $\mathcal{C}$ = outerplanar graphs, then $\mathcal{C} = \text{exc}(K_4, K_{2,3})$.
- If $\mathcal{C}$ = planar graphs, then $\mathcal{C} = \text{exc}(K_5, K_{3,3})$. [Kuratowski. 1930]




- If $\mathcal{C}$ = graphs embeddable in the projective plane, then $|\mathcal{F}_{\mathcal{C}}| = 35$.
- If $\mathcal{C}$ = graphs embeddable in a fixed surface, then $\mathcal{F}_{\mathcal{C}}$ is finite.

[Archdeacon, Huneke. 1989 + Robertson, Seymour. 1990]

# Wagner's conjecture

**Conjecture (Wagner. 1970)**

*For every minor-closed graph class $\mathcal{C}$, there exists a finite set of graphs $\mathcal{F}_{\mathcal{C}}$ such that $\mathcal{C} = \text{exc}(\mathcal{F}_{\mathcal{C}})$.*

# Wagner's conjecture... now Robertson-Seymour's theorem

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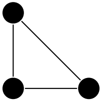
# Treewidth via $k$-trees

Example of a 2-tree:

For $k \geq 1$, a $k$-tree is a graph that can be built starting from a $(k+1)$-clique and then iteratively adding a vertex connected to a $k$-clique.



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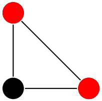

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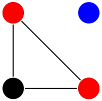

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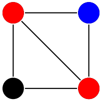

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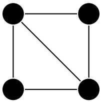

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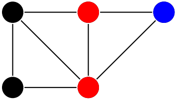

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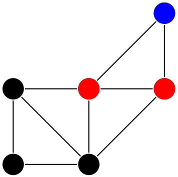

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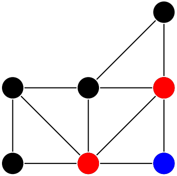

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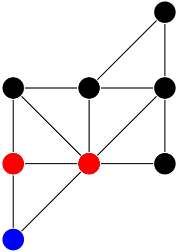

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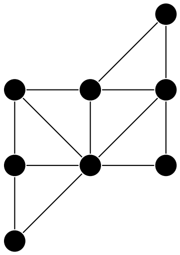

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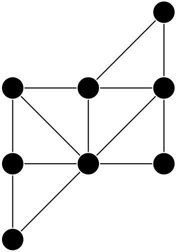

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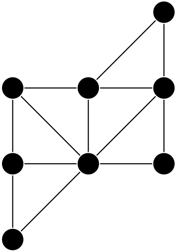

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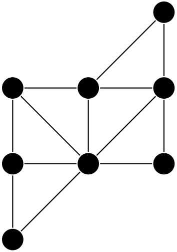

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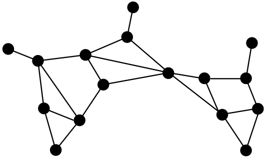
Invariant that measures the topological resemblance of a graph to a forest.

Construction suggests the notion of tree decomposition: small separators.

# Treewidth measures the tree-likeness of a graph



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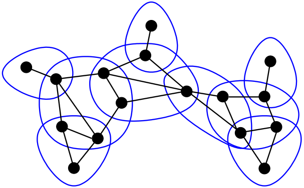

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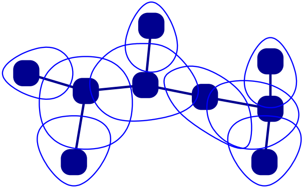

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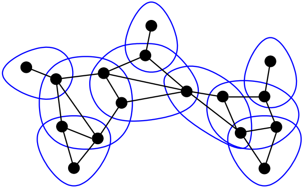

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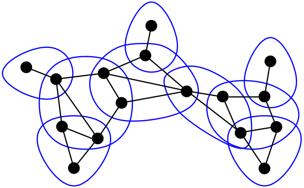

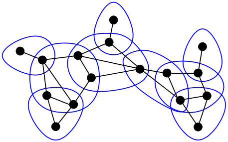

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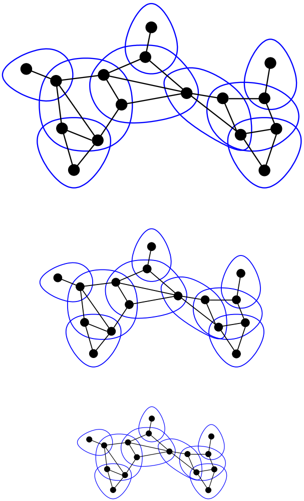

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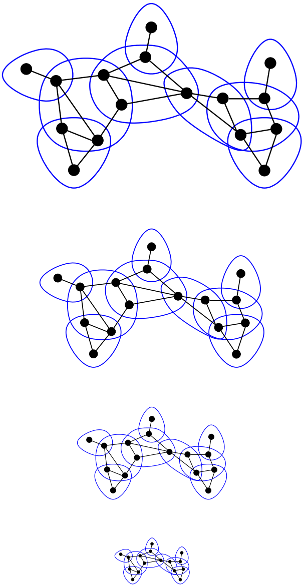

# Back to Vertex Cover

**Theorem (Jansen and Bodlaender, 2011)**

Vertex Cover *admits a polynomial kernel parameterized by the feedback vertex set number of the input graph.*

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**Input**: A graph $G$, an integer $\ell$, and a set $X \subseteq V(G)$ such that $G - X$ is a forest.

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Vertex Cover/$\mathcal{C}$-modulator

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All these graph classes are minor-closed.

# Tree-depth

For a graph $G$, define $\mathrm{td}(G)$ as

$$\begin{cases} 0 & \text{if } G \text{ is the empty graph,} \\ \max_{C_i} \mathrm{td}(C_i) & \text{if } G \text{ has conn. comp. } C_1, \dots, C_c,\ c \geq 2, \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v) & \text{otherwise.} \end{cases}$$

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Treewidth: measures how far it is from being a tree.
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For any graph $G$ it holds that

$$\mathrm{tw}(G) \leq \mathrm{pw}(G) \leq \mathrm{td}(G) - 1.$$

# Only good news?

VERTEX COVER admits a polynomial kernel parameterized by the vertex-deletion distance to a...

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- graphs of maximum degree 2. [Majumdar, Raman, and Saurabh, 2015]
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Vertex Cover does not admit a polynomial kernel parameterized by the vertex-deletion distance to a graph of treewidth 2, unless $\mathsf{NP} \subseteq \mathsf{coNP}/\mathsf{poly}$.
[Cygan, Lokshtanov, Pilipczuk, Pilipczuk, and Saurabh, 2014]

# Only good news? No! Where is the limit?

Vertex Cover admits a polynomial kernel parameterized by the vertex-deletion distance to a...

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★ Which is the most general (minor-closed) graph class $\mathcal{C}$ such that Vertex Cover admits a polynomial kernel parameterized by the vertex-deletion distance to $\mathcal{C}$?

# For minor-closed graph classes, the limit is bridge-depth!

**Theorem (Bougeret, Jansen, and S., 2020)**

*Let $\mathcal{C}$ be a minor-closed graph class, and suppose that* $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$*.* Vertex Cover *admits a polynomial kernel parameterized by the vertex-deletion distance to $\mathcal{C}$* ***if and only if*** *$\mathcal{C}$ has bounded bridge-depth.*

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For any graph $G$, it holds that

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# Back to the statement

**Theorem (Bougeret, Jansen, and S., 2020)**

*Let $\mathcal{C}$ be a minor-closed graph class, and suppose that $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$. Vertex Cover admits a polynomial kernel parameterized by the vertex-deletion distance to $\mathcal{C}$* ***if and only if*** *$\mathcal{C}$ has bounded bridge-depth.*

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Bridge-depth ultimate common generalization of feedback vertex set number and tree-depth (which are incomparable) in the context of polynomial kernels for VERTEX COVER.

Vertex Cover

**Input**: A graph $G$ and an integer $k$.

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Let $\mathcal{F}$ be a fixed finite family of graphs.

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**Input**: A graph $G$ and an integer $k$.
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# What is known about kernels for $\mathcal{F}$-M-Deletion

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Thus, polynomial kernels for structural parameterizations of $\mathcal{F}$-M-Deletion are currently $\boxed{\text{out of reach}}$.

# Some good news: Feedback Vertex Set

**Feedback Vertex Set/$\mathcal{C}$-modulator**

**Input**: A graph $G$, an integer $\ell$, and a set $X \subseteq V(G)$ such that $G - X \in \mathcal{C}$.

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**Theorem (Dekker and Jansen, 2020)**

*Let $\mathcal{C}$ be a minor-closed graph class, and suppose that NP $\not\subseteq$ coNP/poly. Feedback Vertex Set admits a polynomial kernel parameterized by the vertex-deletion distance to $\mathcal{C}$ **if and only if** $\mathcal{C}$ has bounded elimination distance to a forest.*

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For a graph $G$, define $\mathrm{ed}_{\mathrm{for}}(G)$ as

$$\begin{cases} 0 & \text{if } G \text{ is a forest,} \\ \max_{C_i} \mathrm{ed}_{\mathrm{for}}(C_i) & \text{if } G \text{ has conn. comp. } C_1, \ldots, C_c,\ c \geq 2, \\ 1 + \min_{v \in V(G)} \mathrm{ed}_{\mathrm{for}}(G - v) & \text{otherwise.} \end{cases}$$

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By definition, for any graph $G$ is holds that

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**Conjecture (Bougeret, Brandwein, and S.)**

*Let $\mathcal{C}$ be a minor-closed graph class, let $\mathcal{F}$ be a set of 2-connected graphs containing a planar graph, and suppose that* NP $\not\subseteq$ coNP/poly.

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*Let $\mathcal{C}$ be a minor-closed graph class, let $\mathcal{F}$ be a set of 2-connected graphs containing a planar graph, and suppose that NP $\not\subseteq$ coNP/poly. $\mathcal{F}$-M-Deletion admits a poly kernel parameterized by the vertex-deletion distance to $\mathcal{C}$* ***if and only if*** *$\mathcal{C}$ has bounded $\mathrm{exc}(\mathcal{F})$-elimination distance.*

# Next section is...

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Finding the right characterization for VERTEX COVER for hereditary/monotone graph class $\mathcal{C}$ seems currently $\boxed{\text{out of reach}}$.

# Beyond Vertex Cover

Vertex Cover

**Input**: A graph $G$ and an integer $k$.

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**$H$-SUBGRAPH HITTING**

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For every fixed $H$, $H$-(Induced) Subgraph Hitting admits a polynomial kernel parameterized by the solution size. [Abu-Khzam, 2010]

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This seems **really** hard!

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**Theorem (Bougeret, Jansen, and S., 2024)**

*Let $H$ be a graph on $h$ vertices that is not a clique and that has no stable cutset. $H$-Subgraph Hitting and $H$-Induced Subgraph Hitting do not admit a polynomial kernel parameterized by the size of a given vertex set $X$ of the input graph $G$ such that $\mathrm{td}(G - X) = \mathcal{O}(h)$, unless $\mathrm{NP} \not\subseteq \mathrm{coNP/poly}$.*

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$\text{ved}^{+}_{\mathcal{F}^{\text{ind}}_{\bar{H}}}$: the same, but for induced copies of $H$.

# We obtain a new kind of dichotomy: in terms of $H$

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*Let $H$ be a 2-connected graph, let $\lambda \geq 1$ be an integer, and assume that NP $\not\subseteq$ coNP/poly. $H$-Subgraph Hitting (resp. $H$-Induced Subgraph Hitting) admits a polynomial kernel parameterized by the size of a given vertex set $X$ of the input graph $G$ such that $\mathrm{ved}^{+}_{\mathcal{F}_{\bar{H}}}(G - X) \leq \lambda$ (resp. $\mathrm{ved}^{+}_{\mathcal{F}^{\mathrm{ind}}_{\bar{H}}}(G - X) \leq \lambda$)* ***if and only if***

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**What we can remove**: vertex sets $T \subseteq V(G)$ that induce connected subgraphs that do not contain $H$ as a subgraph (or induced subgraph) and that are "weakly attached" to the rest of the graph, meaning that each connected component of $G-T$ has at most one neighbor in $T$.

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We call the resulting parameter $\mathrm{bed}^+_{\mathcal{F}_{\bar{H}}}$ (or $\mathrm{bed}^+_{\mathcal{F}^{\mathrm{ind}}_{\bar{H}}}$),

where 'b' stands for the removal of *blocks*.

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For any two graphs $G$ and $H$, the following holds:

$$\mathrm{bed}^{+}_{\mathcal{F}_{\bar{H}}}(G) \leq \mathrm{ved}^{+}_{\mathcal{F}_{\bar{H}}}(G) \leq \mathrm{ed}_{\mathcal{F}_{\bar{H}}}(G) \leq \mathrm{td}(G).$$

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**Theorem (Bougeret, Jansen, and S., 2024)**

*Let $t \geq 3$ and $\lambda \geq 1$ be fixed integers. The $K_t$-Subgraph Hitting problem admits a polynomial kernel parameterized by the size of a given vertex set $X$ of the input graph $G$ such that $\mathrm{bed}^{+}_{\mathcal{F}_{\bar{K}_t}}(G - X) \leq \lambda$.*

# Next section is...

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# Crucial notion: blocking sets

VERTEX COVER/$\mathcal{C}$-modulator

**Input**: A graph $G$, an integer $\ell$, and a set $X \subseteq V(G)$ such that $G - X \in \mathcal{C}$.

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Parameterized complexity of computing mmbs$(G)$.

[Araújo, Bougeret, Campos, and **S.**, 2023]

# Maximum minimal blocking sets: examples

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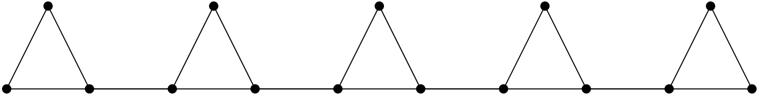
If $G$ is bipartite with at least one edge, then $\mathrm{mmbs}(G) = 2$.

# Maximum minimal blocking sets: examples



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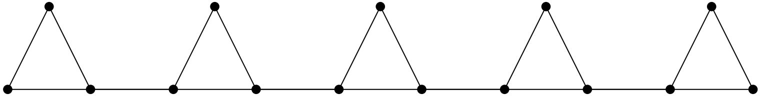

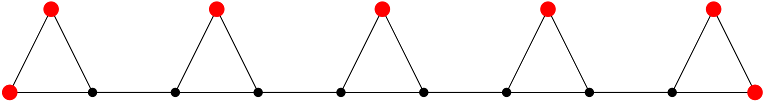

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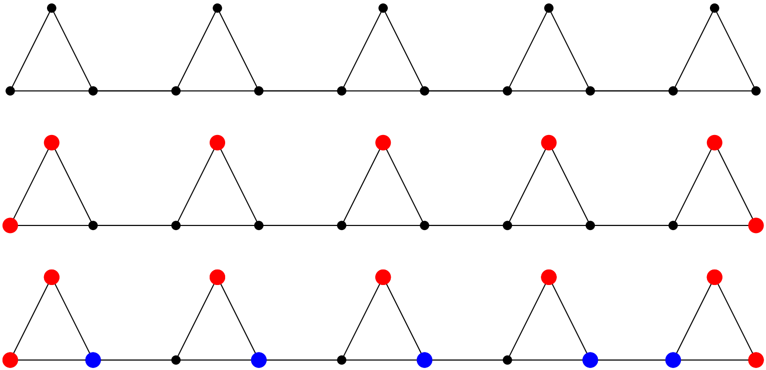

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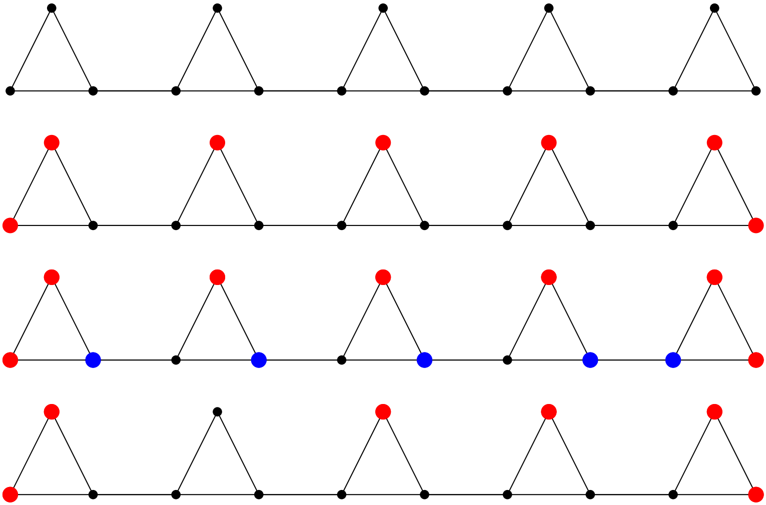

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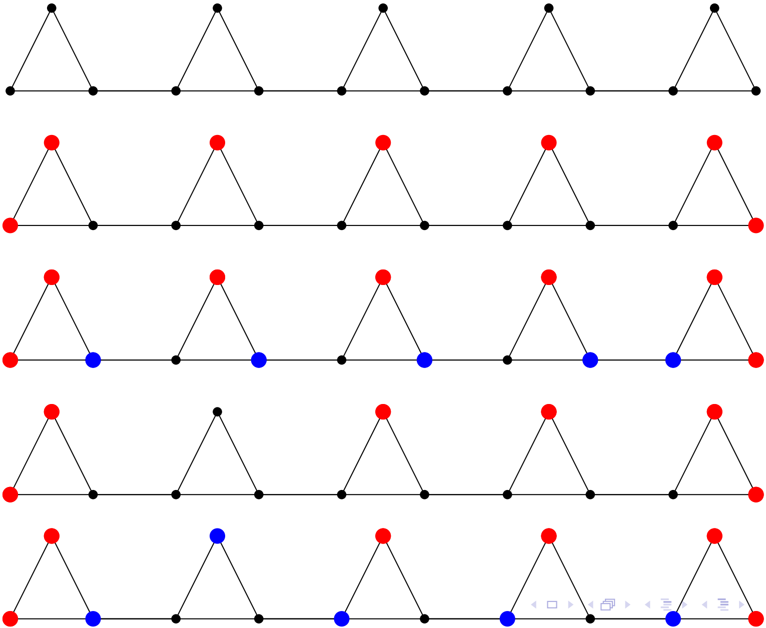

# Exploiting bounded mmbs: conflicts and chunks

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Let $X' \subseteq X$ and $R \subseteq V(G) \setminus X$.

$$\mathrm{conf}_R(X') = \alpha(G[R]) - \alpha(G[R \setminus N_G(X')]).$$

That is, $\mathrm{conf}_R(X')$ measures how much smaller $\alpha(G[R])$ becomes when one is forbidden from picking vertices that are adjacent to $X'$ in $G$.

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$$\text{conf}_R(X') = \alpha(G[R]) - \alpha(G[R \setminus N_G(X')]).$$

That is, $\text{conf}_R(X')$ measures how much smaller $\alpha(G[R])$ becomes when one is forbidden from picking vertices that are adjacent to $X'$ in $G$.

A chunk is a set $X' \subseteq X$ with $|X| \leq c$ such that $G[X]$ is edgeless.

# Exploiting bounded mmbs: conflicts and chunks

Let $\mathcal{C}$ be such that $\text{mmbs}(G) \leq c$ for every $G \in \mathcal{C}$ (like in all known cases).

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**Why chunks are useful:** Let $R \subseteq V(G) \setminus X$.
For every independent set $S_X \subseteq X$ such that $\text{conf}_R(S_X) > 0$ there exists a chunk $X'$, with $X' \subseteq S_X$, such that $\text{conf}_R(X') > 0$.

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The second item needs ad-hoc reduction rules...

# Next subsection is...

1. Introduction to structural parameterizations
2. Graph classes closed under minors
3. Graph classes closed under (induced) subgraphs
4. Some ideas of the techniques
   - Upper bounds
   - Lower bounds

# Useful tool: polynomial parameter transformations

Let $A, B \subseteq \Sigma^* \times \mathbb{N}$ be two parameterized problems.

A polynomial parameter transformation (PPT) from $A$ to $B$ is an algorithm such that:

Instance $(x, k)$ of $A$ → polynomial time → Instance $(x', k')$ of $B$

1. $(x, k)$ is a YES-instance of $A$ $\Leftrightarrow$ $(x', k')$ is a YES-instance of $B$.
2. $k' \leq \text{poly}(k)$.

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$$\boxed{\text{Instance } (x, k) \text{ of } A} \xrightarrow{\text{polynomial time}} \boxed{\text{Instance } (x', k') \text{ of } B}$$

1. $(x, k)$ is a YES-instance of $A$ $\Leftrightarrow$ $(x', k')$ is a YES-instance of $B$.
2. $k' \leq \text{poly}(k)$.

If $A$ does not admit a polynomial kernel (under some hypothesis such as $\text{NP} \not\subseteq \text{coNP}/\text{poly}$), then neither does $B$. [Bodlaender, Thomassé, Yeo. 2011]

# Example: a reduction from CNF-SAT

**Theorem (Bougeret, Jansen, and S., 2024)**

*Let $H$ be a biconnected graph that is not a clique. The $H$-Subgraph Hitting problem does not admit a polynomial kernel parameterized by the size of a given vertex set $X$ of the input graph $G$ such that $\mathrm{ved}^{+}_{\mathcal{F}_{\bar{H}}}(G - X) \leq 1$, unless $\mathrm{NP} \subseteq \mathrm{coNP}/\mathrm{poly}$.*

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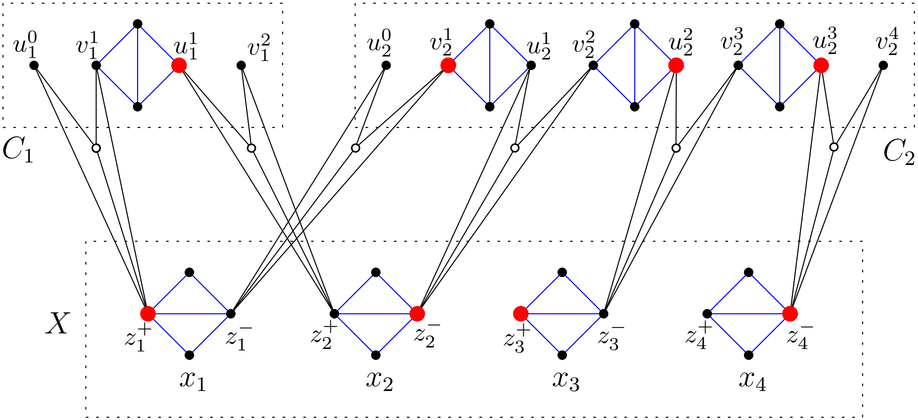
CNF-SAT does not admit a polynomial kernel parameterized by the number of variables of the input formula, unless $\mathrm{NP} \subseteq \mathrm{coNP}/\mathrm{poly}$.

[Dell and van Melkebeek, 2014]

# Idea of the reduction from CNF-SAT to $H$-Subgraph Hitting




$H$ is the diamond.

$\phi$ consists of two clauses $C_1 = (x_1 \vee x_2)$ and $C_2 = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4)$.

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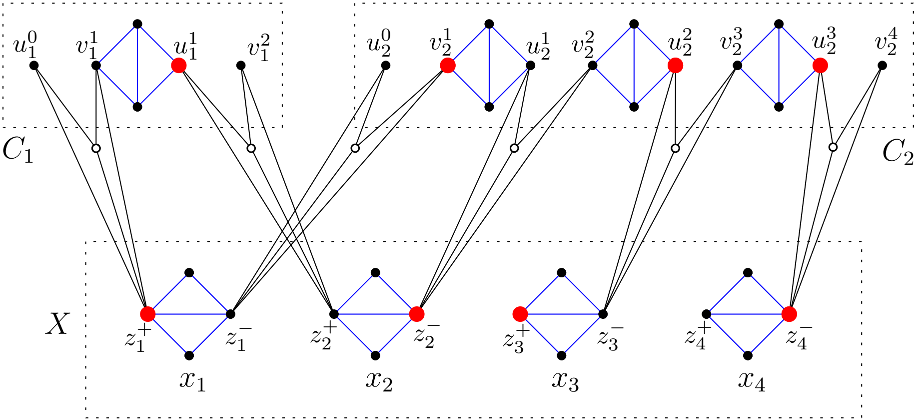


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Satisfying assignment: $\alpha(x_1) = 1$, $\alpha(x_2) = 0$, $\alpha(x_3) = 1$, and $\alpha(x_4) = 0$.

Gràcies!