# On the complexity of finding large odd induced subgraphs and odd colorings


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# Outline of the talk

# Next section is...

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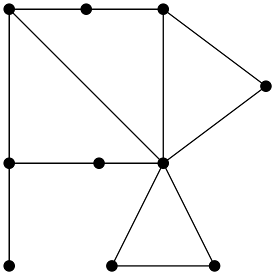

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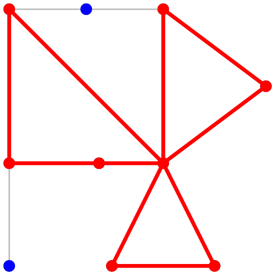

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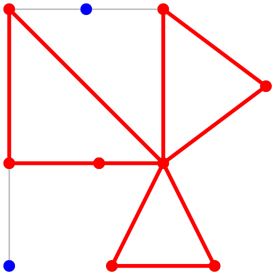

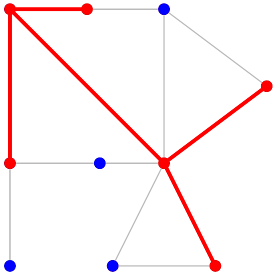

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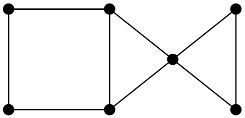

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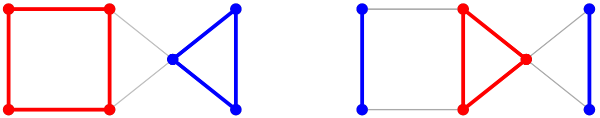

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**Corollary**

*Every graph $G$ contains an even induced subgraph with at least* $|V(G)|/2$ *vertices.*

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$\boxed{\text{What about } \mathrm{mos}(G) \text{ and } \chi_{\mathrm{odd}}(G)\text{?}}$

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The conjecture is still open.

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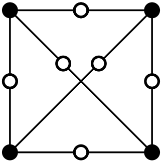

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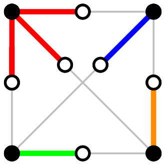

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$\boxed{\text{Our goal}}$ Computational aspects of the parameters mos and $\chi_{\text{odd}}$.

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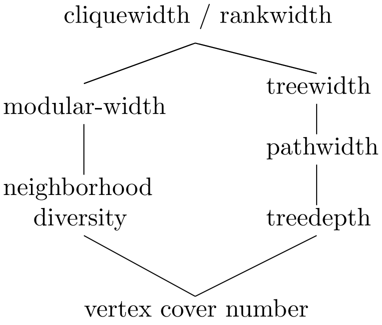

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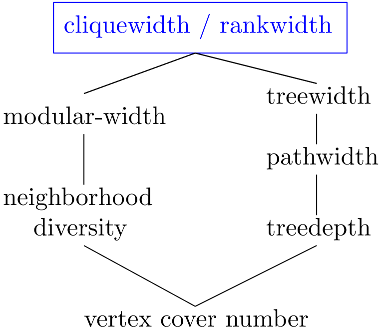

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for an $n$-vertex graph $G$, given a decomposition tree of width at most rw.

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  We show that $\chi_{\mathrm{odd}}$ is unbounded for $P_5$-free graphs.

# Next section is...

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*For an integer* $q \geq 1$*, deciding whether* $\chi_{\text{odd}}(G) \leq q$ *is*

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For $q = 1$ the problem is trivial: $G$ needs to be an odd graph itself.

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- $\chi_{\text{odd}}(G) \leq 2 \iff$ the above system is feasible. □

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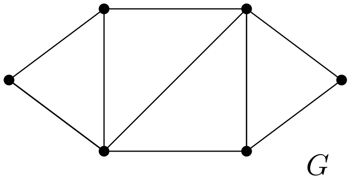

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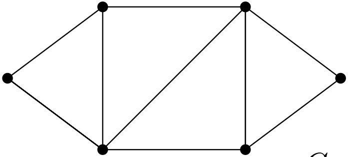

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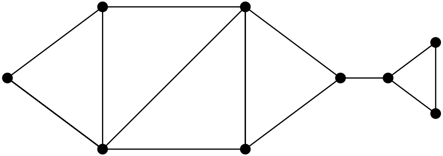

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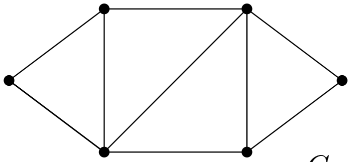

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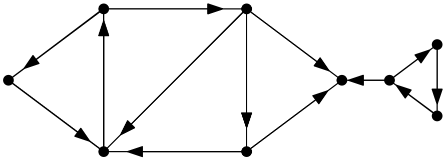

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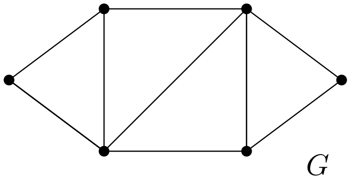


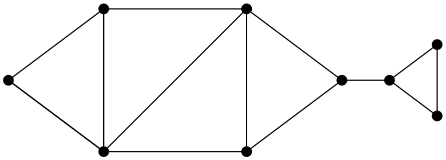

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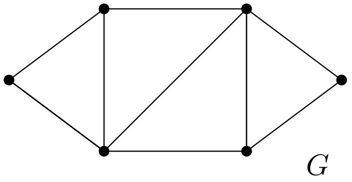


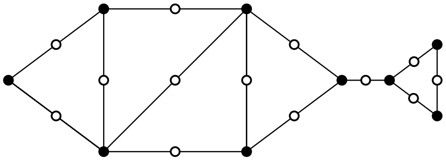

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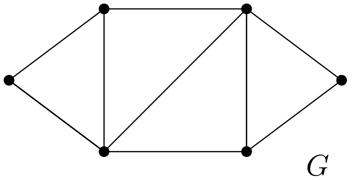


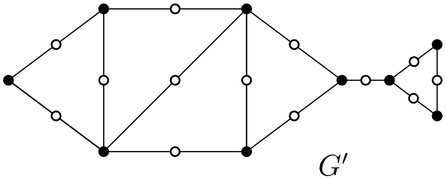

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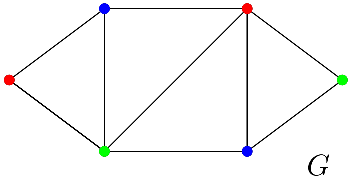


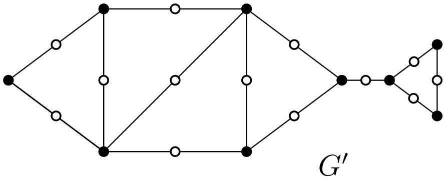

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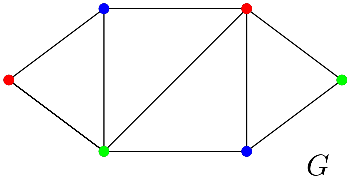


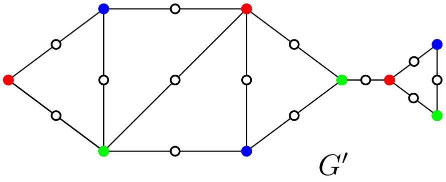

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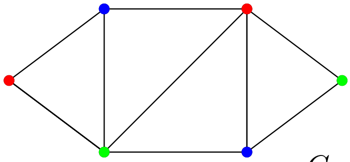

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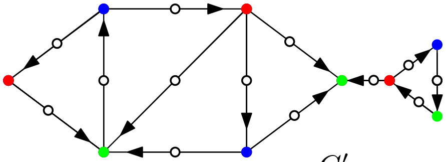

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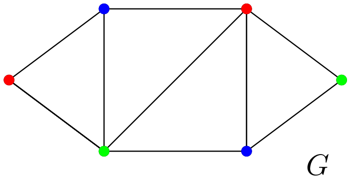


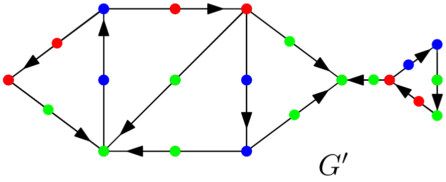

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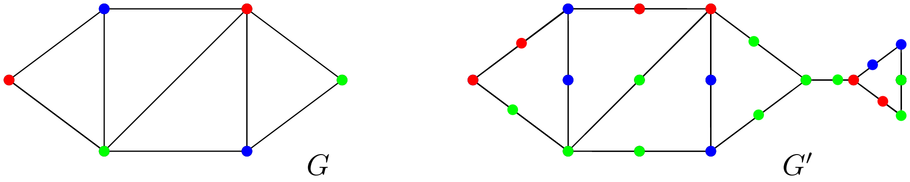

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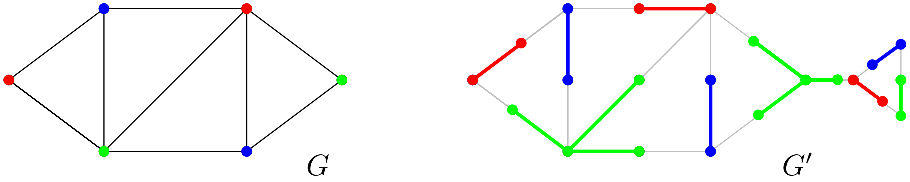

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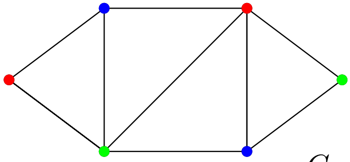


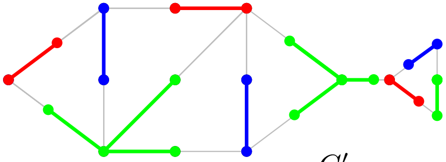


Thus, $G$ is 3-colorable $\iff$ $\chi_{\text{odd}}(G') \leq 3$. ⏩ skip □

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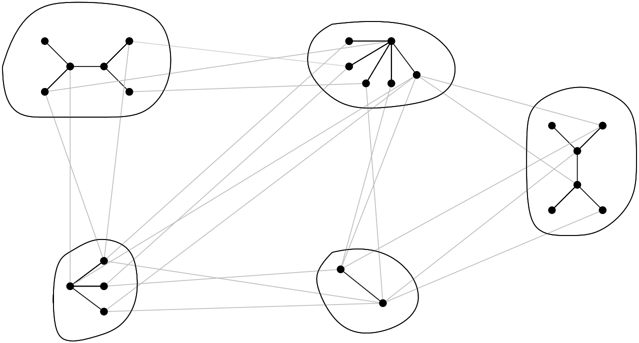

Given $G$, consider a partition of $V(G)$ into induced odd trees.

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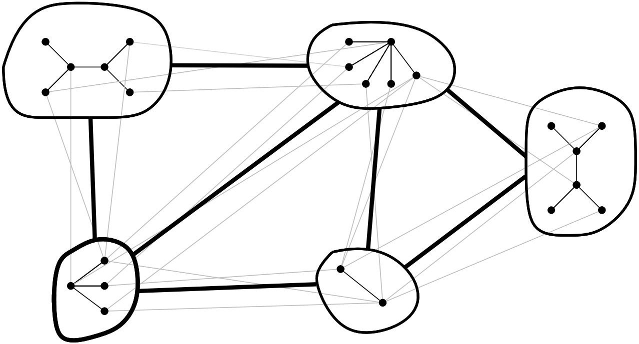

Let $G'$ be obtained from $G$ by contracting each tree to a single vertex.

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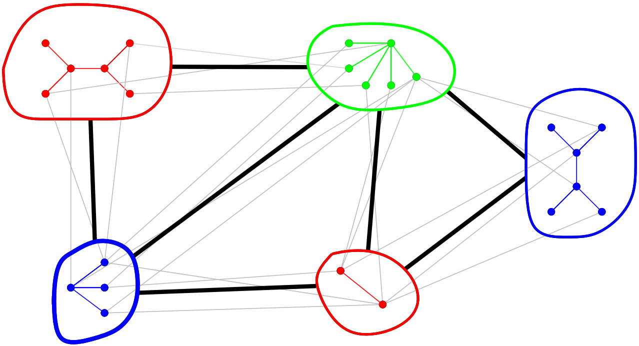

Consider a proper vertex coloring of $G'$ using $\chi(G')$ colors.

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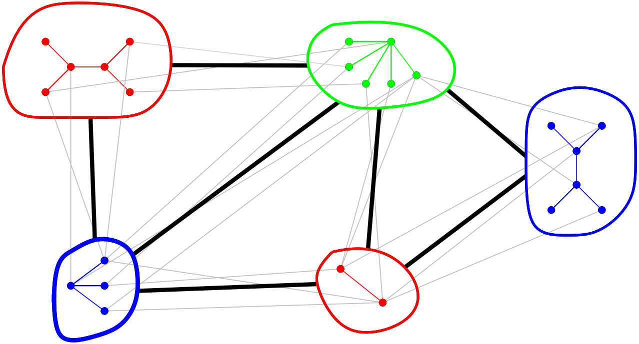

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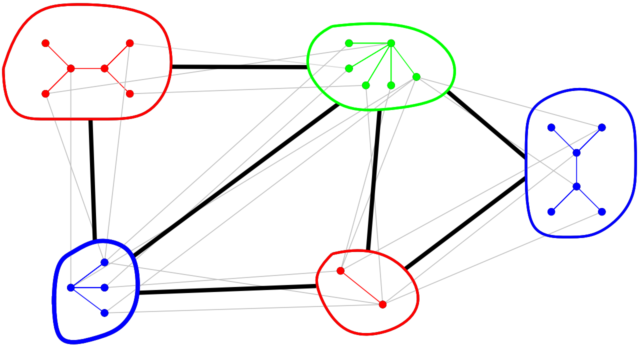

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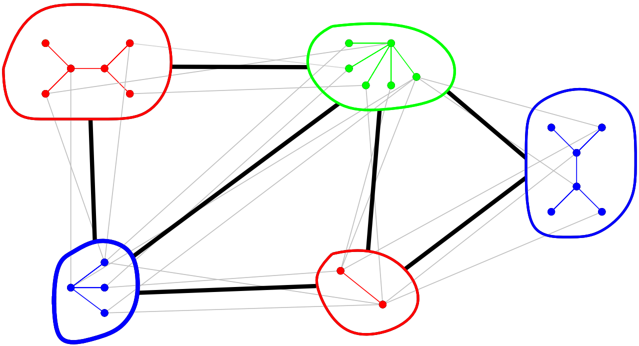

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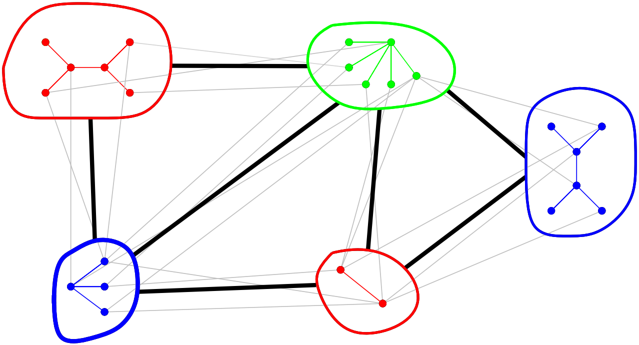

Bound is tight: let $G$ be subdivided $n$-clique with $n \equiv 0, 3 \pmod 4$.

skip

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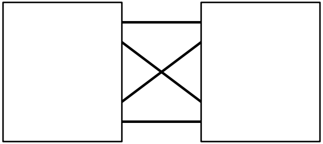

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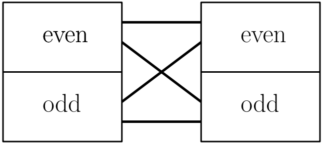

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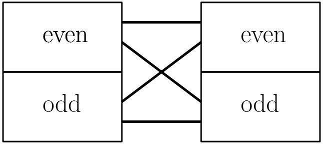


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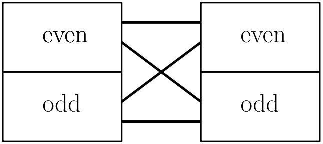


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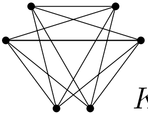
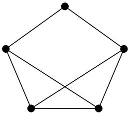


$K_{2,2,2}$



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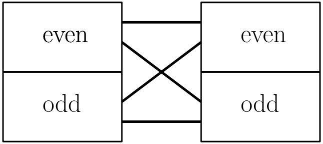


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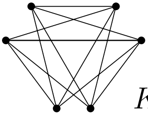

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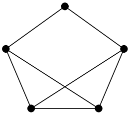

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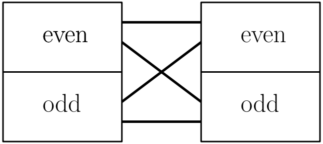


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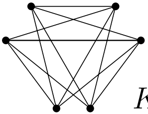

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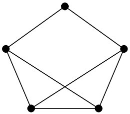

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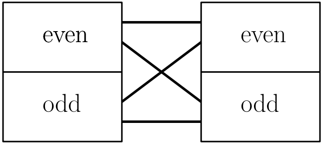


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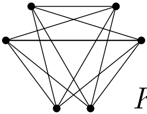

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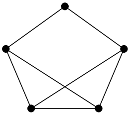

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# Next section is...

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# Gràcies!