Optimal Erdős-Pósa property for pumpkins

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Outline of the talk

1. Motivation
2. Our result
3. Sketch of proof
4. Further research
Next section is...

1. Motivation
2. Our result
3. Sketch of proof
4. Further research
König’s min-max theorem in bipartite graphs:

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\text{Min Vertex Cover} = \text{Max Matching}
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Packing and covering

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König’s min-max theorem in bipartite graphs:

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If there exists such \( f \) for all \( G \), then \( \mathcal{H} \) satisfies the \textbf{Erdős-Pósá property}. If the minimum number of vertices covering all \( H \in \mathcal{H} \) is less than or equal to \( f(\text{max # of disjoint } H \in \mathcal{H}) \)?
Minors and models in graphs

$H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.

A $K_5$-model

The $S_u$'s are called vertex images.
Minors and models in graphs

$H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.

**$H$-model** in $G$: collection $\{S_u : u \in V(H)\}$ s.t.
- the $S_u$’s are vertex-disjoint connected subgraphs of $G$, and
- there is an edge between $S_u$ and $S_v$ in $G$ for every edge $uv \in E(H)$.

A $K_5$-model

The $S_u$’s are called vertex images.
Let $H$ be a fixed graph. For a graph $G$, we define:

$$\nu_H(G) := \text{packing number}$$
$$= \text{max. number of vertex-disjoint } H\text{-models in } G$$

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Clearly, $\tau_H(G) \geq \nu_H(G) \ \forall G.$
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This is called the \textbf{Erdős-Pósa property of } $H$-minors.
Erdős-Pósa property of $H$-minors

Fundamental result:

\[ \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \iff H \text{ is planar} \]

[Robertson, Seymour '86]
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Is it the end of the story?

Known upper bounds $\tau_H(G) \leq f(\nu_H(G))$ are huge: $f(\nu_H(G)) \in \Omega(2^{\nu_H(G)^2})$.

This is because Robertson and Seymour's proof uses the excluded grid theorem from Graph Minors.

Natural objective: optimize $f(\nu_H(G))$. 
Erdős-Pósa property of $H$-minors

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The property does NOT hold if $H$ is not planar

$H = K_5 \times \mathbf{X}$

Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $G$: 

![Diagram of a triangulated toroidal grid with labels a, b, c, d, e, f and numbers 1, 2, 3, 4, 5.](image)
The property does NOT hold if $H$ is not planar

$H = K_5 \times$ 

Take a $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $G$:

$\nu_H(G) = 1$ but $\tau_H(G) = \Theta(\sqrt{n})$
The property does NOT hold if $H$ is not planar

$$H = K_5 \times$$

$H$ not planar

Therefore, the result of Robertson and Seymour is best possible.
Brief state of the art of Erdős-Pósa property for minors

\[ \tau_H(G) \leq f(\nu_H(G)) \quad \forall G \quad \Leftrightarrow \quad H \text{ is planar} \quad \text{[Robertson, Seymour '86]} \]
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- $\tau_H(G) \leq f(\nu_H(G)) \quad \forall G \iff H \text{ is planar}$ [Robertson, Seymour '86]

- Erdős and Pósa seminal result for $H = \text{cycle}$ (optimal):
  $f(k) = O(k \log k)$. [Erdős, Pósa '65]
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- $f(k) = O(k)$ when $H$ is a forest (optimal). \cite{FioriniJoretWood12}
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- $f(k) = O(k)$ when $H$ is a forest (optimal) \cite{Fiorini, Joret, Wood '12}

- $f(k) = O(k)$ when $H$ is planar and $G$ belongs to a minor-closed graph class (optimal) \cite{Fomin, Saurabh, Thilikos '10}
Next section is...
Pumpkins

Can be seen as a natural generalization of a cycle. (N.B: “graph” = multigraph)
$c$-pumpkin:

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Graphs with no $c$-pumpkin minor

- $c = 1$: empty graphs
- $c = 2$: forests
- $c = 3$: no two cycles share an edge

etc.
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- etc.
Covering (or hitting) pumpkins

c-pumpkin hitting set:
vertex subset $X \subseteq V(G)$ s.t. $G - X$ has no c-pumpkin minor

c = 3

$\tau_c(G)$: min. size of a c-pumpkin hitting set
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Hitting set number \( \tau_c(G) \): min. size of a c-pumpkin hitting set
c-pumpkin packing:
collection of vertex-disjoint subgraphs of $G$, each containing a c-pumpkin minor
**Packing pumpkins**

*c*-pumpkin packing:
collection of vertex-disjoint subgraphs of $G$, each containing a *c*-pumpkin minor

Packing number $\nu_c(G)$: max. cardinality of a *c*-pumpkin packing
A recent result and our main theorem

- A recent result on Erdős-Pósa property for pumpkins:

**Theorem (Fomin, Lokshtanov, Misra, Philip, Saurabh ’12)**

For any fixed integer $c \geq 1$ and given an integer $k \geq 1$, every graph $G$ either contains $k$ vertex-disjoint $c$-pumpkins-models, or has a $c$-pumpkin hitting set of size at most $f(k) = O(k^2)$.

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\[ \tau_c \leq \nu_c^2 \]

- Their proof uses tree decompositions and brambles.

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\[ \tau_c \leq \nu_c \log \nu_c \]

- Our proof follows and generalizes Erdős-Pósa’s proof for the case \( c = 2 \).
Optimality of the function $f(k) = O(k \log k)$

**Theorem:** $\exists f(k)$ s.t. $\forall G, k$, either $\nu_c(G) \geq k$ or $\tau_c(G) \leq f(k)$.

The function $f(k) = O(k \log k)$ is asymptotically optimal:
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- Let $G$ be an $n$-vertex cubic graph with $\text{tw}(G) = \Omega(n)$ and $\text{girth}(G) = \Omega(\log n)$. (such graphs are well-known to exist)
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- Any $c$-pumpkin-minor-free graph $H$ satisfies $\text{tw}(H) \leq d$ for some constant $d$, as the $c$-pumpkin is planar. [Robertson, Seymour '86]
- Thus $\text{tw}(G - X) \leq d$ for any $c$-pumpkin hitting set $X$, and therefore $\tau_c(G) = \Omega(n)$. 

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- On the other hand, every subgraph $H$ of $G$ containing a $c$-pumpkin-model has a cycle, so $V(H) = O(\log n)$, and therefore $\nu_c(G) = O(n/\log n)$.
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- This implies that (easy to check) $\exists$ constant $b > 0$ such that $f(k) > b \cdot k \log k$ (i.e., $f(k) = \Omega(k \log k)$).
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2 Our result

3 Sketch of proof

4 Further research
Useful reduction rules

We first need two reduction rules $R1$ and $R2$ dealing with 1-connected and 2-connected components without $c$-pumpkin minors, respectively, that preserve both $\nu_c(G)$ and $\tau_c(G)$:

For $c = 2$:
- $R1$: deleting degree-1 vertices
- $R2$: suppressing degree-2 vertices

Lemma Let $c \geq 2$ be a fixed integer. Suppose that $G^*$ results from the application of $R1$ or $R2$ on a graph $G$. Then $\tau_c(G) = \tau_c(G^*)$ and $\nu_c(G) = \nu_c(G^*)$. 
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Lemma

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We look at a subgraph $H$ with nice properties

- A graph is $c$-reduced if rules $R1$ or $R2$ cannot be applied anymore.
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- A graph is \textit{c-reduced} if rules $\textbf{R1}$ or $\textbf{R2}$ cannot be applied anymore.

- For a graph $G$, we denote by $\overline{G}$ the \textit{c-reduced} graph obtained from $G$ by applying reduction rules $\textbf{R1}$ and $\textbf{R2}$.
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- Given $G$, let $H$ be a maximal subgraph of $G$ (w.r.t. \# vertices and \# edges) such that

$$\Delta(\overline{H}) \leq 3,$$

where $\Delta$ denotes the maximum degree.
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**Main Lemma**

If $|V(\overline{H})| \geq d \cdot k \log k$ for some constant $d$ (depending only on $c$), then $H$ contains $k$ vertex-disjoint $c$-pumpkin-models.
Ingredients in the proof of the Main Lemma

- We prove it by induction on $k$, using that:

**Lemma**

Every $n$-vertex $c$-reduced graph $G$ contains a $c$-pumpkin-model of size $O(\log n)$.

(Generalization of: If $\delta(G) \geq 3$, then $\text{girth}(G) < 2 \log n$)  

[Alon, Hoory, Linial '02]
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- We choose a smallest $c$-pumpkin-model $C$, and to apply induction we need to prove that $\overline{H} - C$ contains a subgraph $F$ such that
  
  $$|V(F)| \geq d \cdot (k - 1) \log(k - 1).$$
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- **Crucial:** \( \forall p \geq 0, \exists f(p) \) s.t. every 3-connected graph with \( \geq f(p) \) vertices has a minor isomorphic to:

  [Oporowski, Oxley, Thomas '93]

\[
W_p \quad K_{3,p}
\]

(Note that for \( p \geq c \), both \( W_p \) and \( K_{3,p} \) contain the \( c \)-pumpkin as a minor)
Outline of the overall proof

- Given $G$, we can prove that $\exists$ a set $X \cup U \subseteq V(H)$, with $|X| = O(k)$, meeting every $c$-pumpkin-model in $G$. As $|X| = O(k)$, it suffices to show that $|U| = O(k \log k)$, unless $H$ contains $k$ disjoint $c$-pumpkin-models. This follows from the Main Lemma applied to the graph $H$. 
Given $G$, we consider the subgraph $H$ defined before:

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  ![Diagram](image)

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Motivation

Our result

Sketch of proof

Further research
Main open problem: $H$ non-acyclic planar, $f_H(k) = O(??)$
Further research

Main open problem: $H$ non-acyclic planar, $f_H(k) = \Omega(k \log k)$

Approximation algorithms

Goal: Given a graph $G$, finding a $c$-pumpkin packing $M$ and a $c$-pumpkin hitting set $X$ s.t. $|X| \leq f(c, n) \cdot |M|$ for some approximation ratio $f(c, n)$ (these problems generalize Vertex Cover, Feedback Vertex Set, ...)

⋆we provided an $O_c(\log n)$-approximation algorithm for $c$-Pumpkin Hitting Set and $c$-Pumpkin Packing. [Joret, Paul, S., Saurabh, Thomassé '11]

⋆constant-factor (deterministic) approximation for the hitting version? (so far, such an algorithm is only known for $c \leq 3$) [Fiorini, Joret, Pietropaoli '10]
Further research

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s.t. $|X| \leq f(c, n) \cdot |\mathcal{M}|$ \text{ for some approximation ratio } $f(c, n)$

( these problems generalize Vertex Cover, Feedback Vertex Set, ... )
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Gràcies!!