Algorithmic aspects of minor-closed graph classes

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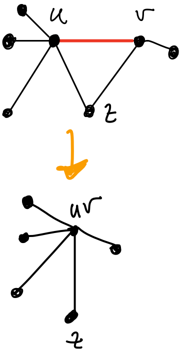
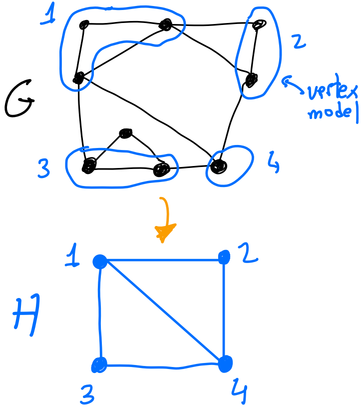
Outline of this mini-course

Next section is...

Graph minors

A graph H is a minor of a graph G, denoted by $H \leqslant_m G$, if H can be obtained by a subgraph of G by contracting edges.





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Note that, in general, this list $\mathcal{F}_{\mathcal{C}} = \{G_1, G_2, \ldots\}$ may be infinite.

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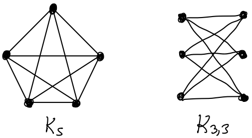

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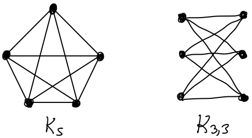

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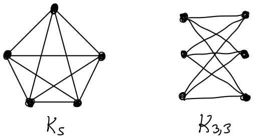

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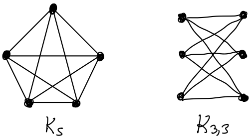

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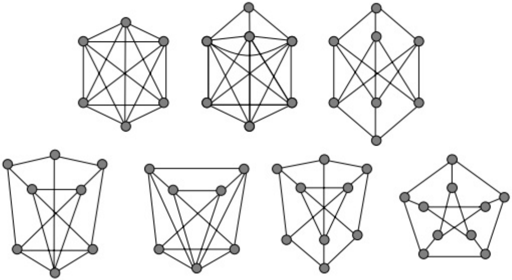

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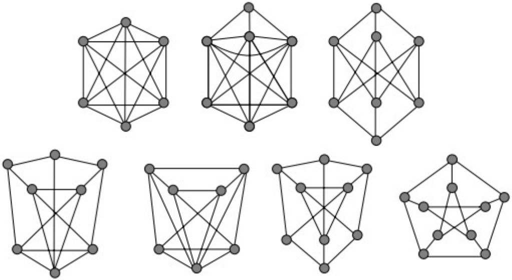

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$\mathcal{F}_{\mathcal{C}}$ seems to get complicated... but always finite!

Wagner's conjecture

Conjecture (Wagner. 1970)

For every minor-closed graph class $\mathcal{C}$, there exists a finite set of graphs $\mathcal{F}_{\mathcal{C}}$ such that $\mathcal{C} = \text{exc}(\mathcal{F}_{\mathcal{C}})$.

Wagner's conjecture... now Robertson-Seymour's theorem

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Yet equivalent: Every infinite set $\{G_1, G_2, \ldots\}$ of finite graphs contains two graphs such that one is a minor of the other (there is no infinite antichain).

Well-quasi orders

A partially ordered set (poset) is a set P with a partial binary relation $\leq$:

1. Reflexive: $a \leq a$.
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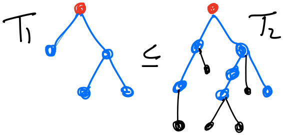
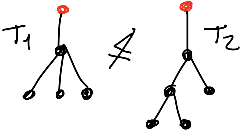
R&S theorem: Finite graphs are wqo with respect to the minor relation.

Illustrative example: rooted trees

Let T_1 and T_2 be two finite rooted trees.
We say that $T_1 \leq T_2$ if there is a subdivision of T_1 that can be embedded into T_2 so that the root of T_1 is mapped onto the root of T_2.





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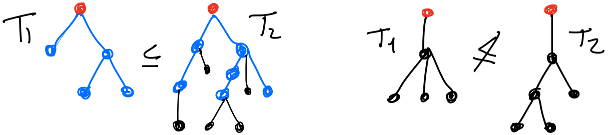

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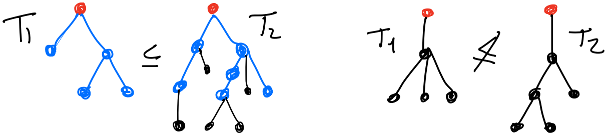

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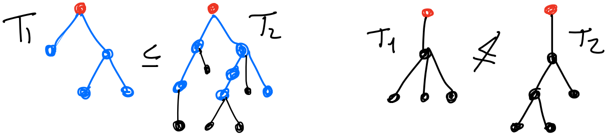

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We will now see a simple proof by [Nash-Williams. 1963]

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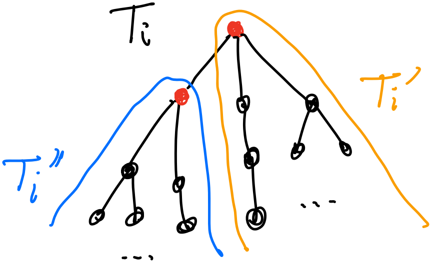
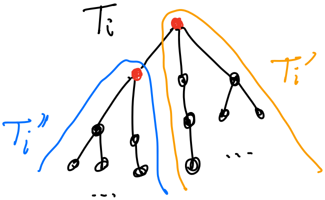
For $k \ge 1$:
Let T_i' be the tree obtained from T_i by deleting any branch from the root.
Let T_i'' be the deleted branch (rooted at a child of the root of T_i).





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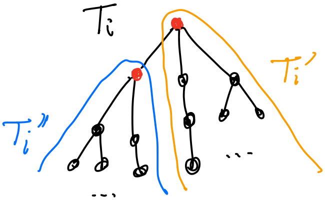

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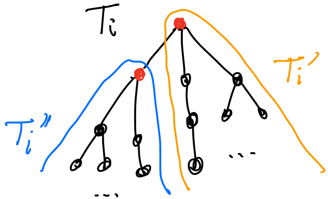

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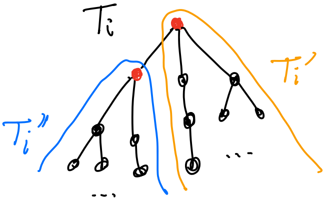

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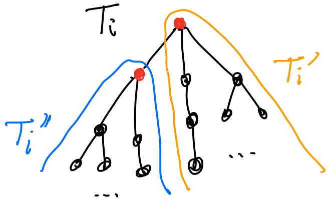

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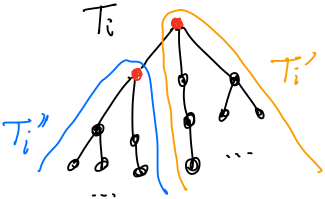
It follows (why?) that $(T'_1, T'_2, \ldots)$ contains an infinite increasing subsequence $T'_{j_1} \leq T'_{j_2} \leq \ldots$



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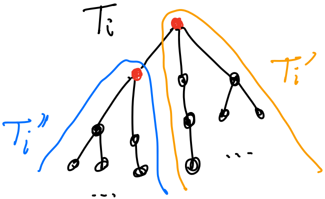

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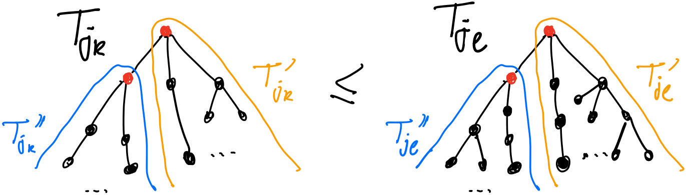

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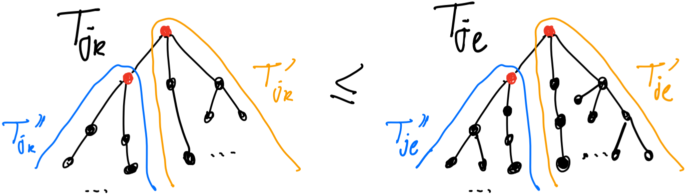

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A notion strongly linked to graph minors

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DISJOINT PATHS

Input: a graph G and $2k$ vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$.

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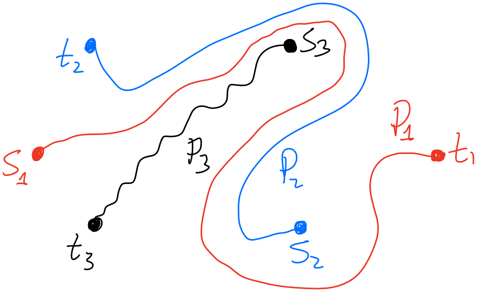

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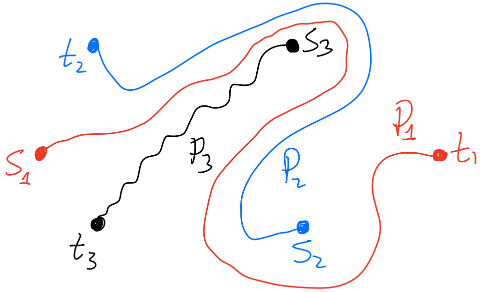

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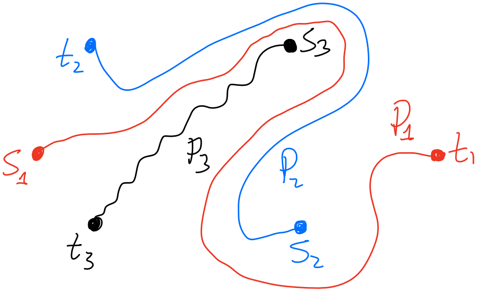

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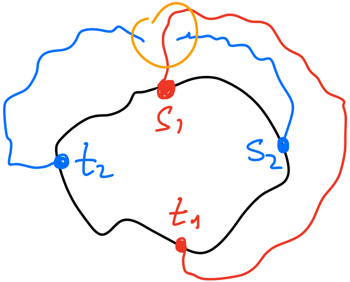
A graph G is k-linked if every instance of DISJOINT PATHS in G with k pairs is positive.

Topology appears naturally in linkages

Theorem (Thomassen and Seymour. 1980)

Let G be a 4-connected graph and $s_1, s_2, t_1, t_2 \in V(G)$. Then (s_1, s_2) and (t_1, t_2) are linked unless G is planar and s_1, s_2, t_1, t_2 are on the boundary of the same face, in this cyclic order.



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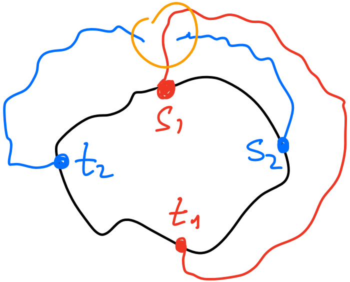

A combinatorial condition (linkage) is translated to a purely topological one (embedding).

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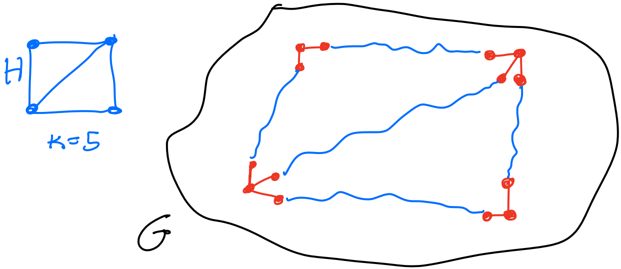

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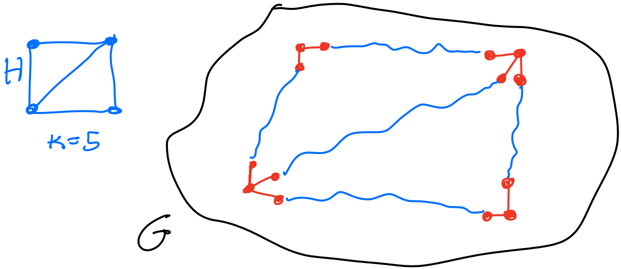

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Idea: if the goal is to decide whether $H \leq_m G$, if G is k-linked, then "yes". Otherwise, we may exploit a topological obstruction to k-linkedness...

Another crucial notion: treewidth

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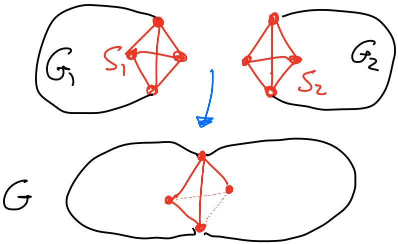

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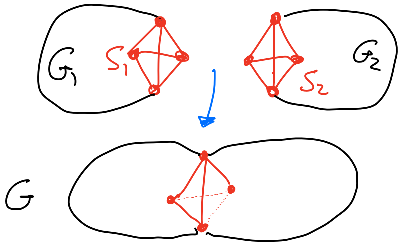

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We say that a graph G has treewidth at most k if it can be obtained by repeatedly taking a k-clique-sum with a graph on at most $k + 1$ vertices.

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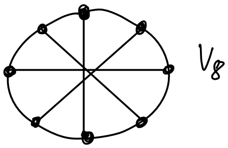
A graph $G \in \mathrm{exc}(K_5)$ if and only if it can be obtained by 0-, 1-, 2- and 3-clique-sums from planar graphs and V_8.



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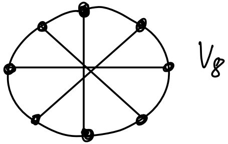

Paradigm: we find "pieces" that exclude K_5 for topological reasons (planarity), add some exceptions (V_8), and then define rules (clique-sums) that preserve being K_5-minor-free.

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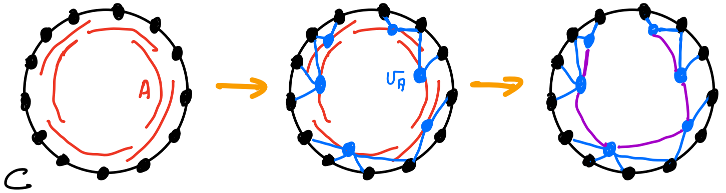
Note: this is an approximate characterization (i.e., not "iff").

Vortices



Adding a vortex of depth h to a cycle C:

- Select arcs on C so that each vertex is contained in at most h arcs.
- For each arc A, create a vertex v_A.
- Connect v_A to some vertices on the arc A.
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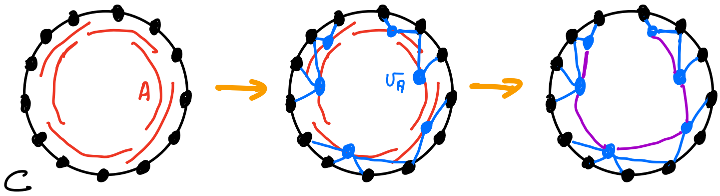

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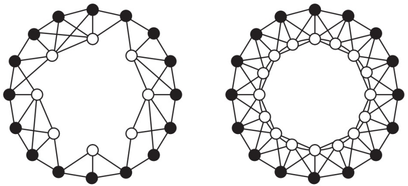

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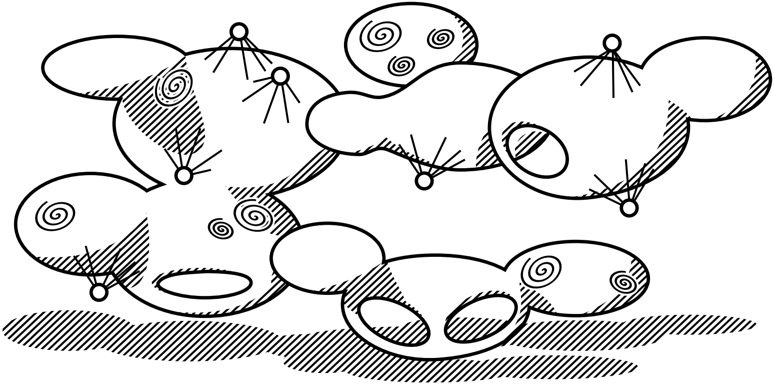
3. *Create at most h new vertices (apices) and connect them to the other vertices arbitrarily.*
4. *Repeatedly construct the h-clique-sum of the current graph with another graph constructed using steps 1-2-3 above.*

A visualization of an H-minor-free graph



[Figure by Felix Riedl]

Sketch of sketch of sketch of proof of Wagner's conjecture

Let's try to mimic the proof for rooted trees by Nash-Williams:

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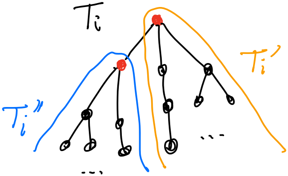

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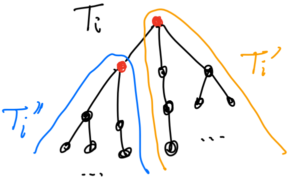

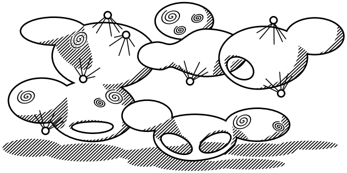

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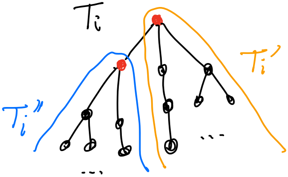

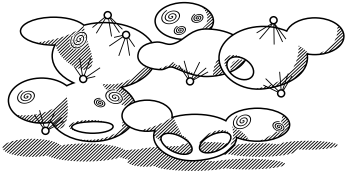

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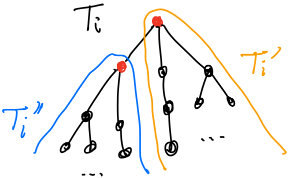

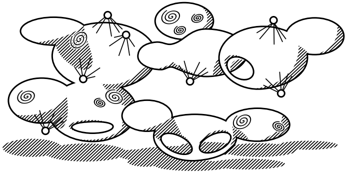

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- If G_1 is planar, every G_i has bounded treewidth: similar to trees.
- Otherwise, by the structure theorem: similar to "extended" surfaces (with apices and vortices), glued in a tree-like way.

Some algorithmic consequences

Disjoint Paths
Input: an n-vertex graph G and vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$.
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Every minor-closed property can be tested in quadratic time.

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Theorem (Robertson, Seymour. 1983-2004)

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This says that there exists an algorithm... no idea how to construct it!!

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Structure of sparse graphs

H-topological-minor-free

$\cup$

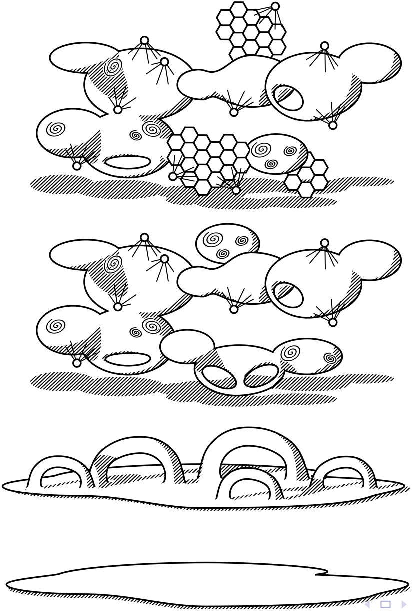
H-minor-free

$\cup$

bounded genus

$\cup$

planar



[Figure by Felix Riedl]

Next section is...

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The multiples origins of treewidth

- 1972: Bertelè and Brioschi (dimension).
- 1976: Halin (S-functions of graphs).
- 1984: Arnborg and Proskurowski (partial k-trees).
- 1984: Robertson and Seymour (treewidth).

A measure of the similarity with a tree

Treewidth measures the (topological) similarity of a graph with a tree.

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Natural candidates:

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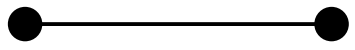

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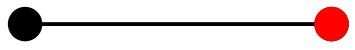

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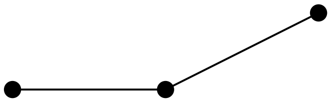

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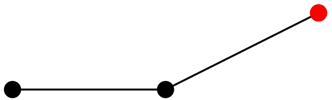

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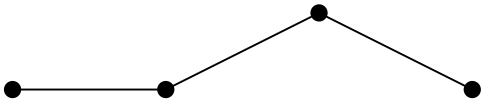

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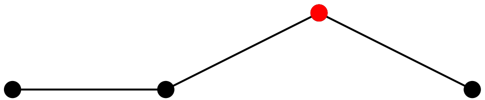

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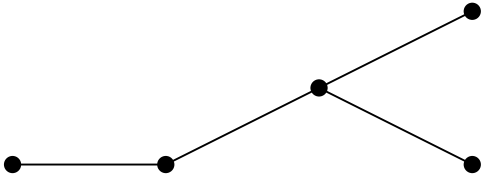

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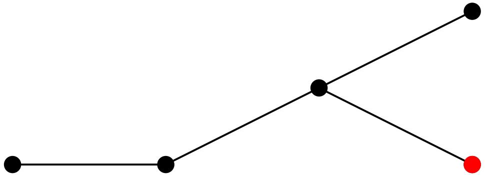

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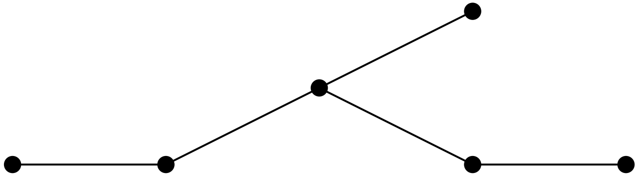
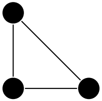


Treewidth via k-trees

Example of a 2-tree:

For $k \geq 1$, a k-tree is a graph that can be built starting from a $(k+1)$-clique and then iteratively adding a vertex connected to a k-clique.



[Figure by Julien Baste]

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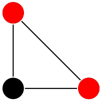

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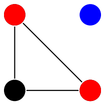

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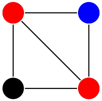

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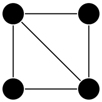

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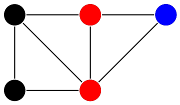

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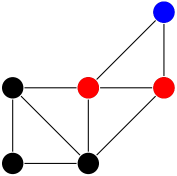

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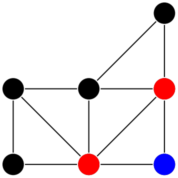

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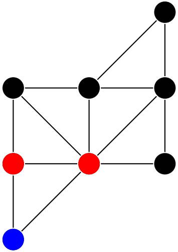

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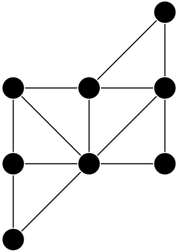

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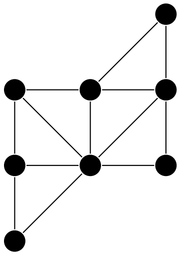

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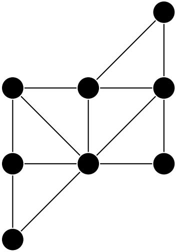

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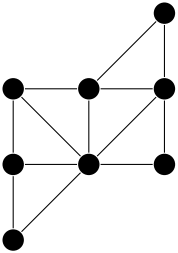

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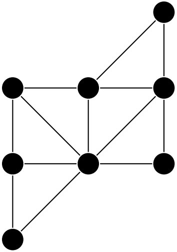
Treewidth of a graph G, denoted $\text{tw}(G)$: smallest integer k such that G is a partial k-tree.

Invariant that measures the topological resemblance of a graph to a forest.

Treewidth via k-trees

Example of a 2-tree:



[Figure by Julien Baste]

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

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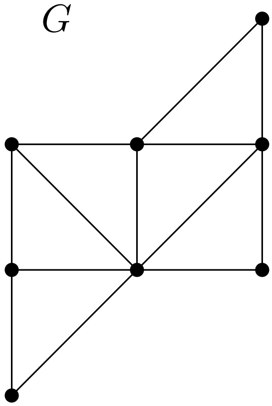

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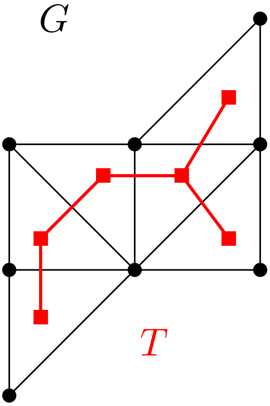

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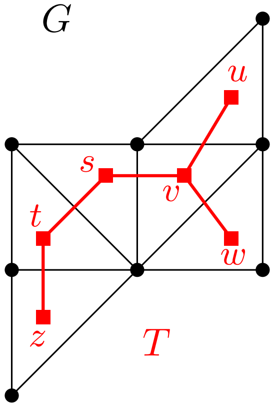

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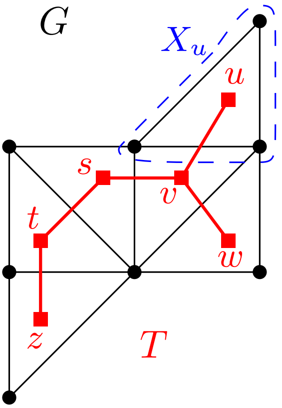

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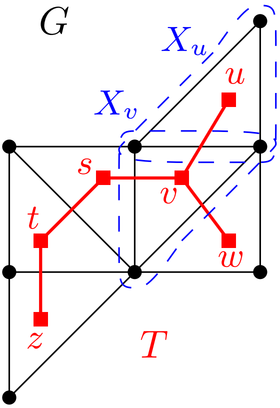

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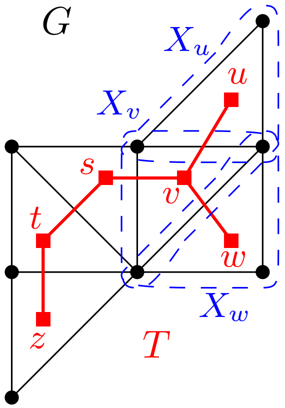

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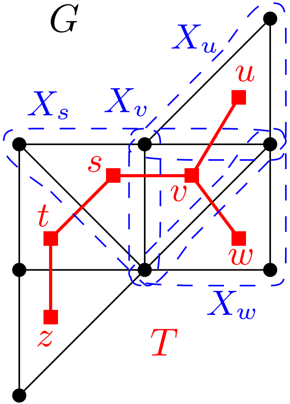

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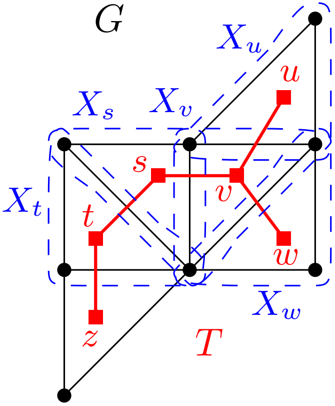

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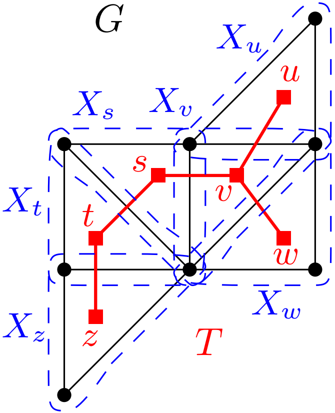

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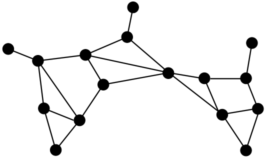

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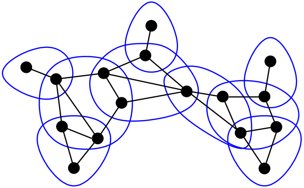

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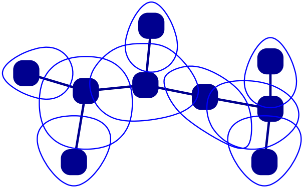

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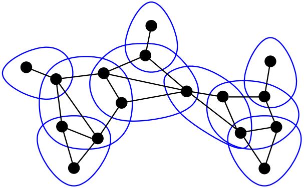

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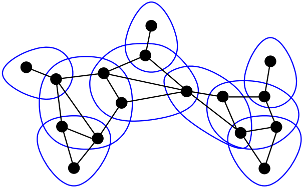

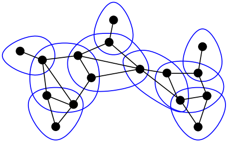

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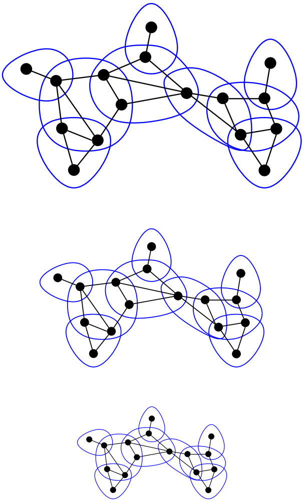

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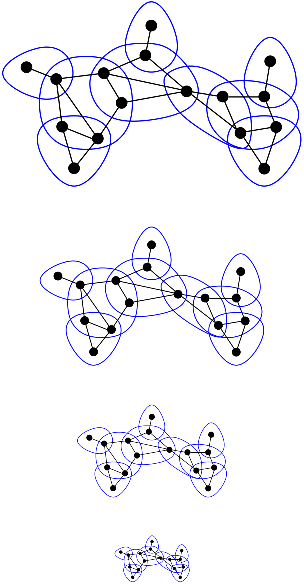

Every bag of a tree decomposition is a separator

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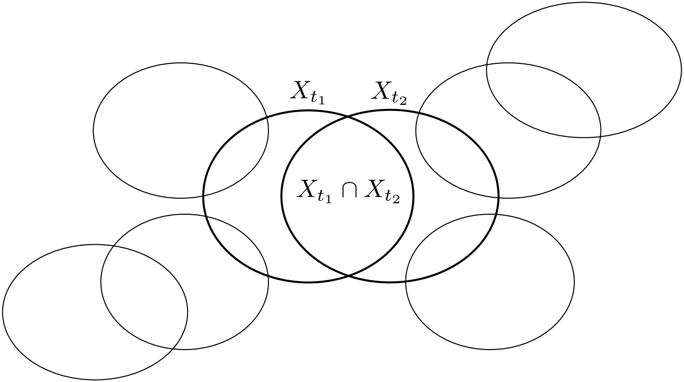

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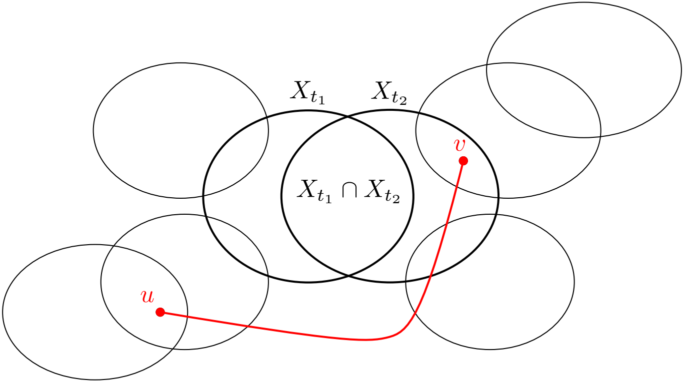

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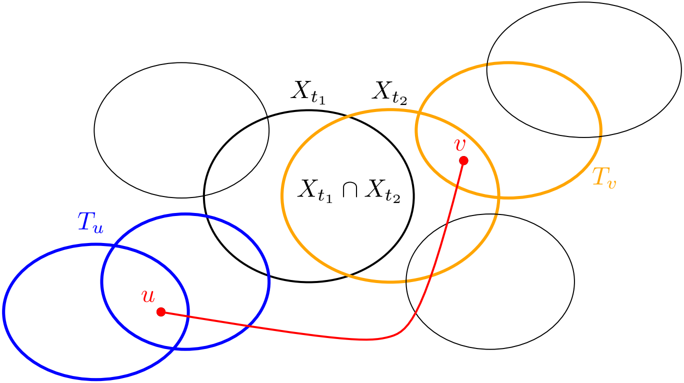

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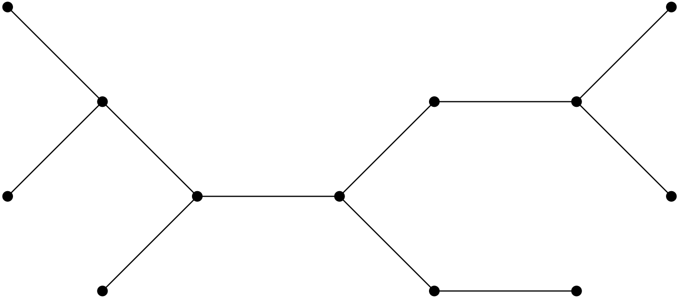

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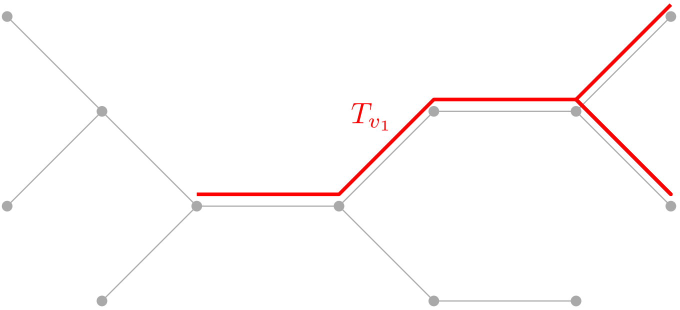

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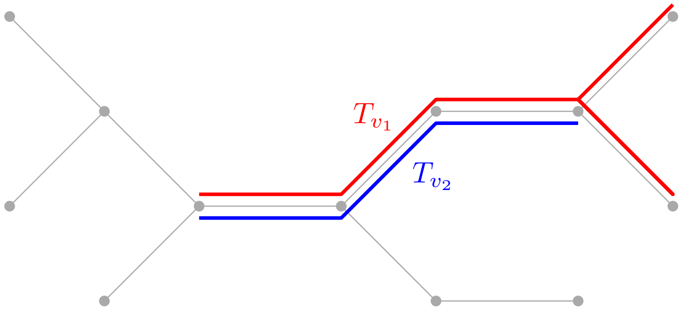

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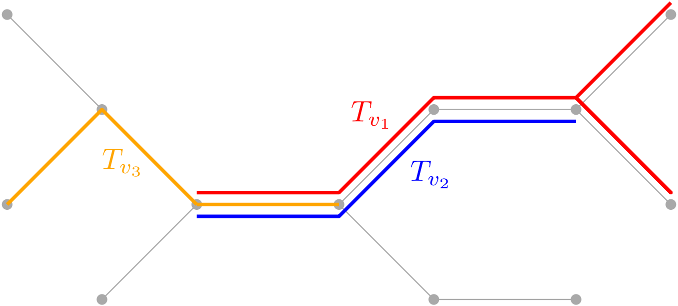

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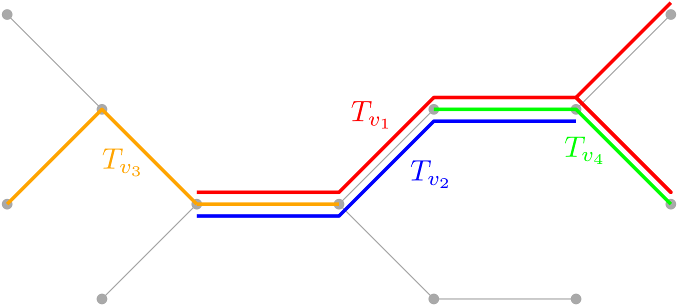

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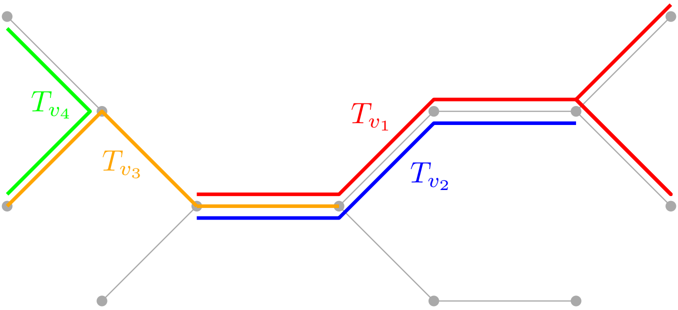

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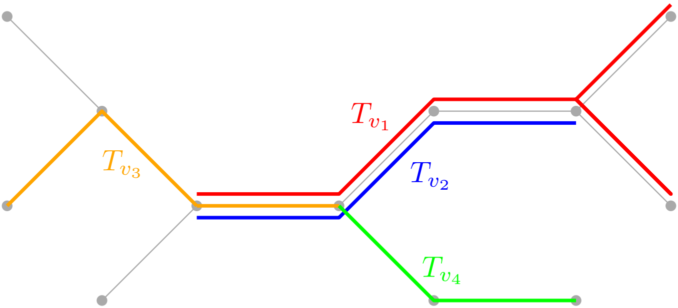

Every clique is entirely contained in some bag

Let G be a graph, $(T, \mathcal{X})$ be a tree decomposition of G, and let $K \subseteq V(G)$ be a clique. Then there exists a bag $X_t \in \mathcal{X}$ such that $K \subseteq X_t$.

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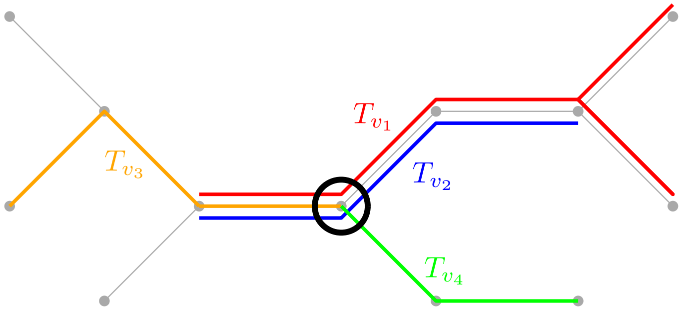

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1. Treewidth is a fundamental combinatorial tool in graph theory: key role in the Graph Minors project of Robertson and Seymour.
2. Treewidth behaves very well algorithmically, and algorithms parameterized by treewidth appear very often in FPT algorithms.
3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

Next subsection is...

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Theorem (Robertson and Seymour. 1993)

For every $k \geq 0$ and graph G, the treewidth of G is at least k if and only if G contains a bramble of order at least $k+1$.

Another dual notion to treewidth: linkedness

[slides borrowed from Christophe Paul]

- Two sets $Y, Z \subseteq V(G)$, with $|Y| = |Z|$, are separable if there is a set $S \subseteq V(G)$ with $|S| < |Y|$ and such that $G - S$ contains no path between $Y \setminus S$ and $Z \setminus S$.

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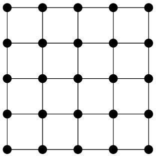

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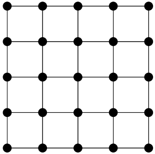

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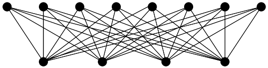

$K_{2k,k}$ is also k-well-linked

Highly linked graphs have large treewidth

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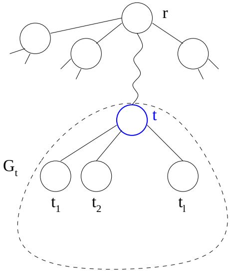

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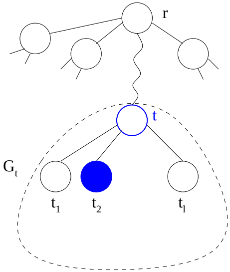

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If $\exists i \in [\ell]$ such that $|V_{t_i} \cap X| \geqslant k$, then we can choose $Y \subseteq V_{t_i} \cap X$, $|Y| = k$ and $Z \subseteq (V \setminus V_{t_i}) \cap X$, $|Z| = k$.

But $S = X_{t_i} \cap X_t$ separates Y and Z and $|S| \leq k - 1$.

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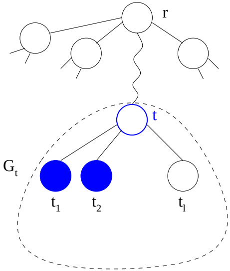

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Otherwise, let $W = V_{t_1} \cup \cdots \cup V_{t_i}$ with $|W \cap X| > k$ and $|(W \setminus V_{t_j}) \cap X| < k$ for $1 \leq j \leq i$.

$Y \subseteq W \cap X$, $|Y| = k+1$ and $Z \subseteq (V \setminus W) \cap X$, $|Z| = k+1$.

But $S = X_t$ separates Y from Z and $|S| \leqslant k$.

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Given a vertex set X of a graph G and $k \leq |X| \leq ck$ for some constant c, it is possible to decide whether X est k-well-linked in time $f(k) \cdot n^{\mathcal{O}(1)}$.

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Remark If X is not k-well-linked we can find, within the same running time,

two separable subsets $Y, Z \subseteq X$.

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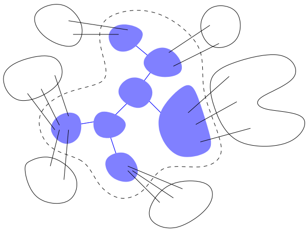
4-approximation of Robertson and Seymour

[slides borrowed from Christophe Paul]



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- We add vertices to a set U in a greedy way, until $U = V(G)$.

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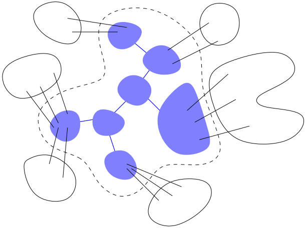

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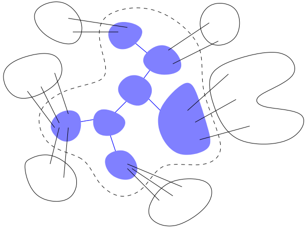

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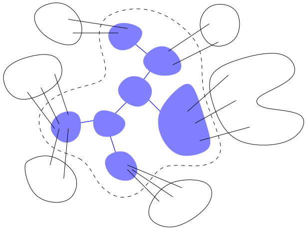

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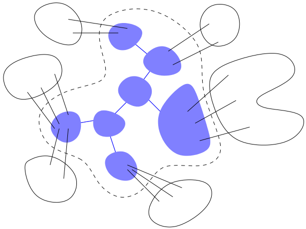

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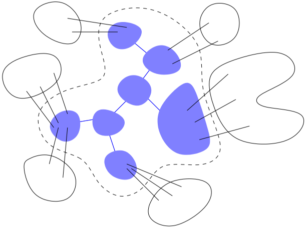

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Initially, we start with U being any set of $3k$ vertices.

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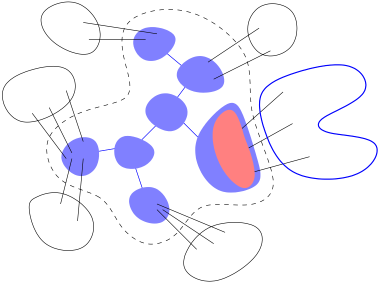

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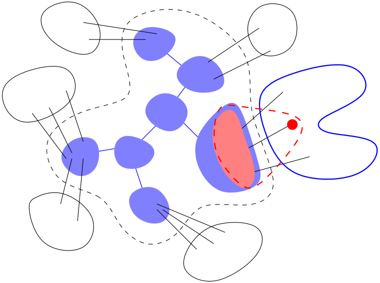

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- If $|X| < 3k$: we add a node t' neighbor of t such that $X_{t'} = \{x\} \cup X$, with $x \in C$ being a neighbor of X_t.

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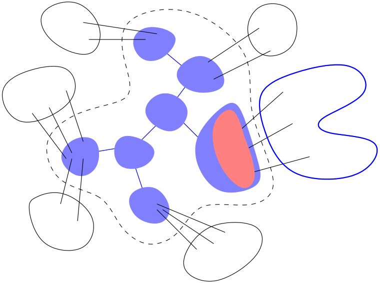

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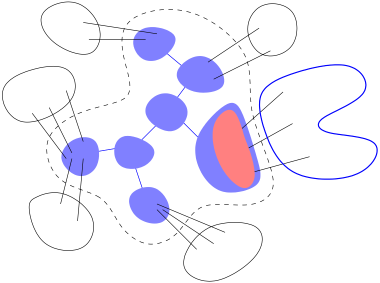

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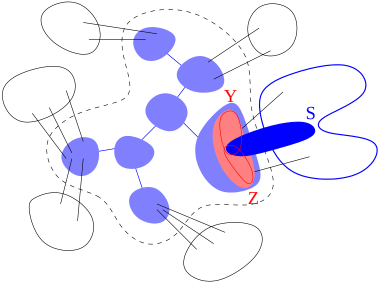

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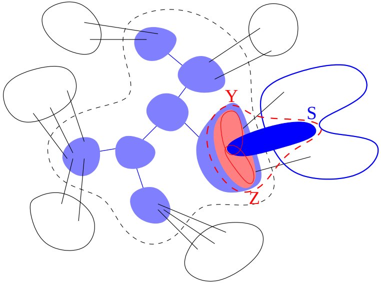

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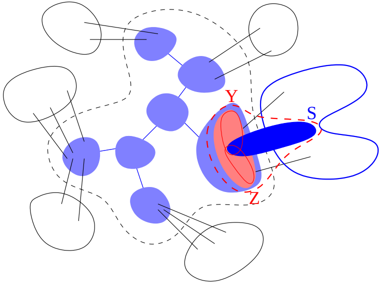

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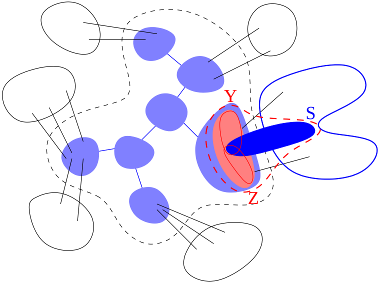

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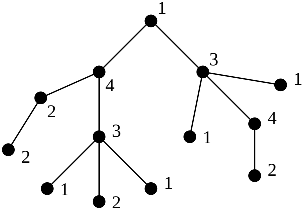
Next subsection is...

WEIGHTED INDEPENDENT SET on trees

[slides borrowed from Christophe Paul]



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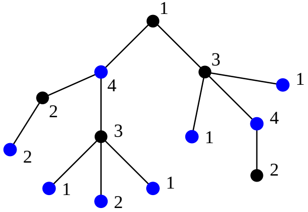

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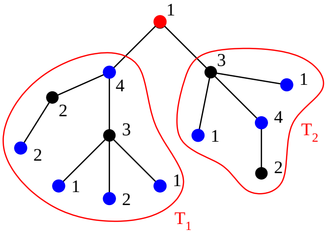
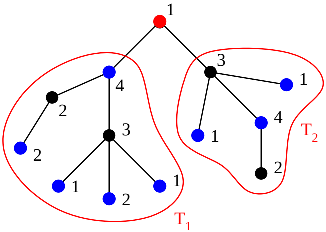


Observations:

1. Every vertex of a tree is a separator.
2. The union of independent sets of distinct connected components is an independent set.

Weighted Independent Set on trees

[slides borrowed from Christophe Paul]



Let x be the root of T, $x_1 \dots x_\ell$ its children, $T_1, \dots T_\ell$ subtrees of $T - x$:

- $wIS(T, x)$: maximum weighted independent set containing x.
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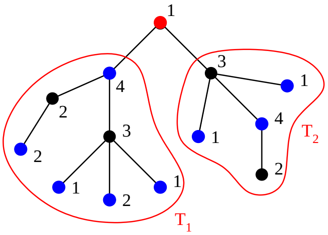

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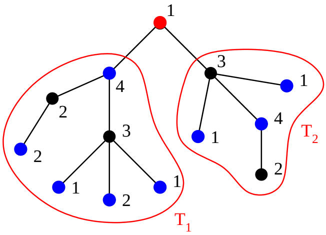

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Dynamic programming on tree decompositions

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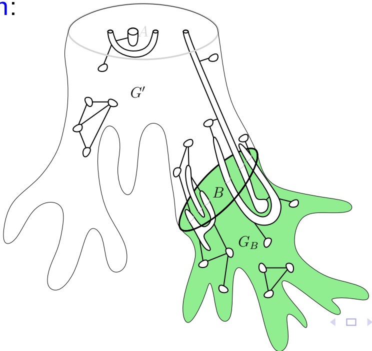
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Back to tree decompositions

Let $(T, \{X_t \mid t \in V(T)\})$ be a tree decomposition of a graph G.

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- V_t: all vertices of G appearing in bags that are descendants of t.
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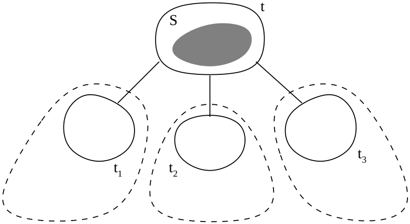

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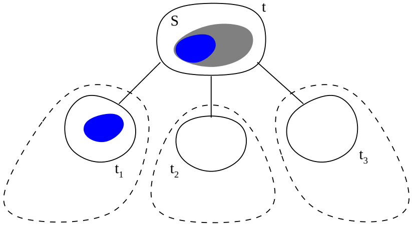

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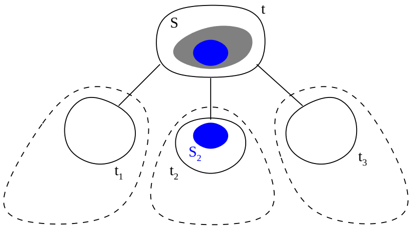

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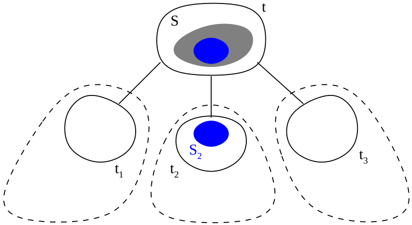

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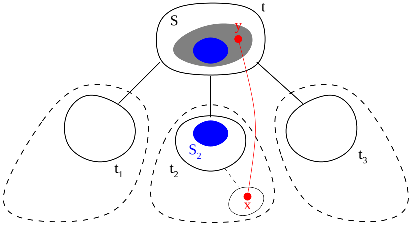

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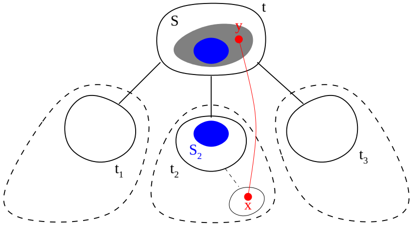

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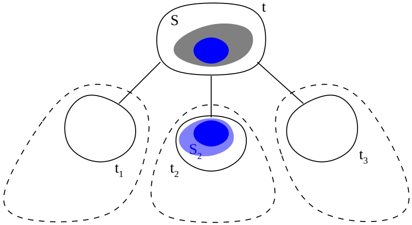
Contradiction! X_{t_j} is not a separator.

INDEPENDENT SET on tree decompositions

Idea of the dynamic programming algorithm:



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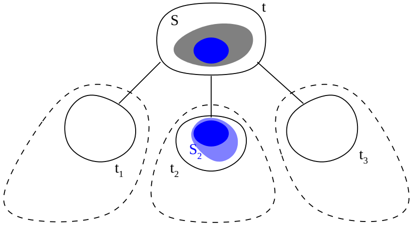

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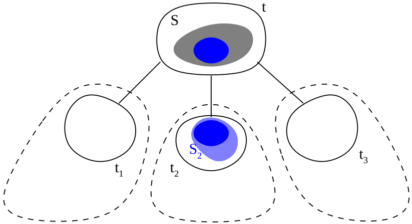

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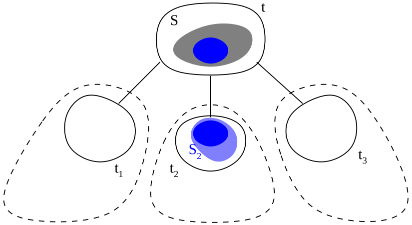

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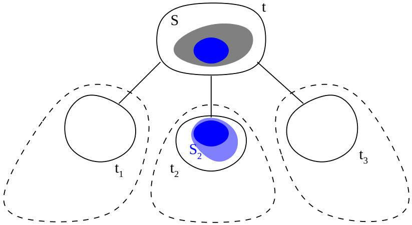

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★ We have to add the time in order to compute a "good" tree decomposition of the input graph (as we have seen before).

Helpful tool: nice tree decompositions

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A rooted tree decomposition $(T, \{X_t : t \in T\})$ of a graph G is nice if every node $t \in V(T) \setminus$ root is of one of the following four types:

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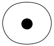

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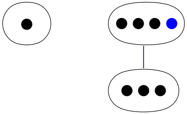

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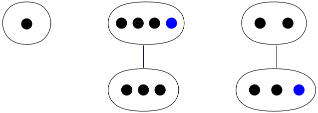

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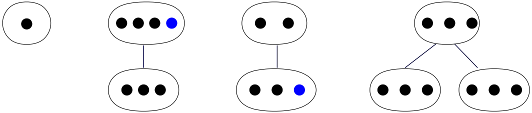

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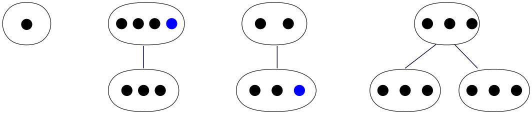

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Lemma

A tree decomposition $(T, \{X_t : t \in T\})$ of width k and x nodes of an n-vertex graph G can be transformed in time $\mathcal{O}(k^2 \cdot n)$ into a nice tree decomposition of G of width k and $k \cdot x$ nodes. (why?)?

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Complexity : $\mathcal{O}(2^k \cdot k^2 \cdot n)$

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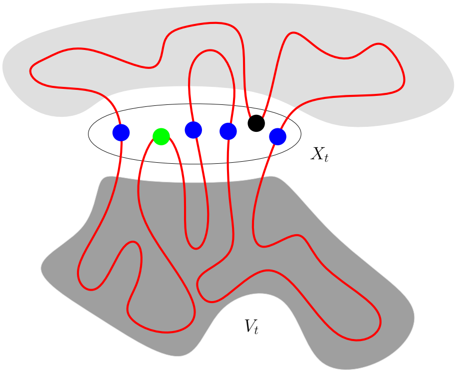
[slides borrowed from Christophe Paul]

Let $\mathcal{C}$ be a Hamiltonian cycle.

- Note that $\mathcal{C} \cap G[V_t]$ is a collection of paths.



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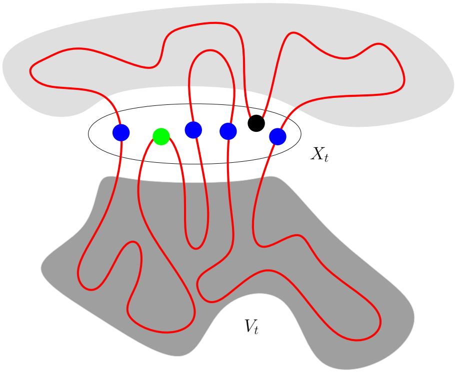

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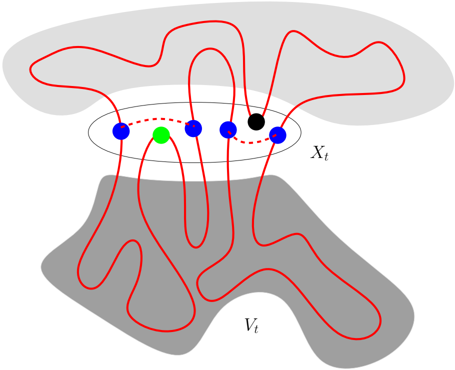

For every node t of the tree decomposition, we need to know if

$$(X_t^0, X_t^1, X_t^2, M)$$

where M is a **matching** on X_t^1, corresponds to a partial solution.

skip

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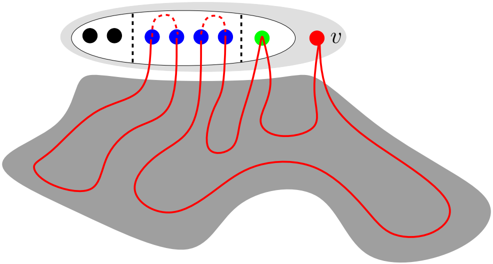

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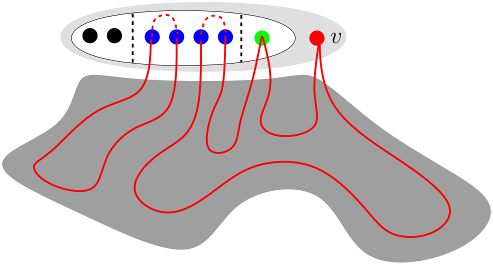

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$(X^0_{t'}, X^1_{t'}, X^2_{t'} \setminus \{v\}, M)$ is a partial solution for t

$\Leftrightarrow$

$(X^0_{t'}, X^1_{t'}, X^2_{t'}, M)$ is a partial solution for t' with $v \in X^2_{t'}$

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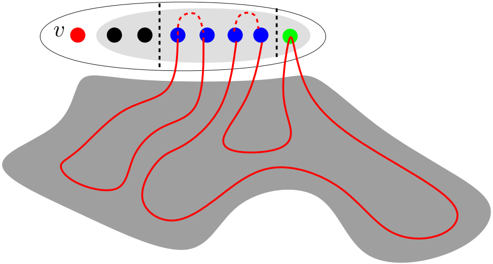

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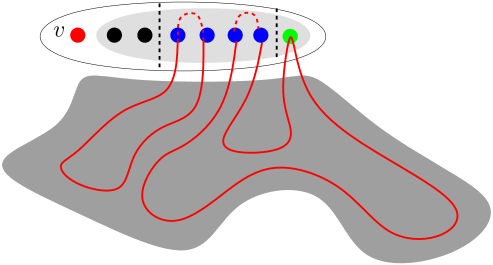

$$(X_{t'}^0 \cup \{v\}, X_{t'}^1, X_{t'}^2, M) \text{ is a partial solution for } t$$
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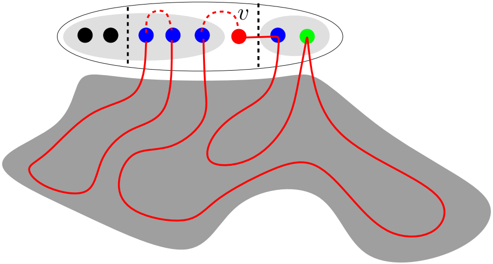

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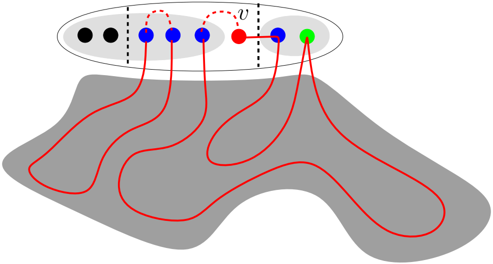

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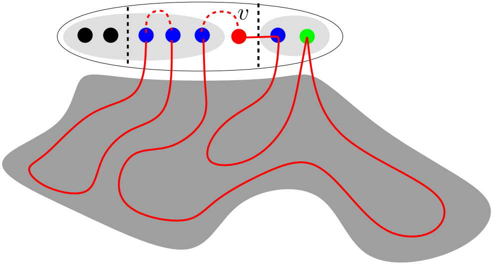

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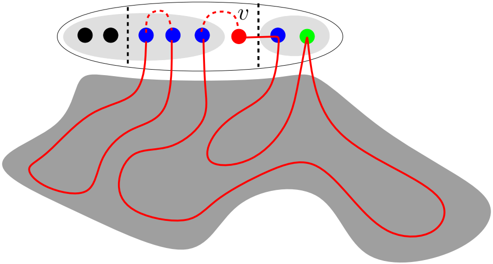

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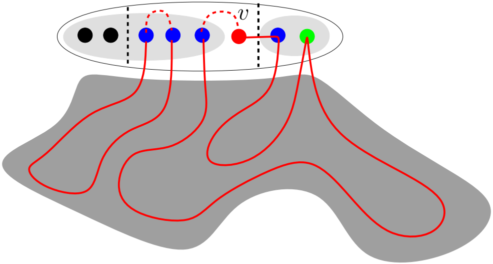

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- or a vertex $w \in X_{t'}^0$ becomes extremity of a path $\Rightarrow$ $w \in X_t^1$ (similar).

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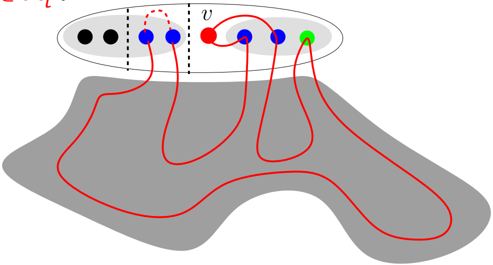

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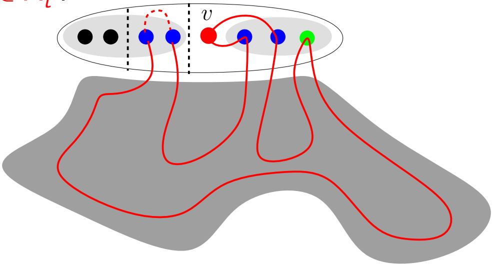

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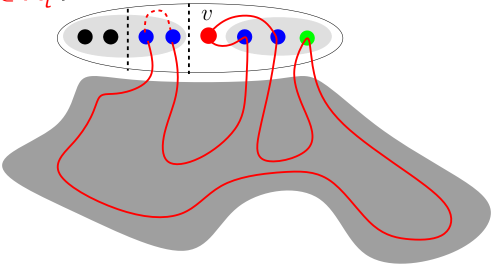

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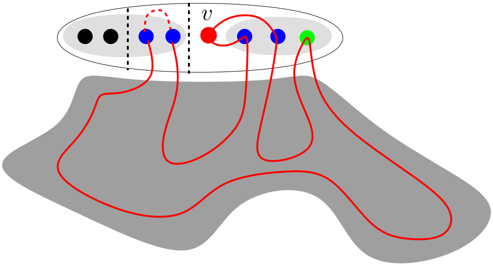

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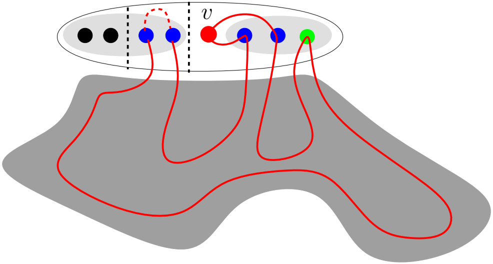

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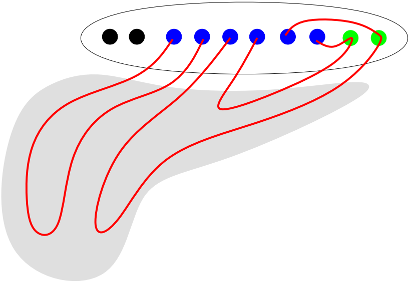

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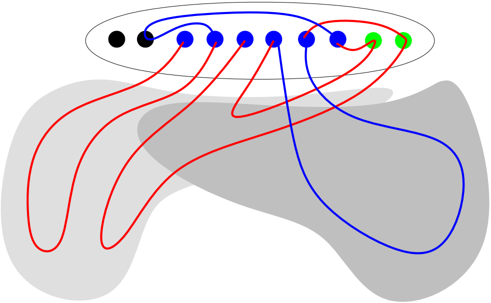

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Can this approach be generalized to more problems?

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- Quantifiers $\exists$, $\forall$ on vertex/edge variables or vertex/edge sets.

($\text{MSO}_1/\text{MSO}_2$)

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Connected: $\forall$ bipartition V_1, V_2, $\exists v_1 \in V_1, \exists v_2 \in V_2, \{v_1, v_2\} \in E(G)$.

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$\forall v \in V, (v \in V_1 \vee v \in V_2) \wedge (v \in V_1 \Rightarrow v \notin V_2) \wedge (v \in V_2 \Rightarrow v \notin V_1).$

Connected: $\forall$ bipartition V_1, V_2, $\exists v_1 \in V_1, \exists v_2 \in V_2, \{v_1, v_2\} \in E(G)$.

Other properties that can be expressed in MSO_2:

- a set being a vertex cover, independent set. (why?)

Monadic second order logic of graphs: examples

Example 1 Expressing that $\{u, v\} \in E(G)$: $\exists e \in E, \mathrm{inc}(u, e) \wedge \mathrm{inc}(v, e)$.

Example 2 Expressing that a set $S \subseteq V(G)$ is a dominating set.

$\mathtt{DomSet}(S)$: $\forall v \in V(G) \setminus S, \exists u \in S : \{u, v\} \in E(G)$.

Example 3 Expressing that a graph $G = (V, E)$ is connected.

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- a graph being k-colorable (for fixed k), having a Hamiltonian cycle.

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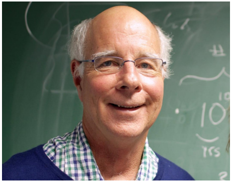
In parameterized complexity: FPT parameterized by treewidth.

Parameterized complexity in a nutshell

Idea Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established and very active area.

Parameterized problems

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$,
where Σ is a fixed, finite alphabet.

For an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, k is called the parameter.

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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$
- k-Clique: Solvable in time $\mathcal{O}(k^2 \cdot n^k)$
- Vertex k-Coloring: NP-hard for fixed $k = 3$.

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skip

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Working hypothesis of parameterized complexity: **k-CLIQUE is not FPT**

(in classical complexity: 3-SAT cannot be solved in poly-time)

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Do all FPT problems admit polynomial kernels? NO!

Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)

Deciding whether a graph has a PATH *with* $\geq k$ *vertices is* FPT *but does not admit a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

Typical approach to deal with a parameterized problem

Parameterized problem L

k-Clique
k-Vertex Cover
k-Path
Vertex k-Coloring

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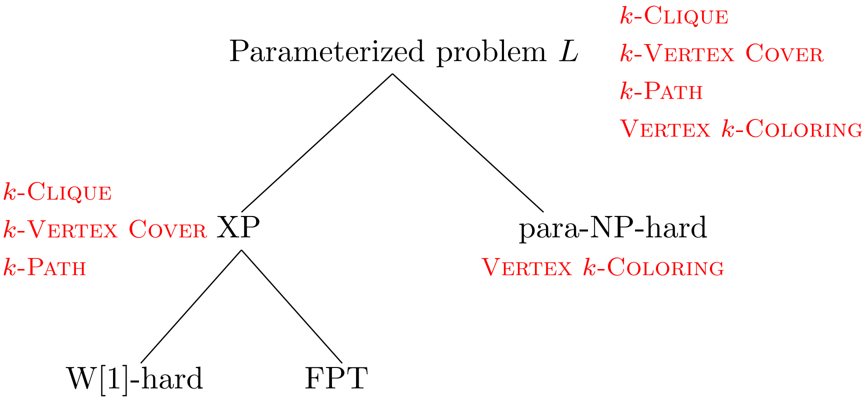

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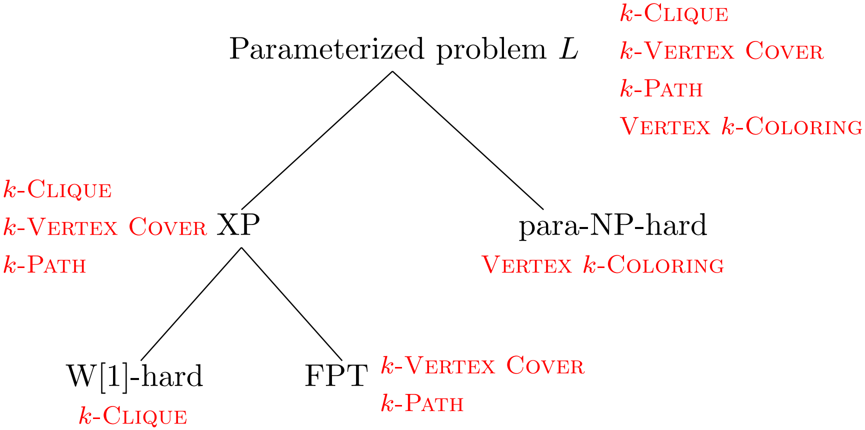

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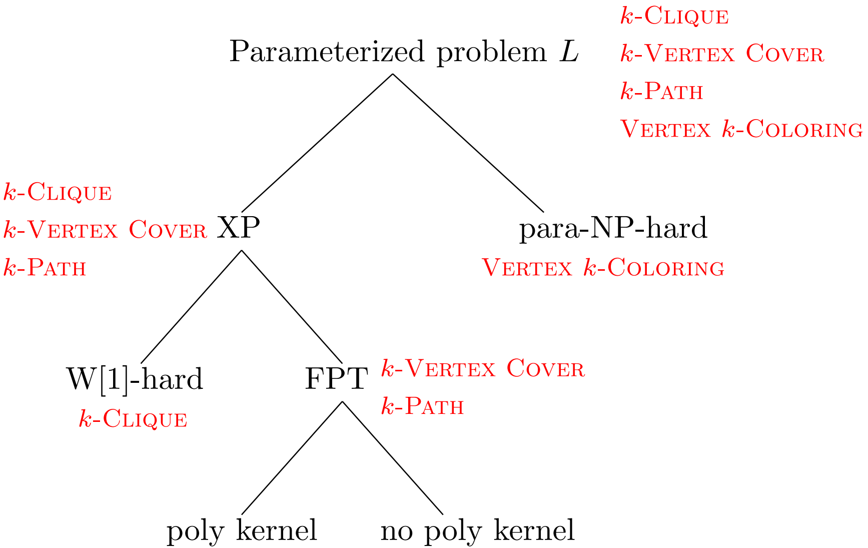

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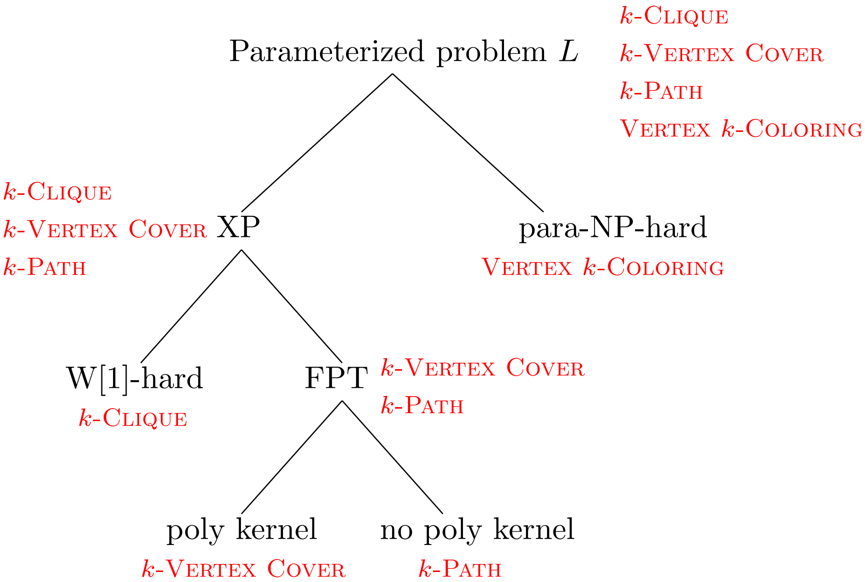

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2. Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

Next subsection is...

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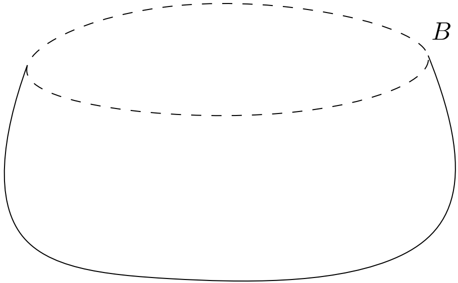
Remark: Algorithms parameterized by treewidth appear very often as a "black box" in all kinds of parameterized algorithms.

Two behaviors for problems parameterized by treewidth

Local problems | Vertex Cover, Dominating Set, Clique, Independent Set, q-Coloring for fixed q.



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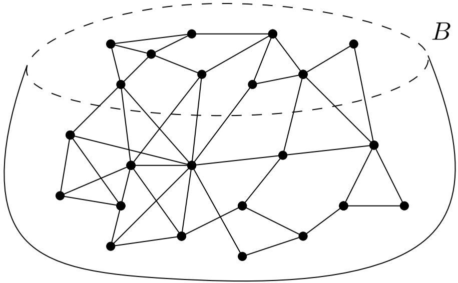

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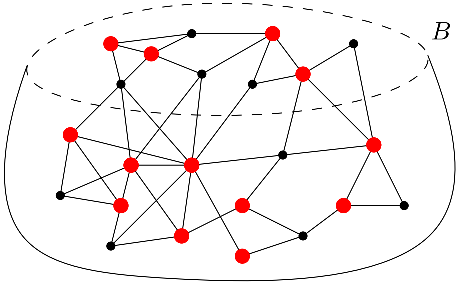

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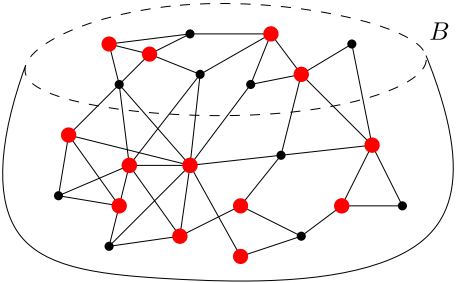

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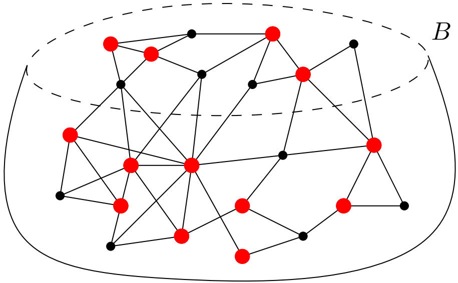

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- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

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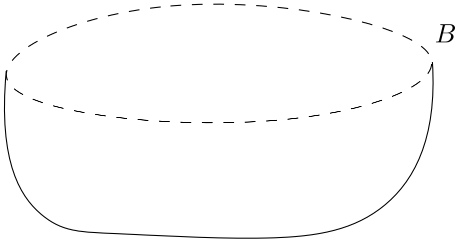
Connectivity problems seem to be more complicated...

Connectivity problems | Hamiltonian Cycle, Longest Path, Steiner Tree, Connected Vertex Cover.



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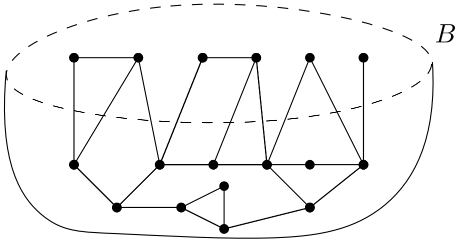

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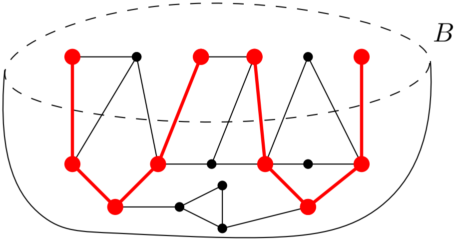

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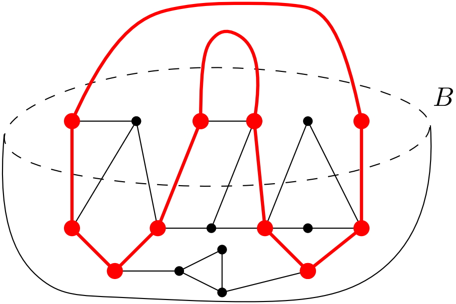

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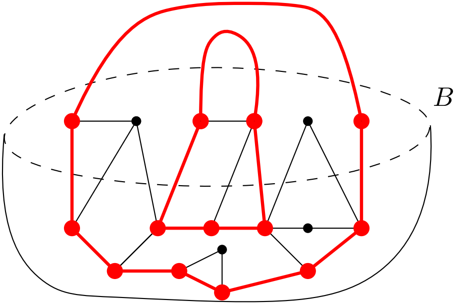

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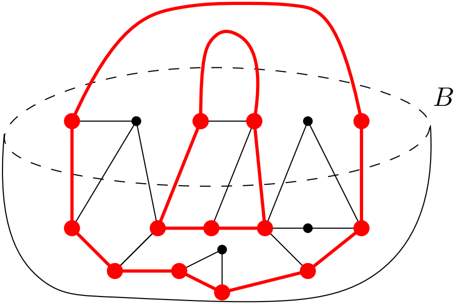

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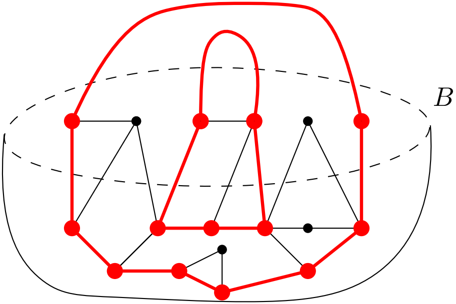

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$2^{\mathcal{O}(\text{tw} \log \text{tw})}$ choices

- The "natural" DP algorithms provide only time $2^{\mathcal{O}(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$.

Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

$$\boxed{2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

Vertex Cover, Dominating Set, ...

- Connectivity problems:

$$\boxed{2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

Longest Path, Steiner Tree, ...

How topology helps for dynamic programming?

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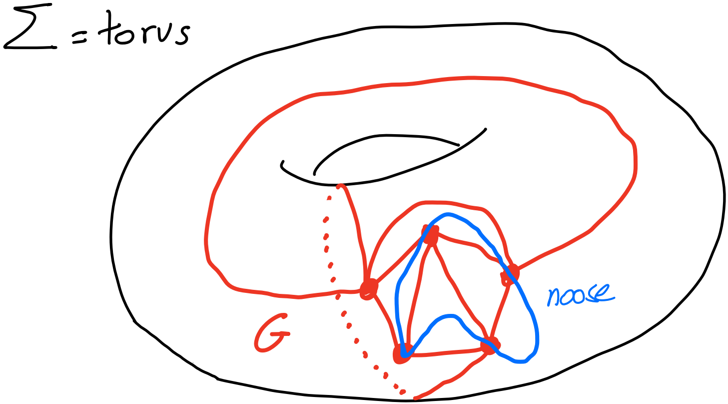
On topologically structured graphs (planar, surfaces, minor-free), it is possible to solve connectivity problems in time $2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}$:

- We consider a special tree-decomposition of a sparse graph, and exploit the structure of the subgraph induced by the bags.
- More precisely, we use the existence of tree decompositions of small width and with nice topological properties.
- These nice properties do not change the DP algorithms, but the analysis of their running time.

Nooses

Let G be a graph embedded in a surface Σ. A noose is a subset of Σ homeomorphic to $\mathbb{S}^1$ that meets G only at vertices.



- Let G be a planar graph. A sphere cut decomposition of G is a tree decomposition $(T, \{X_t : t \in V(T)\})$ of G such that the vertices in each bag X_t are situated around a noose in the plane.

[NB: several details are missing in this definition]

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Theorem (Seymour and Thomas. 1994)

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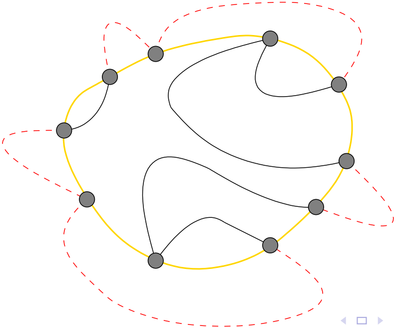

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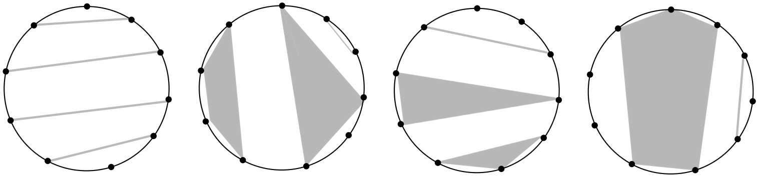

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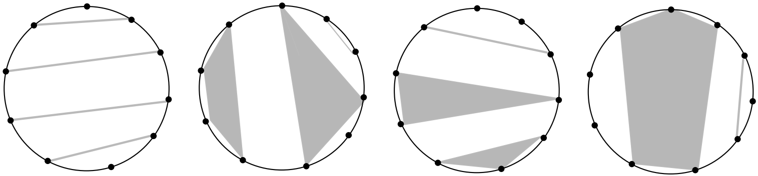

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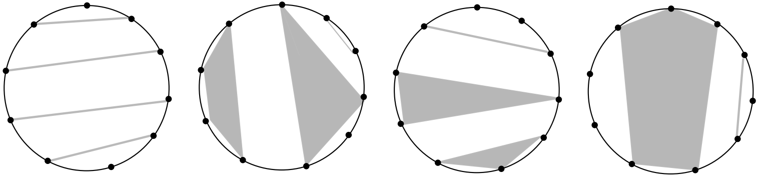

- Exactly the number of *non-crossing partitions* over k elements, which is given by the k-th Catalan number:

$$\mathrm{CN}(k) = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}} \approx 4^k.$$

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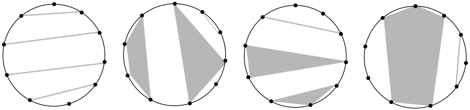
This idea was first used in [Dorn, Penninkx, Bodlaender, Fomin. 2005]

Generalizations to other sparse graph classes

Main idea special type of decomposition with nice topological properties:

partial solutions $\Longleftrightarrow$ non-crossing partitions



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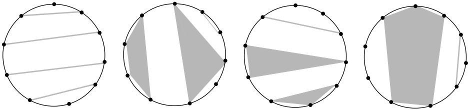

This idea has been generalized to other graph classes and problems:

- Graphs on surfaces: [Dorn, Fomin, Thilikos '06] [Rué, S., Thilikos '10]
- *H*-minor-free graphs: [Dorn, Fomin, Thilikos '08] [Rué, S., Thilikos '12]

The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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There are other examples of such problems...

Next section is...

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A few representative problems

VERTEX COVER
Input: A graph $G = (V, E)$ and a positive integer k.
Parameter: k.
Question: Does there exist a subset $C \subseteq V$ of size at most k such that $G[V \setminus C]$ is an independent set?

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LONG PATH
Input: A graph $G = (V, E)$ and a positive integer k.
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Question: Does there exist a path P in G of length at least k?

A few representative problems (II)

FEEDBACK VERTEX SET
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Dominating Set
Input: A graph $G = (V, E)$ and a positive integers k.
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Question: Does there exist a subset $D \subseteq V$ of size at most k such that for all $v \in V$, $N[v] \cap D \neq \emptyset$?

Minor-closed parameters

- A graph class $\mathcal{G}$ is *minor (contraction)-closed* if any minor (contraction) of a graph in $\mathcal{G}$ is also in $\mathcal{G}$.

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- We say that a parameter P is *closed under taking of minors/contractions* (or, briefly, *minor/contraction-closed*) if for every graph H, $H \preceq_m G$ / $H \preceq_{cm} G$ implies that $P(H) \leq P(G)$.

Examples of minor/contraction closed parameters

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  VERTEX COVER, FEEDBACK VERTEX SET, LONG PATH, TREEWIDTH, ... (why?)

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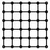
Grid Exclusion Theorem

- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$

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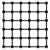
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  ℓ We have $\mathrm{tw}(H_{\ell,\ell}) = \ell$.

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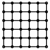
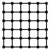
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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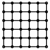
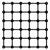
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\text{tw}(H_{\ell,\ell}) = \ell$.
- As Treewidth is minor-closed, if  $_\ell \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$.

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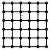
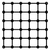
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid: $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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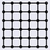
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains $_\ell$ *as a minor.*

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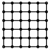
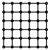
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\text{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\text{tw}(G) \geq \text{tw}(H_{\ell,\ell}) = \ell$.

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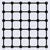
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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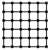
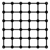
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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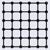
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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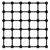
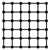
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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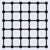
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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- Some improvement: $c(\ell) = 2^{O(\ell \log \ell)}$. [Leaf and Seymour. 2012]

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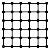
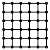
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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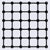
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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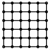
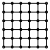
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$. Does the reverse implication hold?

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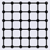
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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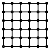
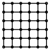
- Let $H_{\ell,\ell}$ be the $(\ell \times \ell)$-grid:  $_\ell$ We have $\mathrm{tw}\,(H_{\ell,\ell}) = \ell$.
- As TREEWIDTH is minor-closed, if  $_\ell \preceq_m G$, then $\mathrm{tw}(G) \geq \mathrm{tw}(H_{\ell,\ell}) = \ell$.

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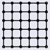
For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains  $_\ell$ *as a minor.*

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Important message grid-minors are the certificate of large treewidth.

Grid Exclusion Theorems on sparse graphs

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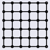
Every planar graph of treewidth $\geq 6 \cdot \ell$ *contains*  $_\ell$ *as a minor.*

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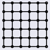
Every planar graph of treewidth $\geq 6 \cdot \ell$ *contains*  $_{\ell}$ *as a minor.*

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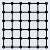
For every fixed g, *there is a constant* c_g *such that every graph of genus* g *and of treewidth* $\geq c_g \cdot \ell$ *contains*  $_{\ell}$ *as a minor.*

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Every planar graph of treewidth $\geq 6 \cdot \ell$ contains $\boxplus_\ell$ as a minor.

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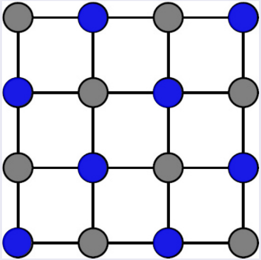
In sparse graphs: linear dependency between treewidth and grid-minors

How to use Grid Theorems algorithmically?

Next subsection is...

Example: FPT algorithm for Planar Vertex Cover

A vertex cover of a graph G is a set of vertices C such that every edge of G has at least one endpoint in C. Min size: $\mathbf{vc}(G)$.



Example: FPT algorithm for Planar Vertex Cover

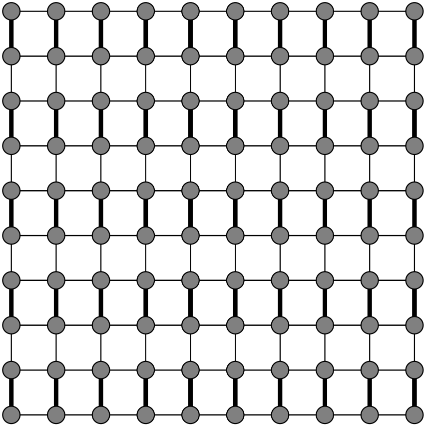
INPUT: Planar graph G on n vertices, and an integer k.

OUTPUT: Either a vertex cover of G of size $\leq k$, or a proof that G has no such a vertex cover.

RUNNING TIME: $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Objective subexponential FPT algorithm for PLANAR VERTEX COVER.

Example: FPT algorithm for Planar Vertex Cover



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- The size of any vertex cover of $H_{\ell,\ell}$ is at least $\ell^2/2$.
- Recall that VERTEX COVER is a minor-closed parameter.
- Since $H_{\ell,\ell} \preceq_m G$, it holds that $\textbf{vc}(G) \geq \textbf{vc}(H_{\ell,\ell}) \geq \ell^2/2$.

We are already very close to an algorithm...

Recall:

- k is the parameter of the problem.
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This gives a subexponential FPT algorithm!

Was Vertex Cover really just an example...?

What is so special in Vertex Cover?

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★ Only the linear Grid Exclusion Theorem!

Arguments go through up to H-minor-free graphs.

Next subsection is...

Minor Bidimensionality:

[Demaine, Fomin, Hajiaghayi, Thilikos. 2005]

Definition

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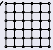
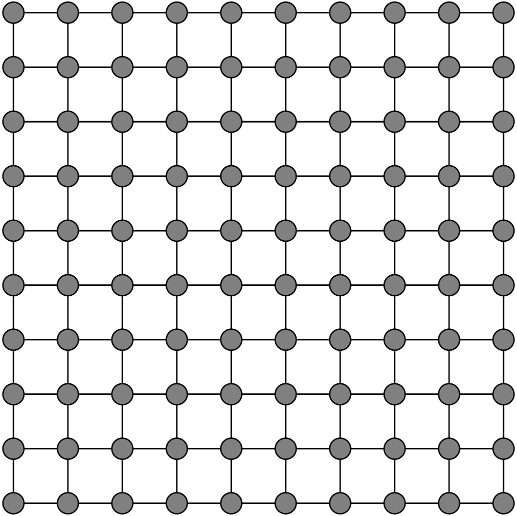
2. **p** ( $_k$) $= \Omega(k^2)$.

Vertex Cover of a Grid



$H_{\ell,\ell}$ for $\ell = 10$

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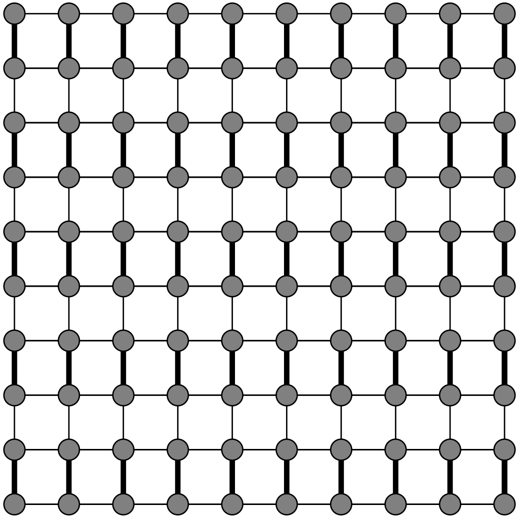

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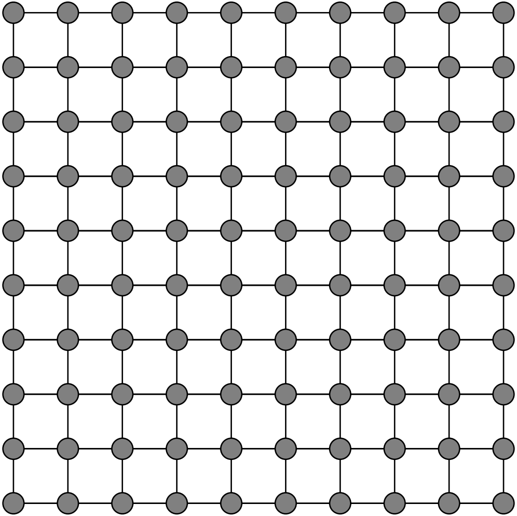
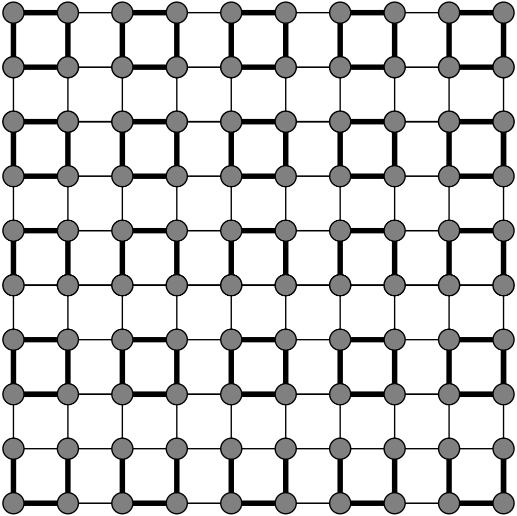


Feedback Vertex Set of a Grid



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- If we have a DP algorithm for bounded treewidth running in time c^t or t^t, then it implies $2^{O(\sqrt{k})}$ or $2^{O(\sqrt{k}\log k)}$ algorithm.

Piecing everything together

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   - Deterministic algorithms based on matrix rank. [Boadlaender et al. 2012]
   - Deterministic algorithms based on matroids. [Fomin et al. 2013]

Minor Bidimensionality provides a meta-algorithm

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- What about contraction-closed parameters??

  DOMINATING SET, CONNECTED VERTEX COVER, r-DOMINATING SET, ...

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What is a $(k \times k)$-grid-like graph...?

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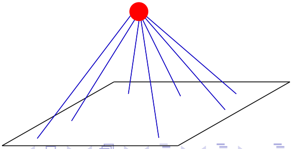
- ★ For apex-minor-free graphs, this is a ($k \times k$)-augmented grid, i.e., partially triangulated grid augmented with additional edges such that each vertex is incident to $O(1)$ edges to non-boundary vertices of the grid.

  [Demaine, Fomin, Hajiaghayi, Thilikos. 2005]

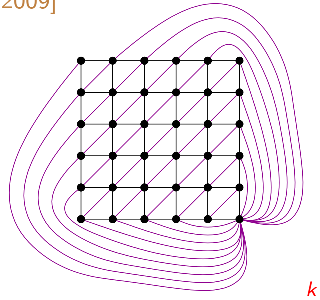
H is an *apex graph* if $\exists v \in V(H)$: $H - v$ is planar



Contraction bidimensionality: new definition

Finally, the right "$(k \times k)$-grid-like graph" was found:

[Fomin, Golovach, Thilikos. 2009]



Contraction bidimensionality: new definition

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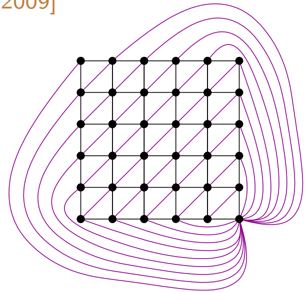
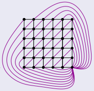


Definition

A parameter **p** is *contraction bidimensional* if the following hold:

1. **p** is contraction-closed, and
2. **p**( $_k$) $= \Omega(k^2)$.

Meta-algorithms for contraction bidimensional parameters

Theorem

Let H be a fixed apex graph, let G be an H-minor free graph, and let $\mathbf{p}$ be a contraction bidimensional parameter computable in $2^{O(\mathbf{tw}(G))} \cdot n^{O(1)}$. Then deciding $\mathbf{p}(G) = k$ can be done in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

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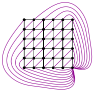
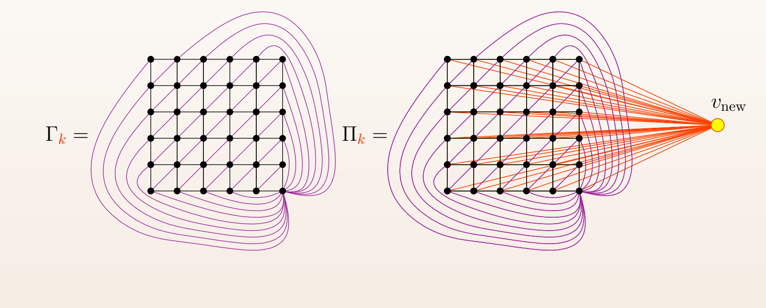
As for minor bidimensionality, we need to prove that

- If $\mathbf{tw}(G) = \Omega(k)$ then G contains  $_k$ as a contraction.

Two important grid-like graphs

Two pattern graphs Γ_k and Π_k:



$\Pi_k = \Gamma_k +$ a new universal vertex $v_{\rm new}$.

The “contraction-certificates” for large treewidth

Theorem (Fomin, Golovach, Thilikos. 2009)

For any integer $\ell > 0$, there is c_ℓ such that every connected graph of treewidth at least c_ℓ contains K_ℓ, Γ_ℓ, or Π_ℓ as a contraction.

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Further applications of Bidimensionality

1. Bidimensionality + DP $\Rightarrow$ Subexponential FPT algorithms

   [Demaine, Fomin, Hajiaghayi, Thilikos. 2004-2005]
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4. Bidimensionality + new Grid Theorems ⇒ Geometric graphs

   [Fomin, Lokshtanov, Saurabh. 2012]
   [Grigoriev, Koutsonas, Thilikos. 2013]

Next section is...

Basic principle of the irrelevant vertex technique

This technique was invented in [Robertson and Seymour. 1995]

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DISJOINT PATHS
Input: a graph G and k pairs of vertices $T = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$.
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   A vertex $v \in V(G)$ such that (G, T, k) and $(G \setminus v, T, k)$ are equivalent instances.

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2. Otherwise, if $\text{tw}(G) \leq f(k)$, solve the problem using dynamic programming (by Courcelle).

How to find an irrelevant vertex when the treewidth is large?

How to find an irrelevant vertex when the treewidth is large?

By using the Grid Exclusion Theorem!

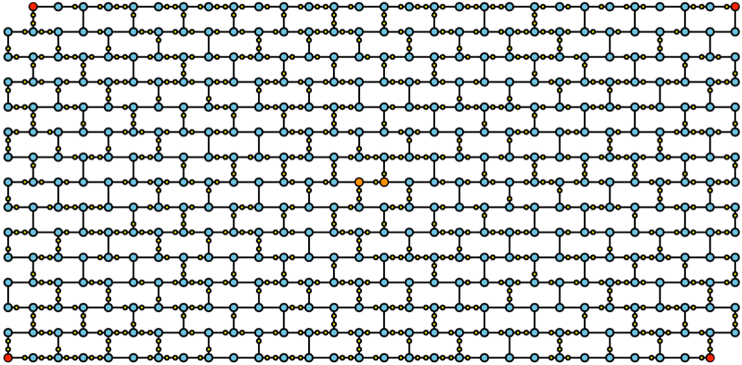
How to find an irrelevant vertex when the treewidth is large?

By using the Wall Exclusion Theorem!

How to find an irrelevant vertex when the treewidth is large?

Theorem (Robertson and Seymour. 1986)

For every integer $\ell > 0$, there is an integer $c(\ell)$ such that every graph of treewidth $\geq c(\ell)$ contains an ℓ-wall as a minor.



[Figure by Dimitrios M. Thilikos]

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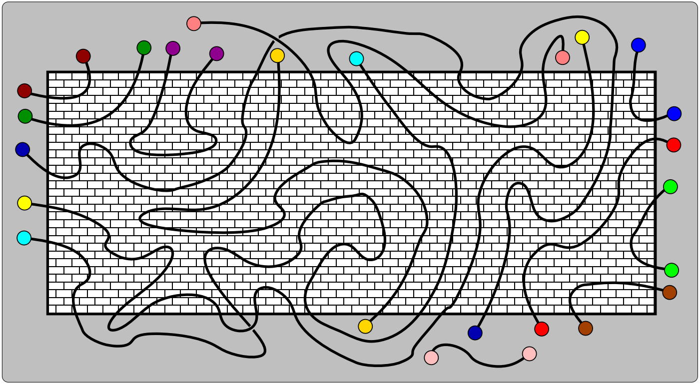
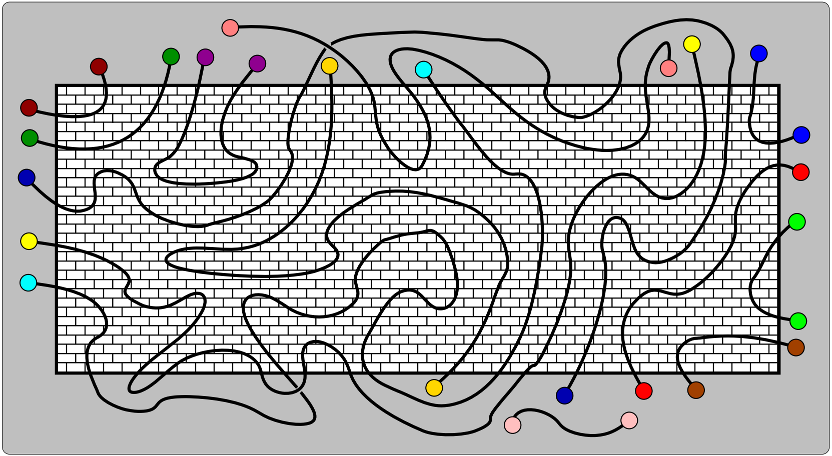


[Figure by Dimitrios M. Thilikos]

Goal: declare one of the central vertices of the wall irrelevant.



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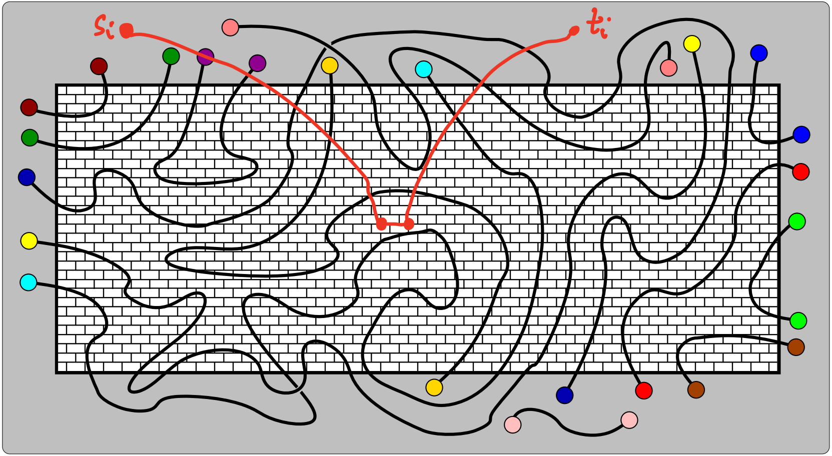
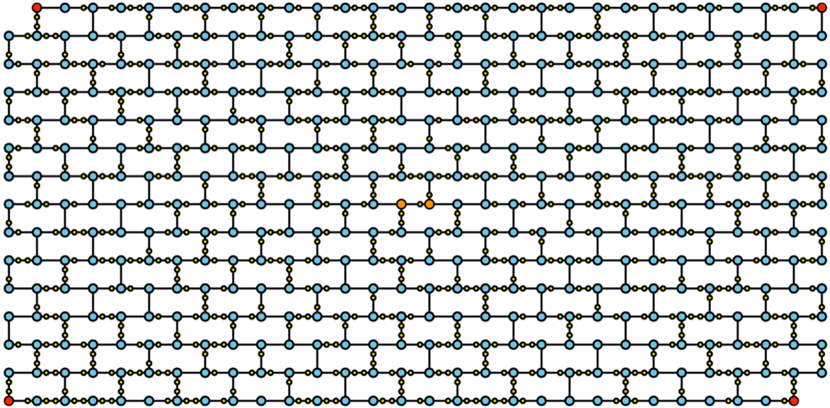


This is only possible if the wall is insulated from the exterior!

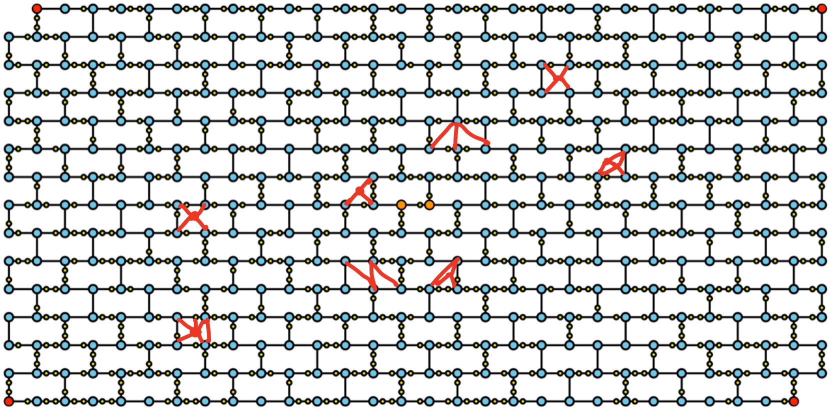
Flat walls

Goal: enrich the notion of wall so that we can insulate it from the exterior.



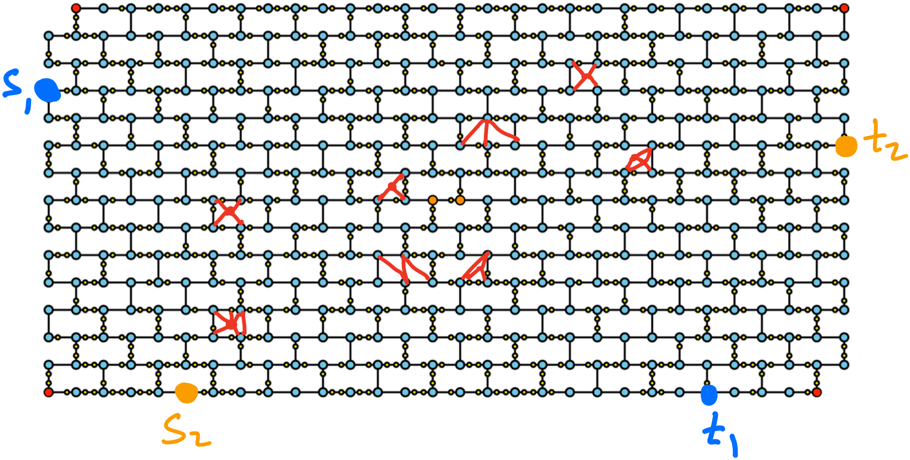
Flat walls

We need to allow some extra edges in the interior of the wall.



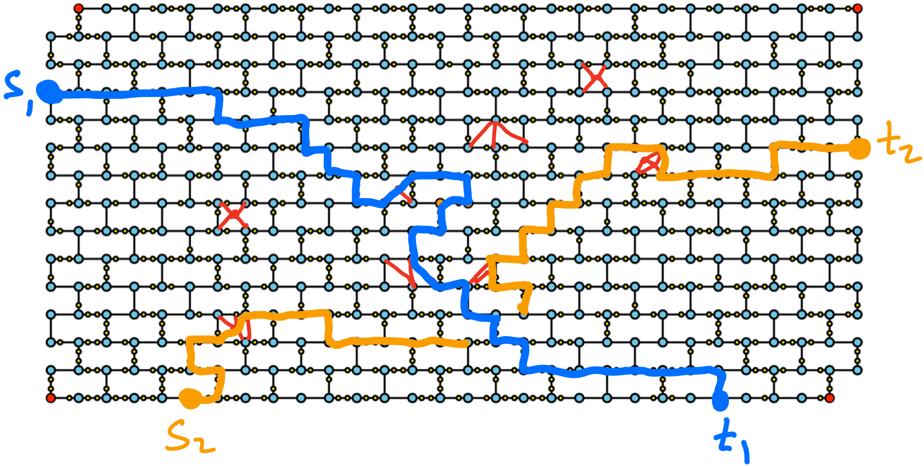
Flat walls

We impose a topological property that defines the "flatness" of the wall.



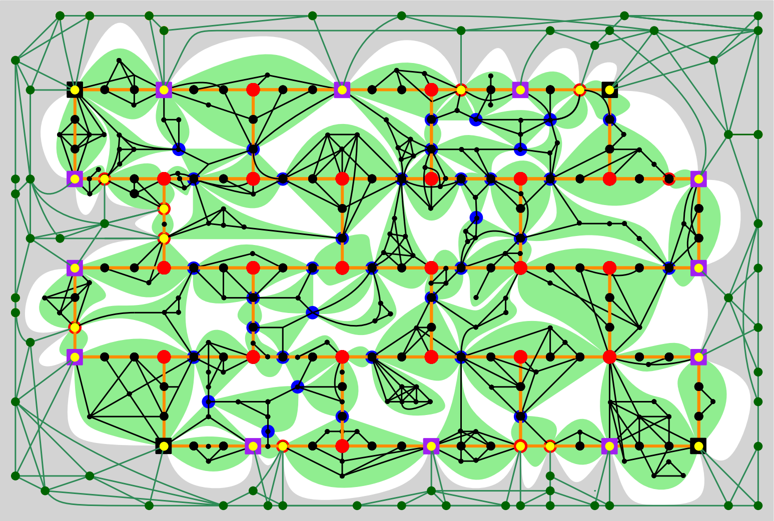
Flat walls

There are no crossing paths $s_1 - t_1$ and $s_2 - t_2$ from/to the perimeter.

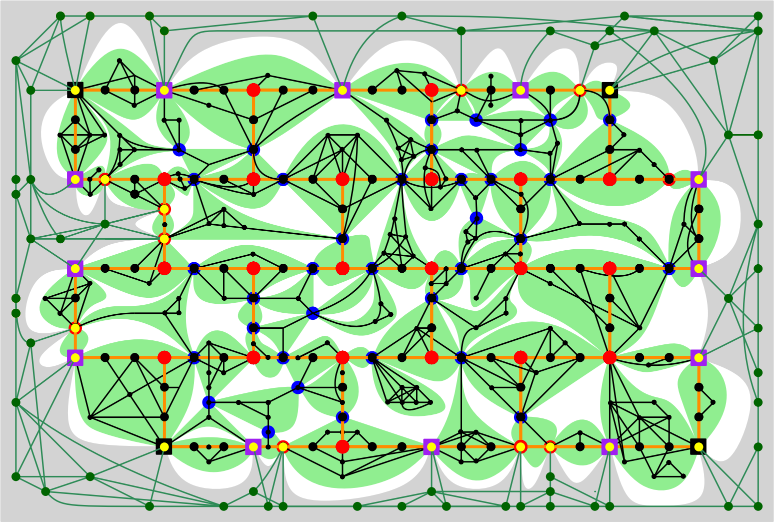


Flat walls

A real flat wall can be quite wild... [Figure by Dimitrios M. Thilikos]



Flat walls: a bit more formal



[Figures by Dimitrios M. Thilikos]

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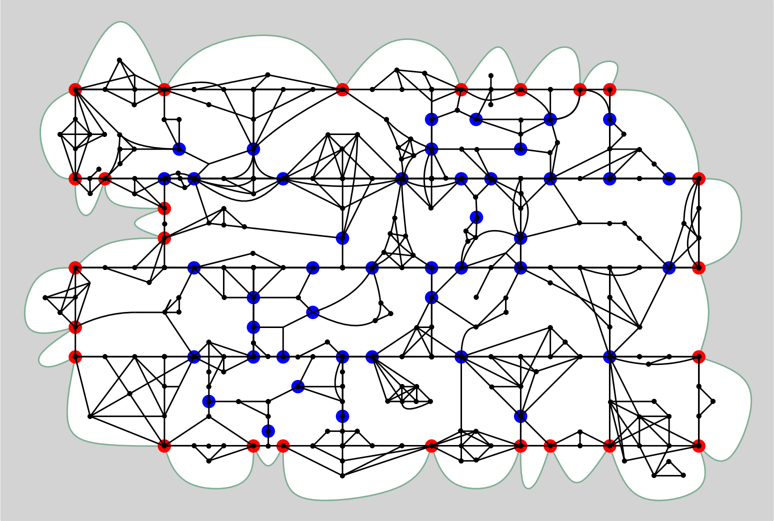

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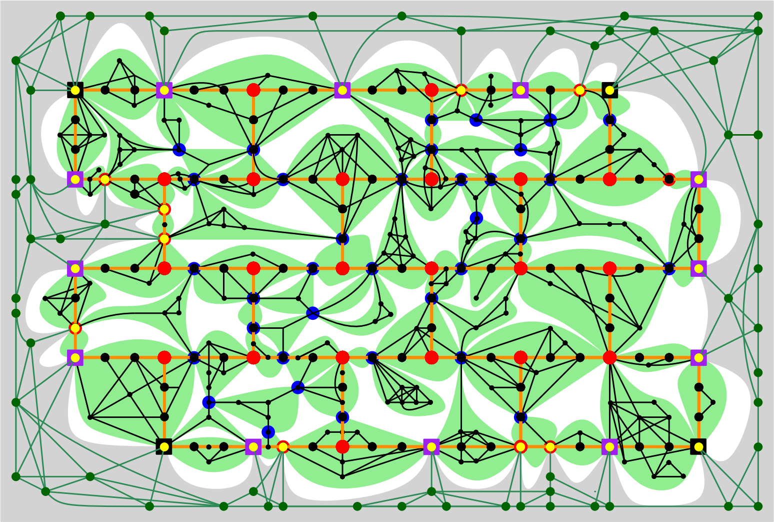

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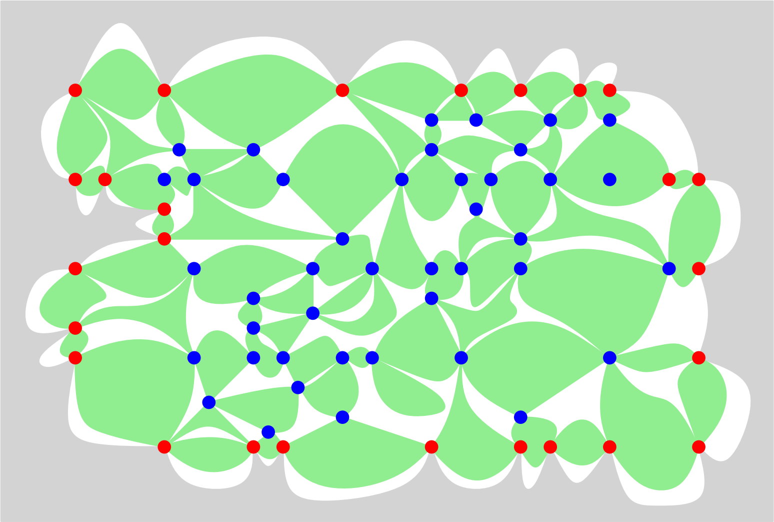

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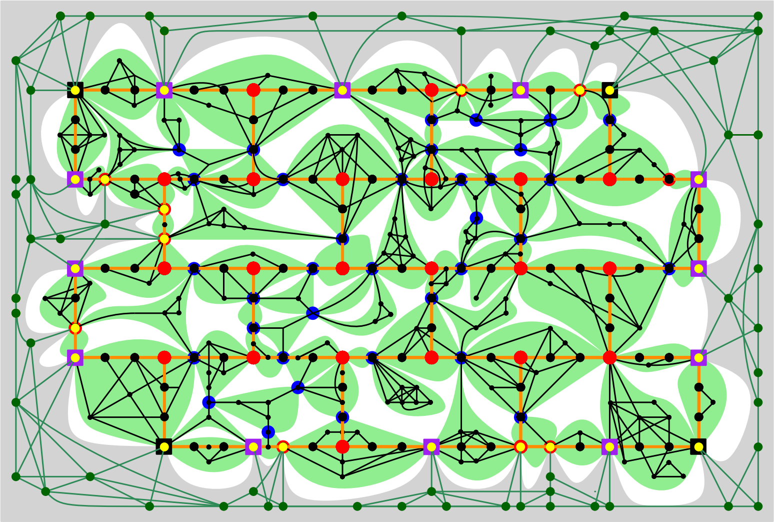

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The Weak Structure Graph Minors Theorem

Theorem (Robertson and Seymour. 1995)

There exist recursive functions $f_1 : \mathbb{N}^2 \to \mathbb{N}$ *and* $f_2 : \mathbb{N} \to \mathbb{N}$, *such that for every graph* G *and every* $q, r \in \mathbb{N}$, *one of the following holds:*

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There are many different variants and optimizations of this theorem...

[Chuzhoy. 2015]

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Important: possible to find one of the outputs in time $f(q, r) \cdot |V(G)|$.

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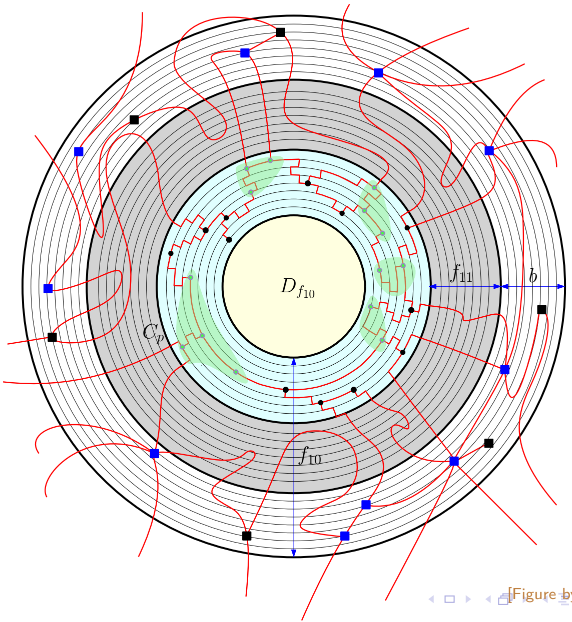
The irrelevant vertex technique has been applied to many problems... usually with a lot of technical pain.

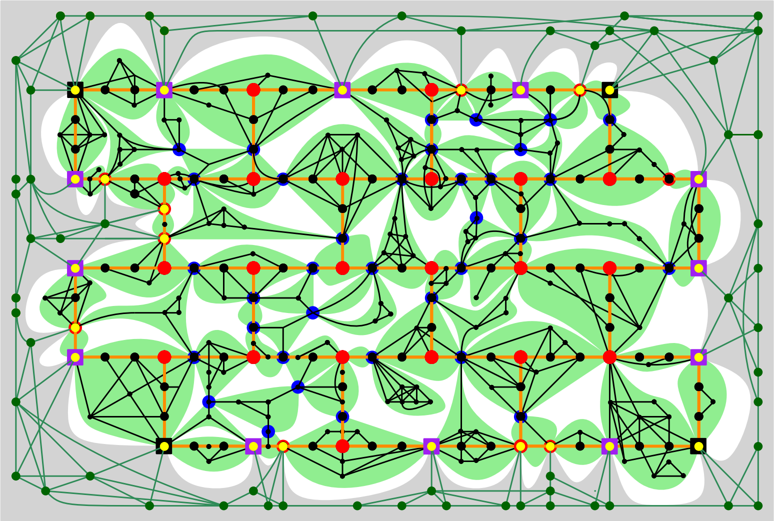
Rerouting inside a big flat wall...



[Figure by Dimitrios M. Thilikos]

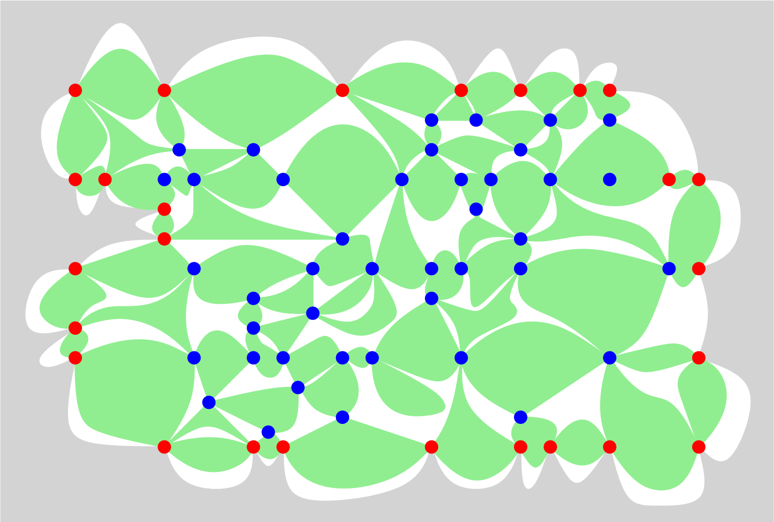
Crucial notion: homogeneity

In order to declare a vertex irrelevant for some problem, usually we need to consider a homogenous flat wall, which we proceed to define.



Crucial notion: homogeneity

We consider a flap-coloring encoding the relevant information of our favorite problem inside each flap (similar to tables of DP).



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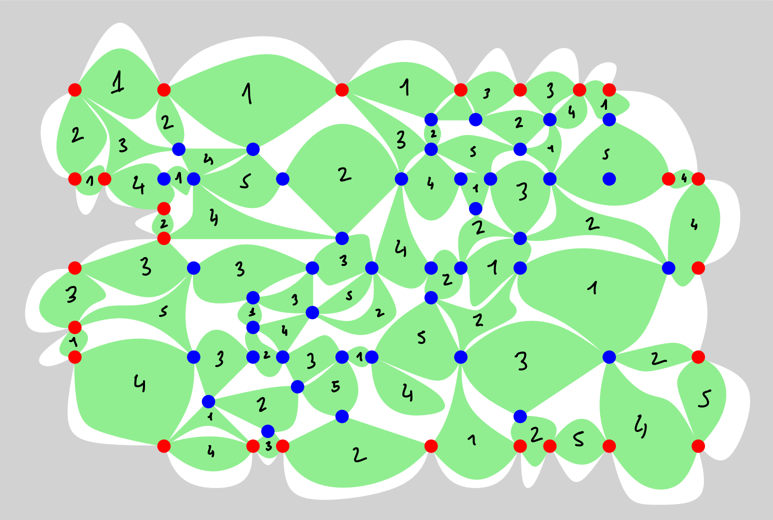
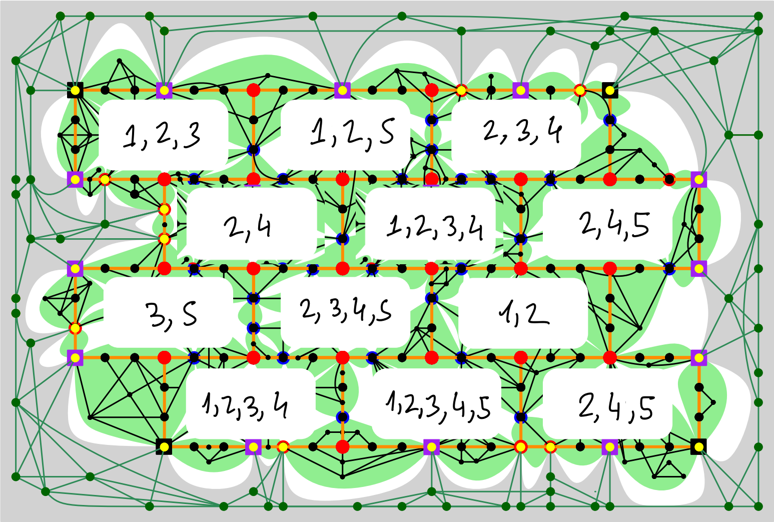


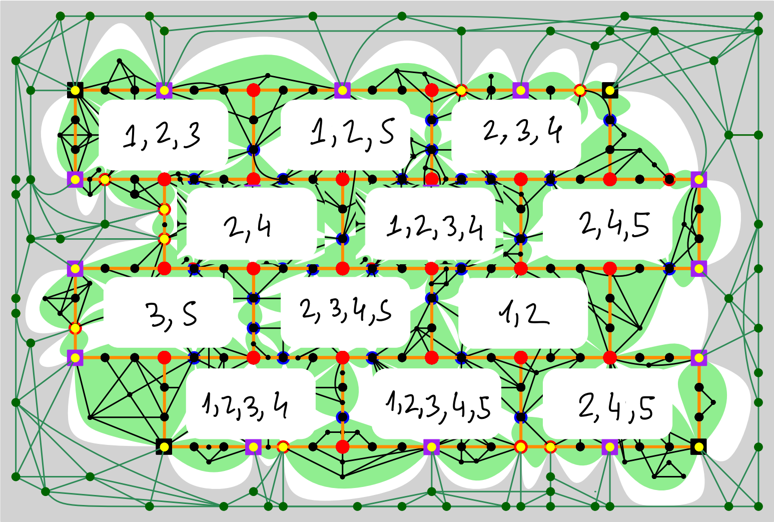
Crucial notion: homogeneity

For every brick of the wall, we define its palette as the colors appearing in the flaps it contains.



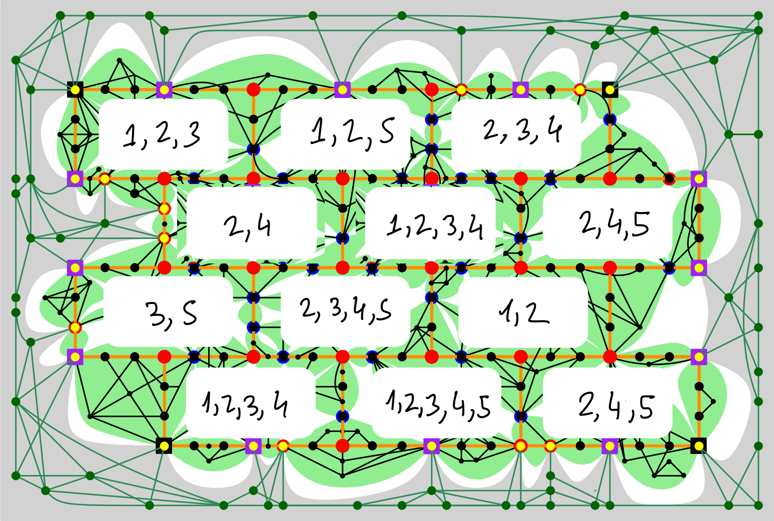
Crucial notion: homogeneity

A flat wall is homogenous if every (internal) brick has the same palette.
Fact: every brick of a homogenous flat wall has the same "behavior".



Crucial notion: homogeneity

Price of homogeneity to obtain a homogenous flat r-wall (zooming):
If we have c colors, we need to start with a flat r^c-wall. (why?)



Next section is...

Hitting forbidden minors

- If $\mathcal{C} = \{\text{edgeless graphs}\}$, then $\mathcal{F} = \{K_2\}$.
- If $\mathcal{C} = \{\text{forests}\}$, then $\mathcal{F} = \{K_3\}$.
- If $\mathcal{C} = \{\text{outerplanar graphs}\}$, then $\mathcal{F} = \{K_4, K_{2,3}\}$.
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$\mathcal{F}$-M-DELETION
Input: A graph G and an integer k.
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Let $\mathcal{F}$ be a fixed finite collection of graphs.

$\mathcal{F}$-M-DELETION

Input: A graph G and an integer k.

Question: Does G contain a set $S \subseteq V(G)$ with $|S| \leqslant k$ such that $G \setminus S$ does not contain any of the graphs in $\mathcal{F}$ as a minor?

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We consider the following two parameterizations of $\mathcal{F}$-M-DELETION:

1. Structural parameter: $\text{tw}(G)$.
2. Solution size: k.

Joint work with Dimitrios M. Thilikos, Julien Baste, Giannos Stamoulis, and Laure Morelle.

Next subsection is...

Theorem (Courcelle. 1990)

Every problem expressible in MSOL can be solved in time $f_{\mathcal{F}}(\mathrm{tw}) \cdot n$ on graphs on n vertices and treewidth at most tw.

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ETH: The 3-SAT problem on n variables cannot be solved in time $2^{o(n)}$.

[Impagliazzo, Paturi. 1999]

What was known for particular collections $\mathcal{F}$

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  [Jansen, Lokshtanov, Saurabh. 2014 + Pilipczuk. 2015]

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-DELETION on n-vertex graphs can be solved in time

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[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. I. General upper bounds**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. II. Single-exponential algorithms**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. III. Lower bounds**. 2020]

[Baste, S., Thilikos. **Hitting minors on bounded treewidth graphs. IV. An optimal algorithm**. 2021]

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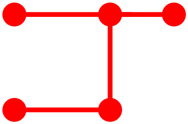
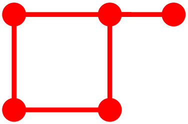
A dichotomy for hitting a connected minor





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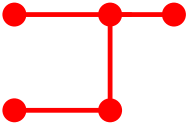

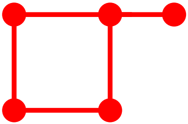

Theorem (Baste, S., Thilikos. 2016-2020)

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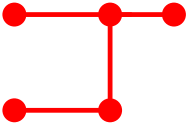

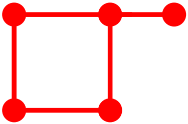

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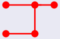
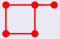
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_c$  *or* $H \leqslant_c$  .

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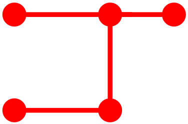

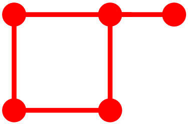

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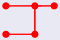
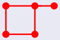
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_c$  *or* $H \leqslant_c$  .
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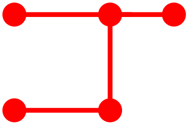 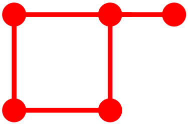

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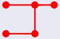
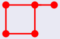
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \leqslant_{\mathsf{c}}$  *or* $H \leqslant_{\mathsf{c}}$  .
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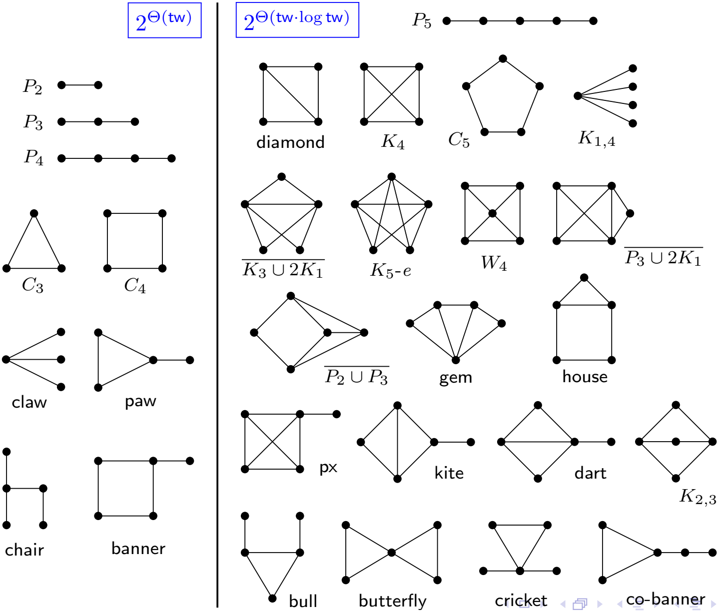
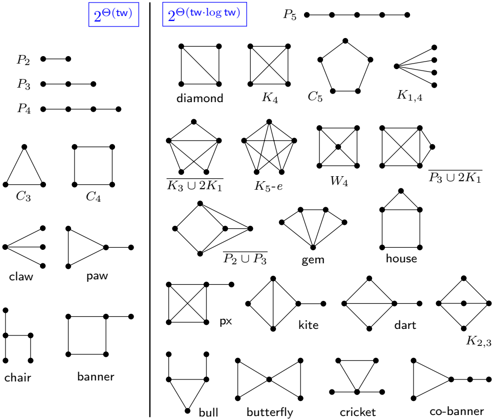
In both cases, the running time is asymptotically optimal under the ETH.

skip

Complexity of hitting a single connected minor H



A compact statement for a single connected graph



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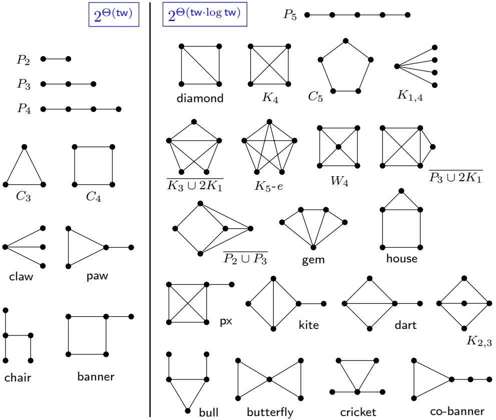

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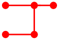
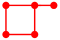
- All graphs on the left are contractions of  or 

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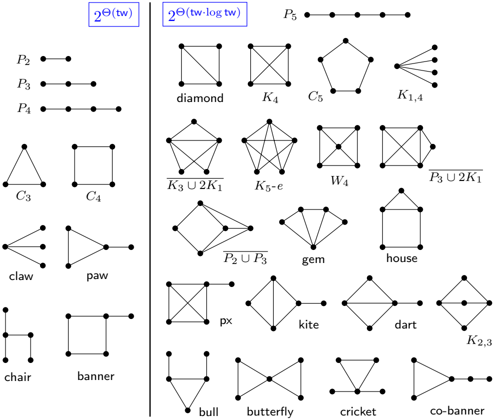

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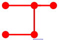
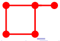
- All graphs on the right are not contractions of  or 

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1. General algorithms
   - For every $\mathcal{F}$: time $2^{2^{\mathcal{O}(\mathrm{tw}\cdot\log \mathrm{tw})}} \cdot n^{\mathcal{O}(1)}$.
   - $\mathcal{F}$ planar: time $2^{\mathcal{O}(\mathrm{tw}\cdot\log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$.
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2. Ad-hoc single-exponential algorithms
   - Some use "typical" dynamic programming.
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1. General algorithms
   - For every $\mathcal{F}$: time $2^{2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})}} \cdot n^{\mathcal{O}(1)}$.
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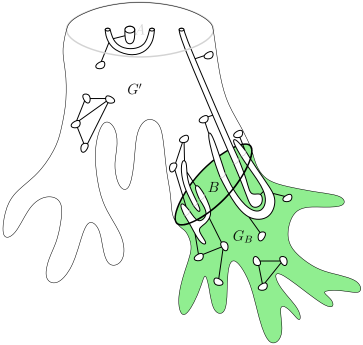
[Fig. by Valentin Garnero]



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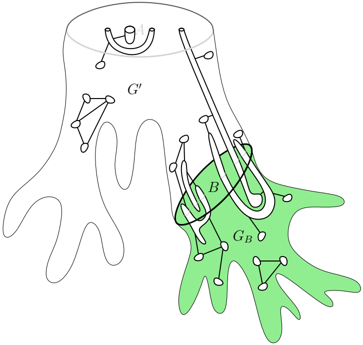

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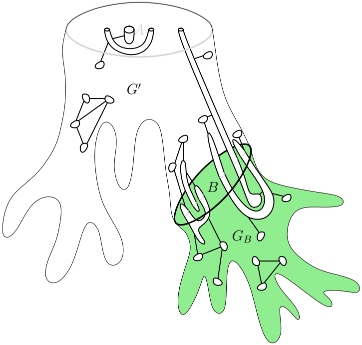

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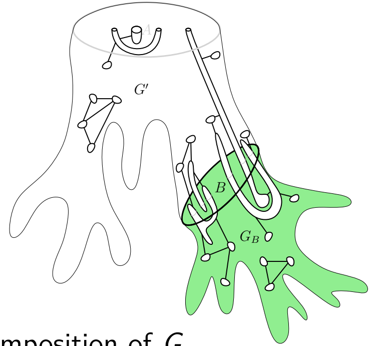

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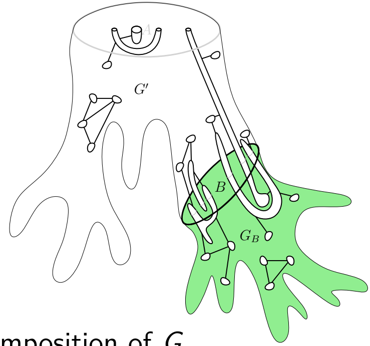

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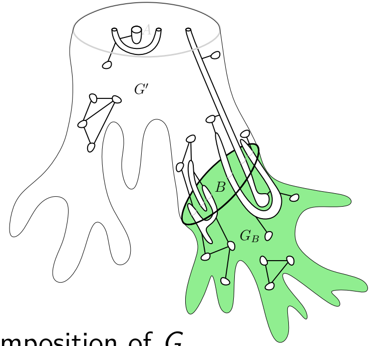

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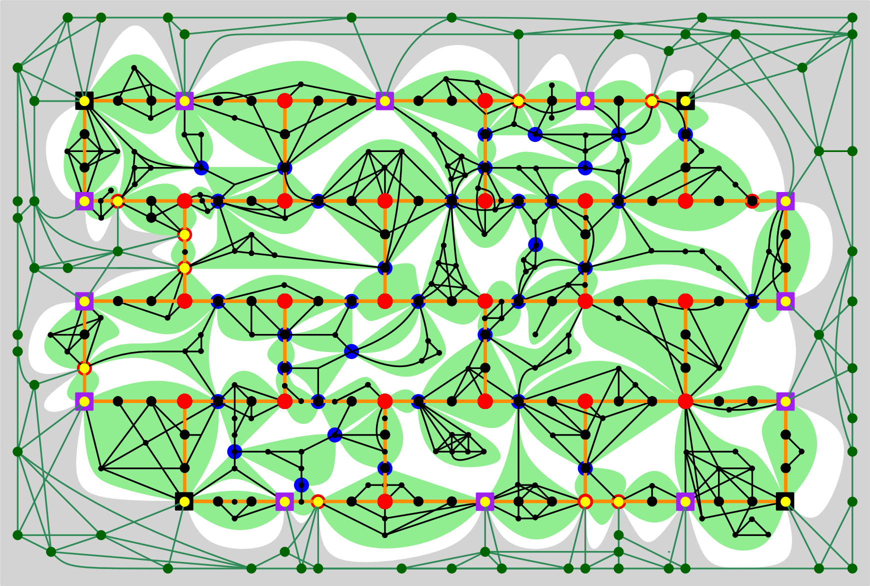
As we know, a flat wall can be quite wild...



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[Figure by Dimitrios M. Thilikos]

Hard part: finding an irrelevant vertex inside a flat wall

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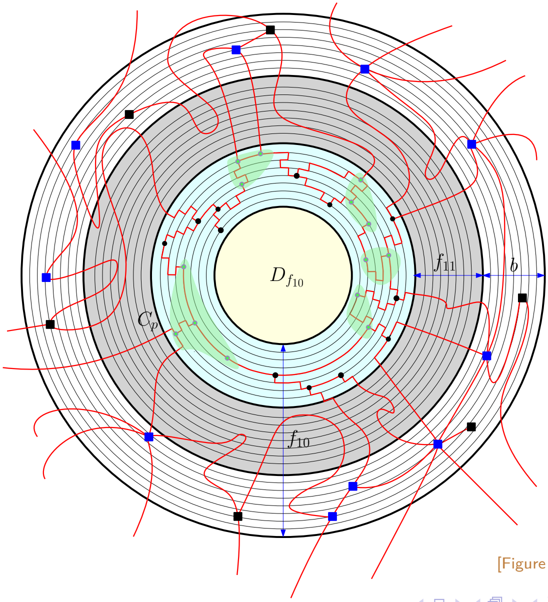

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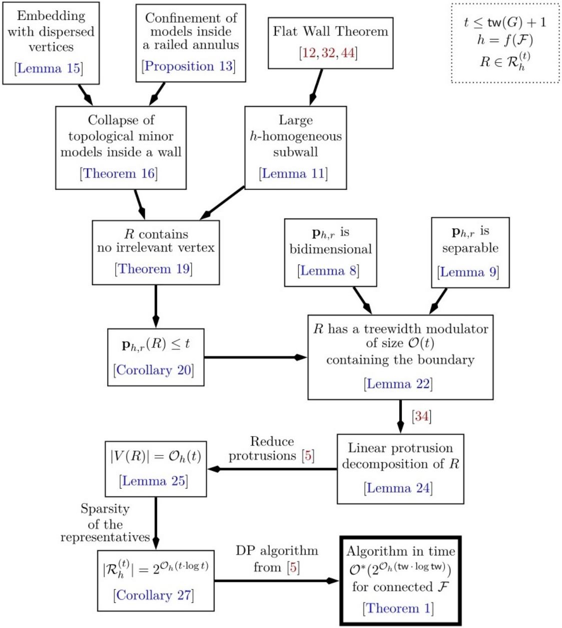
Diagram of the algorithm for a general collection $\mathcal{F}$



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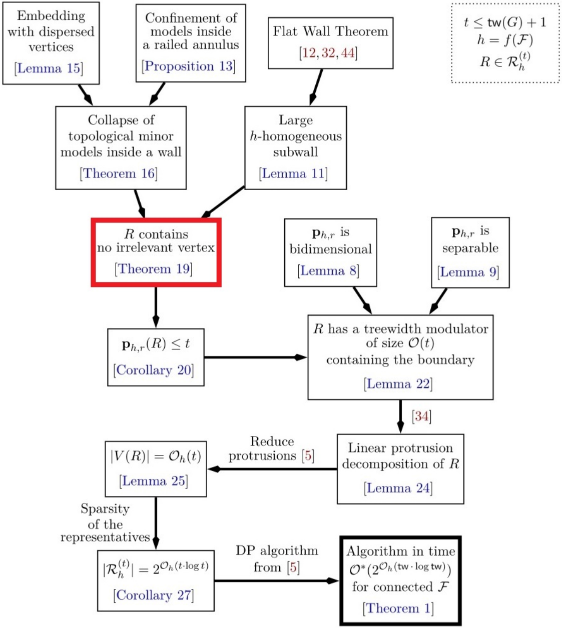

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Next subsection is...

We parameterize by the size of the desired solution

$\mathcal{F}$-M-DELETION

Input: A graph G and an integer k.

Parameter: k.

Question: Does G contain a set $S \subseteq V(G)$ with $|S| \leqslant k$ such that $G \setminus S$ does not contain any of the graphs in $\mathcal{F}$ as a minor?

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$$\mathcal{C}_k = \{G \mid (G, k) \text{ is a positive instance of } \mathcal{F}\text{-M-Deletion}\}$$

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But... only existential, non-uniform, $f(\mathcal{C}_k)$ astronomical.

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Our results

Theorem (S., Stamoulis, Thilikos. 2020)

For all $\mathcal{F}$, the $\mathcal{F}$-M-Deletion problem can be solved in time $2^{\text{poly}(k)} \cdot n^3$.

Here, $\text{poly}(k)$ is a polynomial whose degree depends on $\mathcal{F}$.

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skip

General scheme of the algorithm:

[whole slide shamelessly borrowed from Giannos Stamoulis]

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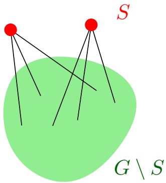

Iterative compression: given solution S of size $k+1$, search solution of size k.

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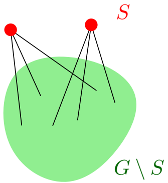

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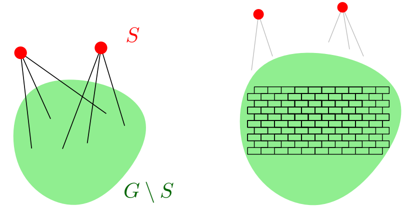

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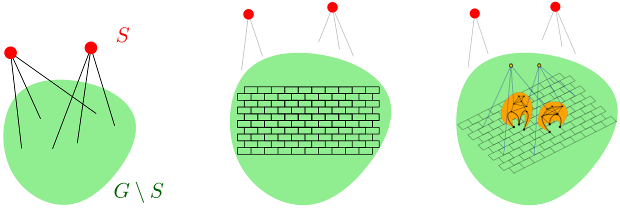

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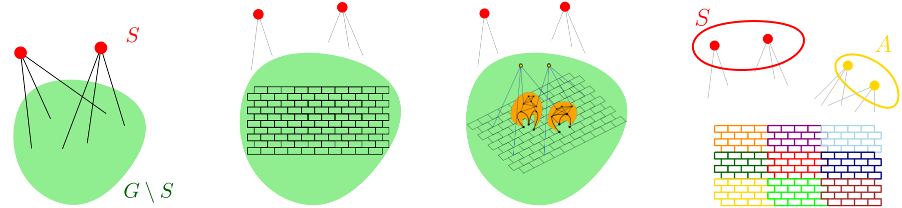

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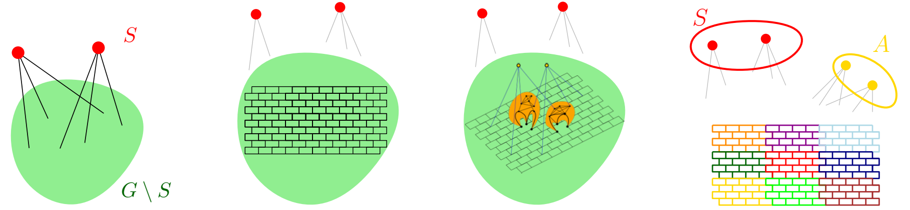

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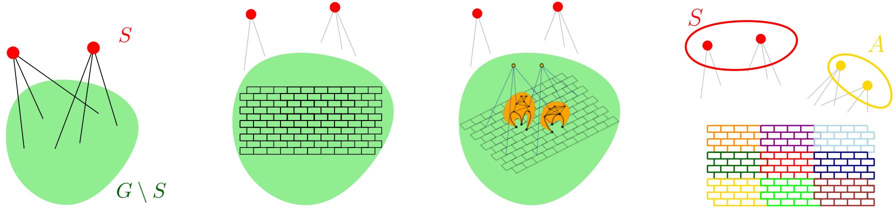

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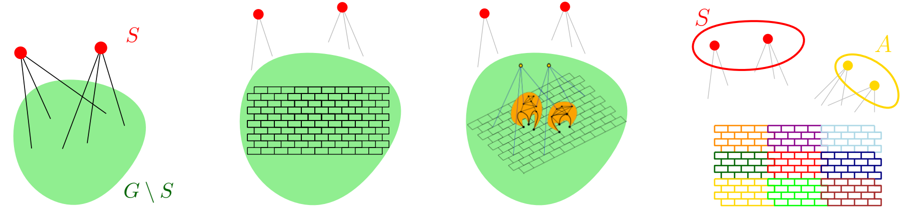
3. Find in W a packing of $\mathcal{O}_{\mathcal{F}}(k^4)$ disjoint "large" subwalls:
   - If every subwall has at least $|A|+1$ neighbors in $S \cup A$:
     Every solution intersects $S \cup A$ $\rightarrow$ we can branch!

General scheme of the algorithm:

[whole slide shamelessly borrowed from Giannos Stamoulis]



Iterative compression: given solution S of size $k+1$, search solution of size k.

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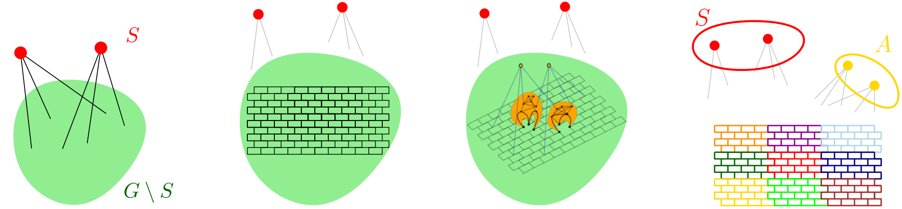

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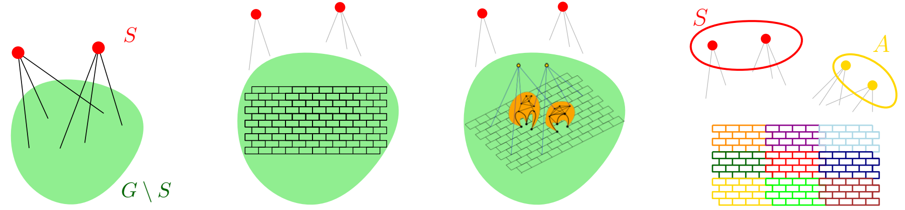

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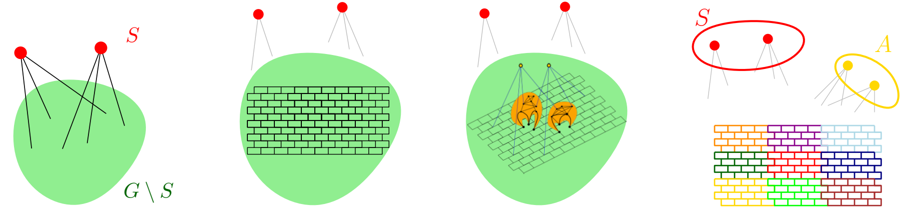

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Thus, $\mathrm{tw}(G \setminus S) = k^{\mathcal{O}_{\mathcal{F}}(1)}$: our previous FPT algo gives $2^{k^{\mathcal{O}_{\mathcal{F}}(1)}} \cdot n^2$.

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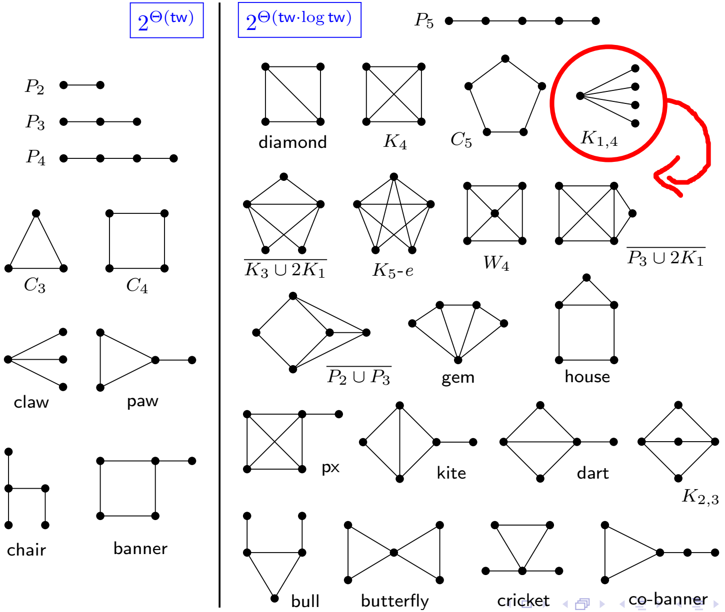
Is the price of homogeneity unavoidable?

skip

For topological minors, there is (at least) one change



Gràcies!