Finding subdivisions of spindles on digraphs

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[arXiv 1706.09066]

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Outline of the talk

Next section is...

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In this talk we focus on directed graphs, or digraphs.

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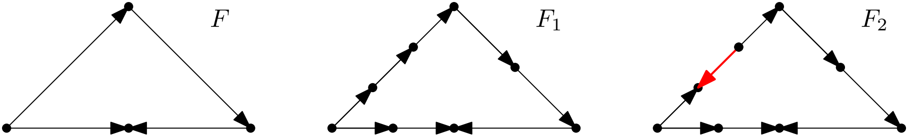

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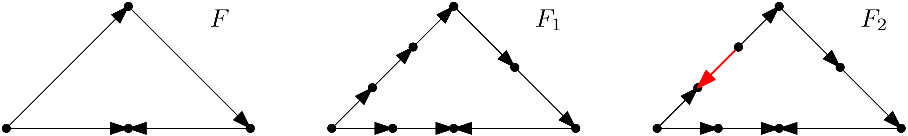

We are interested in the following problem:

Digraph Subdivision	
Instance:	Two digraphs G and F.
Question:	Does G contain a subdivision of F as a subdigraph?

Recent work on finding digraph subdivisions

This problem has been introduced by [Bang-Jensen, Havet, Maia. 2015]

Let F be a fixed digraph.

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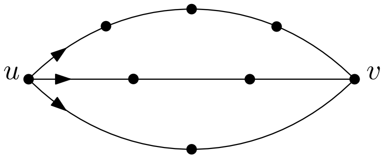
When $|V(F)| = 4$, there are only 5 open cases. [Havet, Maia, Mohar. 2017]

We focus on finding subdivisions of spindles



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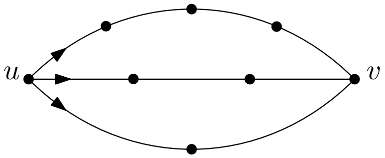

For k positive integers $\ell_1, \ldots, \ell_k$, a $(\ell_1, \ldots, \ell_k)$-spindle is the digraph containing k paths $P_1, \ldots, P_k$ from a vertex u to a vertex v, such that $|E(P_i)| = \ell_i$ for $1 \leq i \leq k$ and $V(P_i) \cap V(P_j) = \{u, v\}$ for $1 \leq i \neq j \leq k$.

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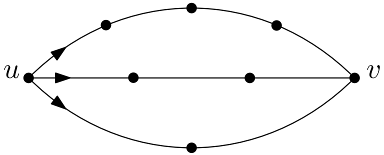

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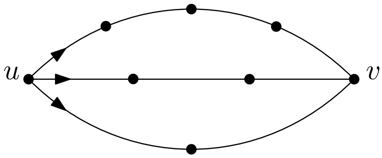

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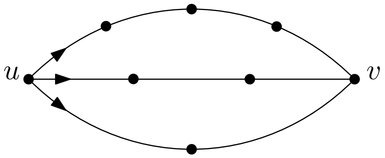

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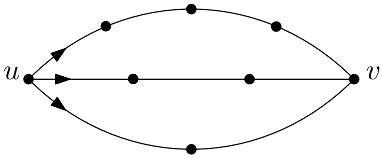

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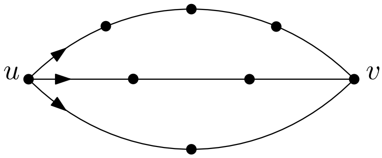

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G contains a subdivision of a $(1, \ell)$-spindle $\iff$
the length of a Longest Path in G is at least ℓ.

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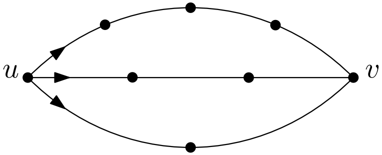

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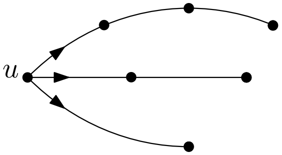

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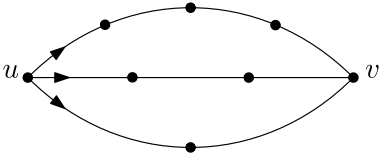

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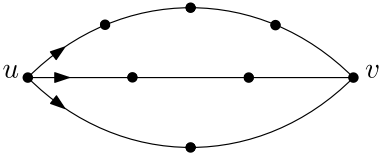

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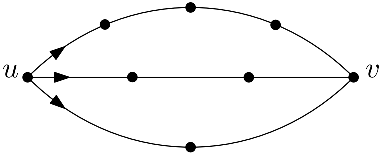

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Is a running time $\boxed{f(|V(F)|) \cdot n^{O(1)}}$ possible, for some function f?

This question had been asked by [Bang-Jensen, Havet, Maia. 2015]

Parameterized complexity in one slide

- The area of parameterized complexity was introduced in the 90's by Downey and Fellows.

- $\boxed{\text{Idea}}$ given an NP-hard problem with input size n, fix one parameter k of the input to see whether the problem gets more "tractable".

  Example: $k =$ length of a LONGEST PATH.

- Given a (NP-hard) problem with input of size n and a parameter k, a fixed-parameter tractable (FPT) algorithm runs in time

$$f(k) \cdot n^{O(1)}, \quad \text{for some function } f.$$

  Examples: k-VERTEX COVER, k-LONGEST PATH.

Next section is...

Our results (I): optimization problems

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Our results (II): FPT algorithms for finding 2-spindles

2-spindle: spindle with exactly two paths.

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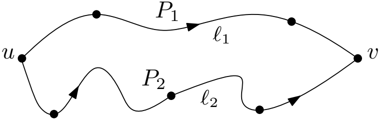

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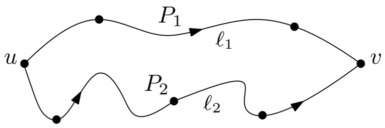

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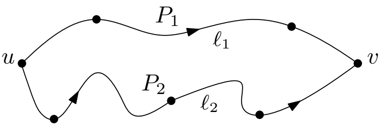

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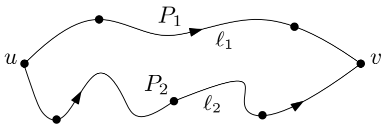

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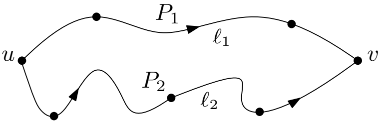

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Both FPT algorithms are asymptotically optimal under the ETH.

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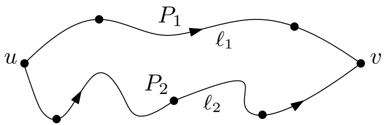

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ETH: $\nexists$ algorithm solving 3-SAT on a formula with n variables in time $2^{o(n)}$.

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NP-hardness reduction

Max $(\bullet \times \ell)$-Spindle Subdivision
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We prove the case $\ell = 4$, by reduction from 3-DIMENSIONAL MATCHING:

Given three sets A, B, C of the same size and a set of triples $\mathcal{T} \subseteq A \times B \times C$, decide whether there exists a set $\mathcal{T}' \subseteq \mathcal{T}$ of pairwise disjoint triples with $|\mathcal{T}'| = |A|$.

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Our reduction is strongly inspired by [Brewster, Hell, Pantel, Rizzi, Yeo. 2003]

Reduction for $\ell = 4$

Given $(A, B, C, \mathcal{T})$ of 3-Dimensional Matching, with $|A| = n$ and $\mathcal{T} = m$, we construct G of Max $(\bullet \times \ell)$-Spindle Subdivision as follows:

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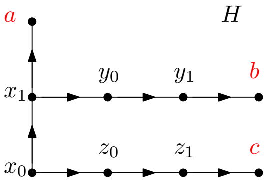

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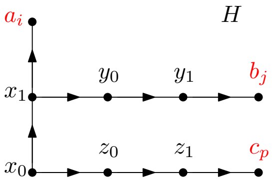

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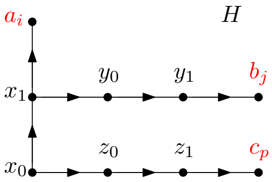

We add a new vertex s (source) and a vertex t (sink) that we connect to every other vertex. They will be the endpoints of the desired spindle.

Reduction for $\ell = 4$

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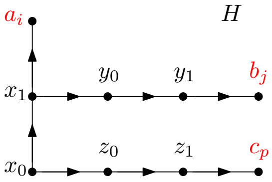

Claim $(A, B, C, \mathcal{T})$ is a Yes-instance of 3-Dim. Matching $\iff$ G contains a subdivision of a $(n + 2m \times 4)$-spindle.

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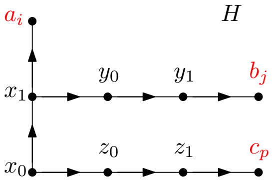

By construction of G, a $(n + 2m \times 4)$-spindle covers all $V(G)$, so it is equivalent to partitioning $G \setminus \{s, t\}$ into 2-paths (paths with 2 arcs).

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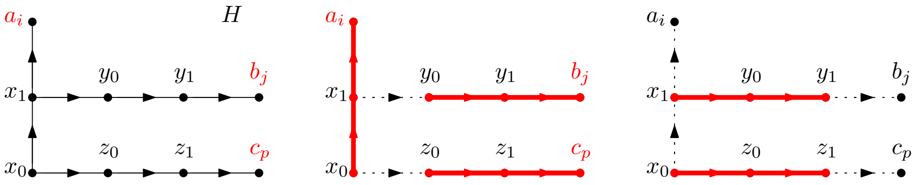

Key property: for every copy of H, there are exactly two ways the 2-paths can intersect H. This defines whether each triple $T \in \mathcal{T}$ is chosen or not.

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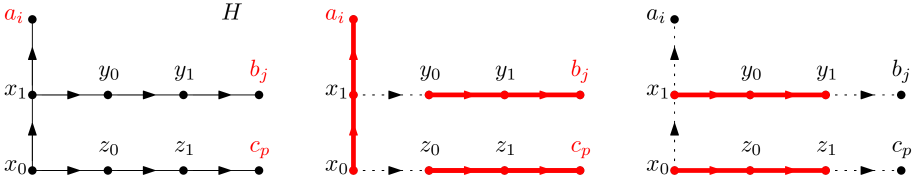

Modification for $\ell > 4$: we define the digraph G in the same way, except that we subdivide the arcs outgoing from s exactly $\ell - 4$ times.

Next section is...

Cases that can be solved in polynomial time

MAX $(\bullet \times \ell)$-SPINDLE SUBDIVISION
For a fixed $\ell \geq 1$, given an input digraph G, find the largest k such that G contains a subdivision of a $(k \times \ell)$-spindle.

Theorem

Let $\ell \geq 1$ be fixed. MAX $(\bullet \times \ell)$-SPINDLE SUBDIVISION *is in* $\boxed{\text{P } \textit{if } \ell \leq 3}$, *and* NP-*hard if* $\ell \geq 4$, *even restricted to DAGs.*

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- $\ell = 1$: can be solved by a flow algorithm.
- $\ell = 2$: guess two vertices, delete arcs between them, and then flow.
- $\ell = 3$: we reduce the problem to computing a maximum matching in an auxiliary undirected graph, as follows...

Idea of the case $\ell = 3$

A directed path P is nontrivial if its endpoints are distinct.

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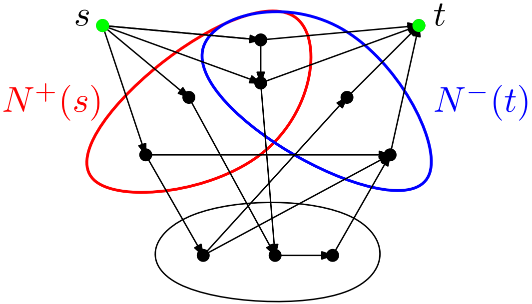

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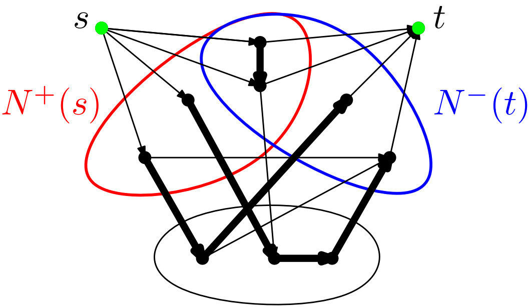

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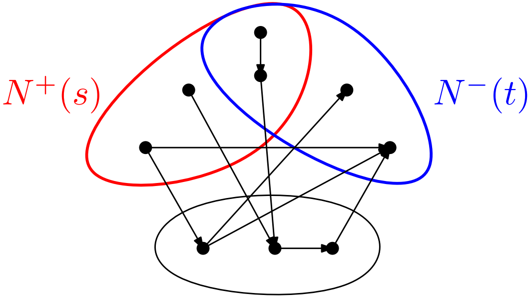

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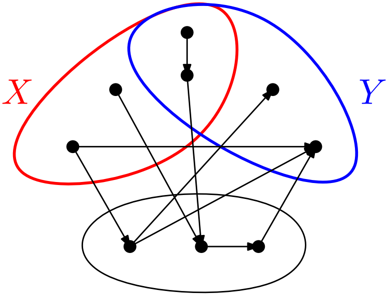
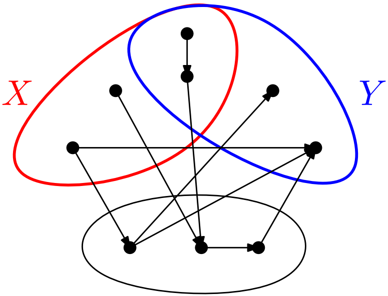


Main ingredient for the case $\ell = 3$

Proposition

Let G be a digraph and $X, Y \subseteq V(G)$. The maximum number of vertex-disjoint directed nontrivial paths from X to Y can be computed in polynomial time.



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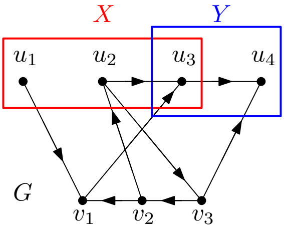
Given a digraph G and $X, Y \subseteq V(G)$, we build an undirected graph G':



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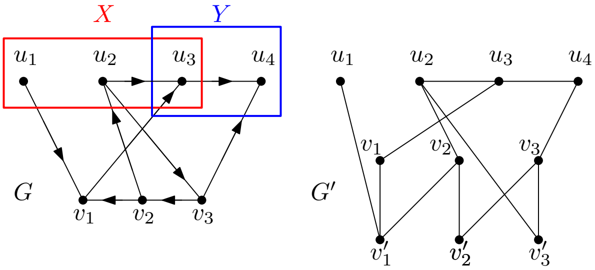

$V(G') = V(G)$ + a copy v' of each vertex $v \notin X \cup Y$.

$E(G')$: For each $v \notin X \cup Y$, add to G' the edge $\{v, v'\}$.
For each (u, v), add $\{u, v\}$ if $v \in X \cup Y$, and $\{u, v'\}$ otherwise.

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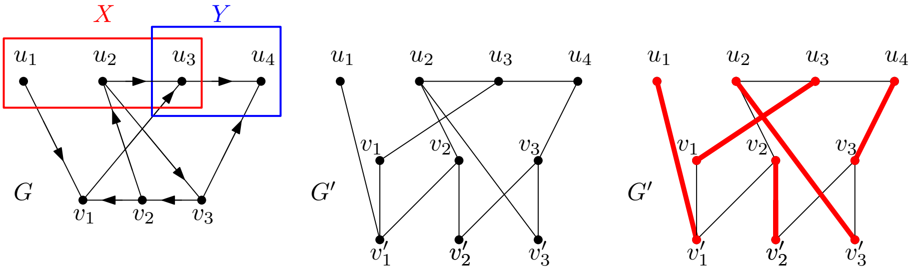

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$\boxed{\text{Claim}}$ G contains k vertex-disjoint directed nontrivial paths from X to Y
$\iff$ G' has a matching of size $k + |V(G) \setminus (X \cup Y)|$.

Next section is...

Matroids

A pair $\mathcal{M} = (E, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets of E, is a matroid if it satisfies the following three axioms:

1. $\emptyset \in \mathcal{I}$.
2. If $A' \subseteq A$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$.
3. If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called the independent sets of the matroid.

Representative sets in matroids

Two independent sets A, B of $\mathcal{M}$ fit if $A \cap B = \emptyset$ and $A \cup B$ is independent.

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Let $\mathcal{A}$ be a family of sets of size p in a matroid $\mathcal{M}$. A subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is said to q-represent $\mathcal{A}$, denoted $\mathcal{A}' \subseteq^q_{\text{rep}} \mathcal{A}$, if for every set B of size q such that there is an $A \in \mathcal{A}$ that fits B, there is an $A' \in \mathcal{A}'$ that also fits B.

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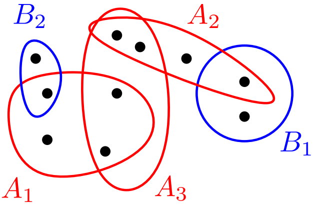

$$\mathcal{A} = \{A_1, A_2, A_3\}, \quad p = 4, q = 2$$
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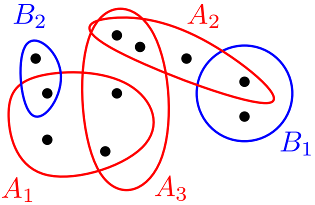

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We consider the uniform matroid with ground set $V(G)$ and rank $\ell + q$, with $0 \leq q \leq 2\ell$.

Finding a 2-spindle of large total size

If a subdigraph S of G is a subdivision of a (ℓ_1, ℓ_2)-spindle, with $\min\{\ell_1, \ell_2\} \geq 1$ and $\ell_1 + \ell_2 = \ell$, we say that S is a good spindle.

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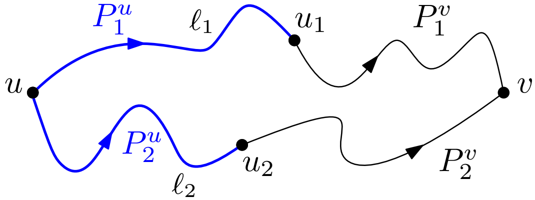
$\boxed{\text{Idea}}$ We will q-represent the "first part" of the desired spindle (paths P_u^1 and P_u^2), for every $u, u_1, u_2 \in V(G)$, $\ell_1, \ell_2 \leq \ell$, and $0 \leq q \leq 2\ell$.



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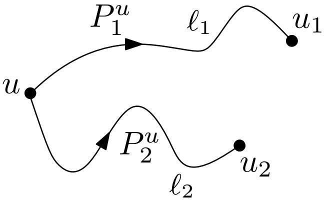

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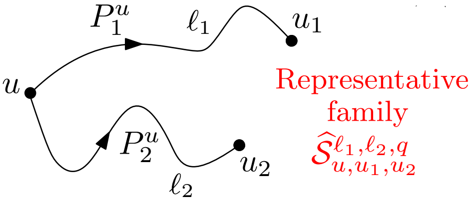

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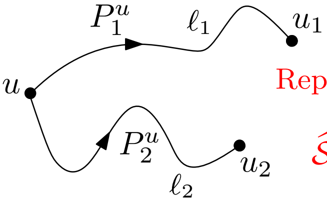


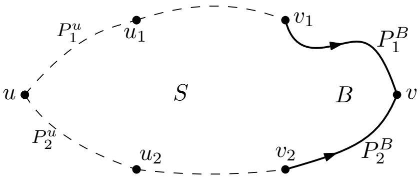
Representative family $\widehat{\mathcal{S}}^{\ell_1,\ell_2,q}_{u,u_1,u_2}$

Using the recent techniques of [Fomin, Lokshtanov, Panolan, Saurabh. 2016], $|\widehat{\mathcal{S}}^{\ell_1,\ell_2,q}_{u,u_1,u_2}| = 2^{O(\ell)}$ and can be computed in time $2^{O(\ell)} \cdot n^{O(1)}$.

Key property: these families indeed represent the solutions

Consider a good spindle S with minimum number of vertices:

$$|B| = \ell - 1$$



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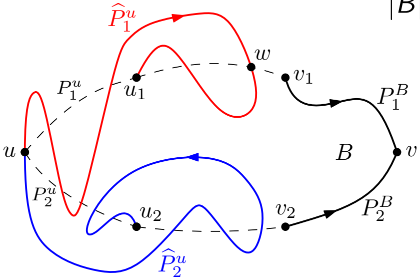

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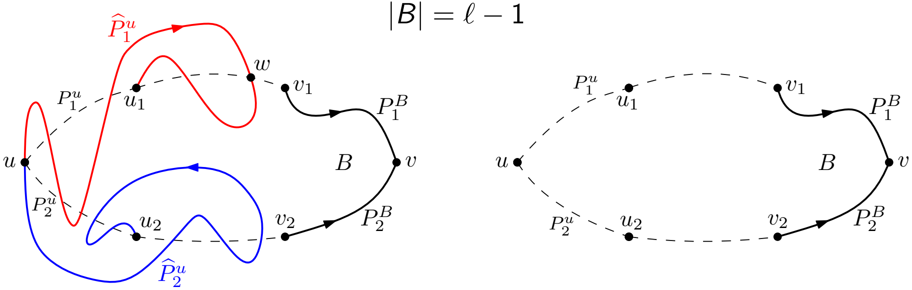

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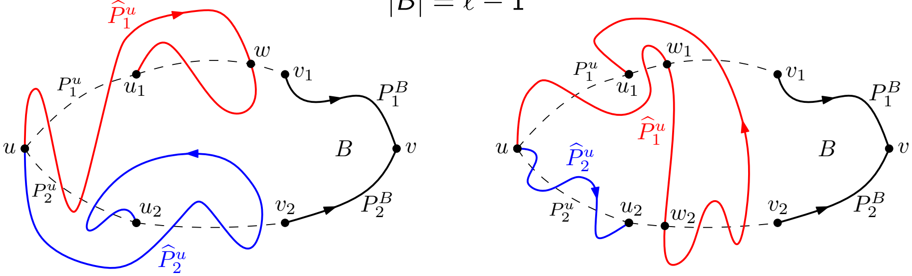

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Wrapping up the algorithm

1. For every $u, u_1, u_2 \in V(G)$, $\ell_1, \ell_2 \leq \ell$, and $0 \leq q \leq 2\ell$, we compute a q-representative family $\widehat{\mathcal{S}}^{\ell_1,\ell_2,q}_{u,u_1,u_2}$ in time $2^{O(\ell)} \cdot n^{O(1)}$.

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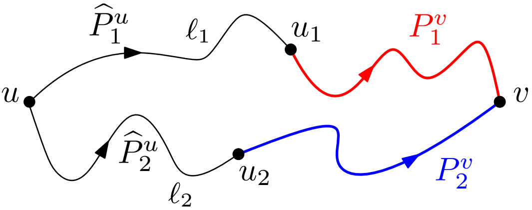

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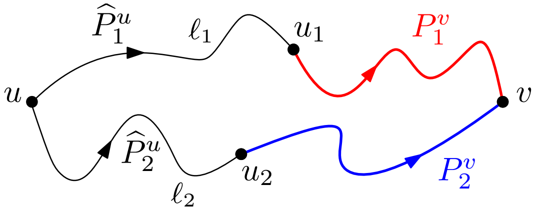

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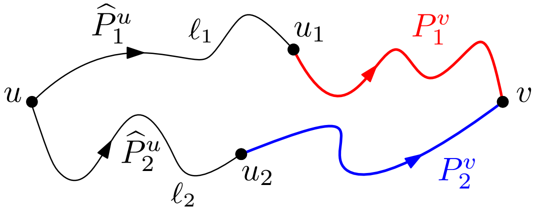

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Overall running time: $2^{O(\ell)} \cdot n^{O(1)}$.

Next section is...

Further research

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Theorem

Given an acyclic digraph G and integers k, ℓ, deciding whether G contains a subdivision of a $(k \times \ell)$-spindle can be solved in time $O(\ell^k \cdot n^{2k+1})$.

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If k is a constant: the problem is polynomial on acyclic digraphs (this generalizes the case $k = 1$, that is, Longest Path on DAGs).

Further research

Main open question:

Finding a subdivision of a spindle F is FPT parameterized by $|V(F)|$?

We do not know the answer even if F is a 2-spindle.

When G is an acyclic digraph, we can prove the following:

Theorem

Given an acyclic digraph G and integers k, ℓ, deciding whether G contains a subdivision of a $(k \times \ell)$-spindle can be solved in time $O(\ell^k \cdot n^{2k+1})$.

If k is a constant: the problem is polynomial on acyclic digraphs (this generalizes the case $k = 1$, that is, Longest Path on DAGs).

But is the problem FPT on acyclic digraphs? That is, in time $f(k, \ell) \cdot n^{O(1)}$?

Ça fait déjà 10 jours que ces deux personnes sont en prison à Madrid: