Finding subdivisions of spindles on digraphs

Júlio Araújo\textsuperscript{1}  Victor A. Campos\textsuperscript{2}  Ana Karolinna Maia\textsuperscript{2}  Ignasi Sau\textsuperscript{1,3}  Ana Silva\textsuperscript{1}

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\textsuperscript{1} Departamento de Matemática, UFC, Fortaleza, Brazil.
\textsuperscript{2} Departamento de Computação, UFC, Fortaleza, Brazil.
\textsuperscript{3} CNRS, LIRMM, Université de Montpellier, Montpellier, France.
Outline of the talk

1. Introduction
2. Our results
3. NP-hardness reduction
4. Polynomial-time algorithm
5. Sketch of the FPT algorithms
6. Conclusions
Introduction

Our results

NP-hardness reduction

Polynomial-time algorithm

Sketch of the FPT algorithms

Conclusions
In this talk we focus on directed graphs, or digraphs.
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A subdivision of a digraph $F$ is a digraph obtained from $F$ by replacing each arc $(u, v)$ of $F$ by a directed $(u, v)$-path.
Digraph subdivisions

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A subdivision of a digraph $F$ is a digraph obtained from $F$ by replacing each arc $(u, v)$ of $F$ by a directed $(u, v)$-path.

We are interested in the following problem:

**Digraph Subdivision**

**Instance:** Two digraphs $G$ and $F$.

**Question:** Does $G$ contain a subdivision of $F$ as a subdigraph?
Recent work on finding digraph subdivisions

This problem has been introduced by [Bang-Jensen, Havet, Maia. 2015]

Let $F$ be a fixed digraph.

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Conjecture (Bang-Jensen, Havet, Maia. 2015)

For every fixed digraph $F$, $F$-Subdivision is either in $P$ or $NP$-complete.

This conjecture is wide open, and only examples of both cases are known.

When $|V(F)| = 4$, there are only 5 open cases. [Havet, Maia, Mohar. 2017]
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We focus on finding subdivisions of spindles.

For positive integers $\ell_1, \ldots, \ell_k$, a $(\ell_1, \ldots, \ell_k)$-spindle is the digraph containing $k$ paths $P_1, \ldots, P_k$ from a vertex $u$ to a vertex $v$, such that $|E(P_i)| = \ell_i$ for $1 \leq i \leq k$ and $V(P_i) \setminus V(P_j) = \{u, v\}$ for $1 \leq i \neq j \leq k$.

If $\ell_i = \ell$ for $1 \leq i \leq k$, a $(\ell_1, \ldots, \ell_k)$-spindle is also called a $(k \times \ell)$-spindle.

$G$ contains a subdivision of a $(k, 1)$-spindle $\iff \exists u, v \in V(G)$: the Maximum Flow from $u$ to $v$ is at least $k$.

$G$ contains a subdivision of a $(1, \ell)$-spindle $\iff$ the length of a Longest Path in $G$ is at least $\ell$. 
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If \( \ell_i = \ell \) for \( 1 \leq i \leq k \), a \((\ell_1, \ldots, \ell_k)\)-spindle is also called a \((k \times \ell)\)-spindle.

\( G \) contains a subdivision of a \((k, 1)\)-spindle \iff \( \exists u, v \in V(G) : \text{the Maximum Flow from } u \text{ to } v \text{ is at least } k \).

\( G \) contains a subdivision of a \((1, \ell)\)-spindle \iff the length of a Longest Path in \( G \) is at least \( \ell \).
What is known about subdivisions of spindles

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The running time of this algorithm is $n^{O(|V(F)|)}$, where $n = |V(G)|$. 
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The running time of this algorithm is $O(|V(F)|)$, where $n = |V(G)|$.

Is a running time $f(|V(F)|) \cdot n^{O(1)}$ possible, for some function $f$?

This question had been asked by [Bang-Jensen, Havet, Maia. 2015]
Parameterized complexity in one slide

- The area of **parameterized complexity** was introduced in the 90’s by Downey and Fellows.

- **Idea** given an NP-hard problem with input size $n$, fix one parameter $k$ of the input to see whether the problem gets more “tractable”.

  **Example**: $k =$ length of a Longest Path.

- Given a (NP-hard) problem with input of size $n$ and a parameter $k$, a fixed-parameter tractable (FPT) algorithm runs in time

  $$f(k) \cdot n^{O(1)}$$

  for some function $f$.

  **Examples**: $k$-Vertex Cover, $k$-Longest Path.
**Max \((k \times \bullet)\)-Spindle Subdivision**

For a fixed \(k \geq 1\), given an input digraph \(G\), find the largest \(\ell\) such that \(G\) contains a subdivision of a \((k \times \ell)\)-spindle.

**Theorem**

Let \(k \geq 1\) be fixed. \(\text{Max} \ (k \times \bullet)\)-Spindle Subdivision is \(\text{NP}\)-hard.

**Theorem**

Let \(\ell \geq 1\) be fixed. \(\text{Max} \ (\bullet \times \ell)\)-Spindle Subdivision is in \(\mathcal{P}\) if \(\ell \leq 3\), and \(\text{NP}\)-hard if \(\ell \geq 4\), even restricted to acyclic digraphs (DAGs).
Our results (I): optimization problems

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Our results (II): FPT algorithms for finding 2-spindles

2-spindle: spindle with exactly two paths.  

\[ u \rightarrow P_1 \rightarrow \ell_1 \rightarrow P_2 \rightarrow \ell_2 \rightarrow v \]

[Benhocine, Wojda. 1983]  
[Cohen, Havet, Lochet, Nisse. 2016]  
[Kim, Kim, Ma, Park. 2016]

Theorem  
Given a digraph $G$ and $\ell \geq 1$, deciding whether there exist $\ell_1 \geq \ell_2 \geq 1$ such that $G$ contains a subdivision of a $(\ell_1, \ell_2)$-spindle is NP-hard and FPT parameterized by $\ell$, with running time \(2^{O(\ell)} \cdot n^{O(1)}\).

Theorem  
Given a digraph $G$ and $\ell_1, \ell_2$ with $\ell_2 \geq \ell_1 \geq 1$, deciding whether $G$ contains a subdivision of a $(\ell_1, \ell_2)$-spindle can be solved in time \(2^{O(\ell_2)} \cdot n^{O(\ell_1)}\). When $\ell_1$ is a constant, the problem remains NP-hard.

Both FPT algorithms are asymptotically optimal under the ETH.
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![Diagram of a 2-spindle](image)

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ETH: \# algorithm solving 3-SAT on a formula with $n$ variables in time $2^{o(n)}$. 
3 NP-hardness reduction
NP-hardness reduction

**Max ($\bullet \times \ell$)-Spindle Subdivision**

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Let $\ell \geq 1$ be fixed. Max ($\bullet \times \ell$)-Spindle Subdivision is in $\text{P}$ if $\ell \leq 3$, and $\text{NP-hard}$ if $\ell \geq 4$, even restricted to DAGs.
**Max (● × ℓ)-Spindle Subdivision**

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**Theorem**

Let $\ell \geq 1$ be fixed. Max (● × ℓ)-Spindle Subdivision is in P if $\ell \leq 3$, and **NP-hard if $\ell \geq 4$**, even restricted to DAGs.

We prove the case $\ell = 4$, by reduction from 3-Dimensional Matching:

Given three sets $A, B, C$ of the same size and a set of triples $T \subseteq A \times B \times C$, decide whether there exists a set $T' \subseteq T$ of pairwise disjoint triples with $|T'| = |A|$.
**NP-hardness reduction**

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Our reduction is strongly inspired by [Brewster, Hell, Pantel, Rizzi, Yeo. 2003]
Reduction for $\ell = 4$

Given $(A, B, C, T)$ of 3-Dimensional Matching, with $|A| = n$ and $T = m$, we construct $G$ of Max $(\bullet \times \ell)$-Spindle Subdivision as follows:
Reduction for $\ell = 4$

For $i \in [n]$, we add to $G$ three vertices $a_i, b_i, c_i$ (element of sets $A, B, C$).
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For $i \in [n]$, we add to $G$ three vertices $a_i, b_i, c_i$ (element of sets $A, B, C$).

For $T \in \mathcal{T}$, with $T = (a_i, b_j, c_p)$, we add to $G$ a copy of $H$ and we identify vertex $a$ with $a_i$, vertex $b$ with $b_j$, and vertex $c$ with $c_p$. 

![Graph](graph.png)
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![](image)
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We add a new vertex $s$ (source) and a vertex $t$ (sink) that we connect to every other vertex. They will be the endpoints of the desired spindle.
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Claim $(A, B, C, T)$ is a $\text{YES}$-instance of $3$-$\text{DIM. MATCHING}$ $\iff$ $G$ contains a subdivision of a $(n + 2m \times 4)$-spindle.
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By construction of $G$, a $(n + 2m \times 4)$-spindle covers all $V(G)$, so it is equivalent to partitioning $G \setminus \{s, t\}$ into 2-paths (paths with 2 arcs).
Reduction for $\ell = 4$

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**Key property:** for every copy of $H$, there are exactly two ways the 2-paths can intersect $H$. This defines whether each triple $T \in \mathcal{T}$ is chosen or not.
Reduction for $\ell = 4$

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Modification for $\ell > 4$: we define the digraph $G$ in the same way, except that we subdivide the arcs outgoing from $s$ exactly $\ell - 4$ times.
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Let \(\ell \geq 1\) be fixed. **Max \((\bullet \times \ell)\)-Spindle Subdivision** is in \(P\) if \(\ell \leq 3\), and **NP-hard** if \(\ell \geq 4\), even restricted to DAGs.
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Max ($\bullet \times \ell$)-Spindle Subdivision
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- $\ell = 1$: can be solved by a flow algorithm.
- $\ell = 2$: guess two vertices, delete arcs between them, and then flow.
- $\ell = 3$: we reduce the problem to computing a maximum matching in an auxiliary undirected graph, as follows...
Idea of the case $\ell = 3$

A directed path $P$ is nontrivial if its endpoints are distinct.
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We first guess vertices $s, t \in V(G)$ as endpoints of the spindle.

Largest $k$ such that $G$ contains a $(k \times 3)$-spindle from $s$ to $t$. 

$$s \quad t$$

$$N^+(s) \quad N^-(t)$$

$$17/28$$
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maximum number of vertex-disjoint nontrivial directed paths from $N^+(s)$ to $N^-(t)$ in the digraph $G \setminus \{s, t\}$. 

![Diagram showing a directed graph with vertices $s$ and $t$ connected by multiple paths, highlighting $N^+(s)$ and $N^-(t)$]
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Proposition

Let $G$ be a digraph and $X, Y \subseteq V(G)$. The maximum number of vertex-disjoint directed nontrivial paths from $X$ to $Y$ can be computed in polynomial time.
Given a digraph $G$ and $X, Y \subseteq V(G)$, we build an undirected graph $G'$:

Claim $G$ contains $k$ vertex-disjoint directed nontrivial paths from $X$ to $Y$ $\iff$ $G'$ has a matching of size $k + |V(G) \setminus (X \cup Y)|$. 

[Diagram of $G$ and $G'$ showing the vertices and edges with labels $u_1, u_2, u_3, u_4, v_1, v_2, v_3$.]
Idea of the proof

Given a digraph $G$ and $X, Y \subseteq V(G)$, we build an undirected graph $G'$:

$V(G') = V(G) + \text{a copy } v' \text{ of each vertex } v \notin X \cup Y.$

$E(G')$: For each $v \notin X \cup Y$, add to $G'$ the edge $\{v, v'\}$.
For each $(u, v)$, add $\{u, v\}$ if $v \in X \cup Y$, and $\{u, v'\}$ otherwise.
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Claim: $G$ contains $k$ vertex-disjoint directed nontrivial paths from $X$ to $Y$ if and only if $G'$ has a matching of size $k + |V(G) \setminus (X \cup Y)|$. 
A pair $\mathcal{M} = (E, \mathcal{I})$, where $E$ is a ground set and $\mathcal{I}$ is a family of subsets of $E$, is a matroid if it satisfies the following three axioms:

1. $\emptyset \in \mathcal{I}$.
2. If $A' \subseteq A$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$.
3. If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called the independent sets of the matroid.
Representative sets in matroids

Two independent sets $A, B$ of $\mathcal{M}$ fit if $A \cap B = \emptyset$ and $A \cup B$ is independent.
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Let $\mathcal{A}$ be a family of sets of size $p$ in a matroid $\mathcal{M}$. A subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is said to $q$-represent $\mathcal{A}$, denoted $\mathcal{A}' \subseteq_{\text{rep}} \mathcal{A}$, if for every set $B$ of size $q$ such that there is an $A \in \mathcal{A}$ that fits $B$, there is an $A' \in \mathcal{A}'$ that also fits $B$. 

\[
\left\{ A_3 \right\} \subseteq_{2\text{rep}} \left\{ A_1, A_2, A_3 \right\}
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\[
\mathcal{A} = \{A_1, A_2, A_3\}, \quad p = 4, q = 2
\]

\[
\{A_3\} \subseteq^2_{\text{rep}} \{A_1, A_2, A_3\}
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Finding a 2-spindle of large total size

If a subdigraph $S$ of $G$ is a subdivision of a $(\ell_1, \ell_2)$-spindle, with $\min\{\ell_1, \ell_2\} \geq 1$ and $\ell_1 + \ell_2 = \ell$, we say that $S$ is a good spindle.

Using the recent techniques of [Fomin, Lokshtanov, Panolan, Saurabh. 2016], $|\hat{S}_{\ell_1, \ell_2, u, u_1, u_2}| = 2^{O(\ell)}$ and can be computed in time $2^{O(\ell)} \cdot n^{O(\frac{1}{\ell})}$. 
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Idea: We will $q$-represent the “first part” of the desired spindle (paths $P_u^1$ and $P_u^2$), for every $u, u_1, u_2 \in V(G)$, $\ell_1, \ell_2 \leq \ell$, and $0 \leq q \leq 2\ell$. 

![Diagram of a 2-spindle with paths $P_u^1$, $P_u^2$, $P_v^1$, and $P_v^2$.]
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Using the recent techniques of [Fomin, Lokshtanov, Panolan, Saurabh. 2016], $|\hat{S}_{\ell_1, \ell_2, q_u, u_1, u_2}| = 2 \mathcal{O}(\ell)$ and can be computed in time $2 \mathcal{O}(\ell) \cdot n \mathcal{O}(1)$. 

![Diagram of a 2-spindle](image-url)
**Finding a 2-spindle of large total size**

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Finding a 2-spindle of large total size

If a subdigraph $S$ of $G$ is a subdivision of a $(\ell_1, \ell_2)$-spindle, with $\min\{\ell_1, \ell_2\} \geq 1$ and $\ell_1 + \ell_2 = \ell$, we say that $S$ is a good spindle.

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Key property: these families indeed represent the solutions

Consider a good spindle $S$ with minimum number of vertices:

![Diagram of a spindle with vertices $u$, $u_1$, $u_2$, $v$, $v_1$, and $v_2$, and paths $P^u_1$, $P^u_2$, $P^B_1$, and $P^B_2$. The paths $P^u_1$ and $P^u_2$ are disjoint from the rest of the spindle $S$.]
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Consider a **good spindle** $S$ with minimum number of vertices:

The representatives $\hat{P}_1^u$ and $\hat{P}_2^u$ are **disjoint** from the rest of the spindle $S$. 
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Wrapping up the algorithm

For every \( u, u_1, u_2 \in V(G) \), \( \ell_1, \ell_2 \leq \ell \), and \( 0 \leq q \leq 2\ell \), we compute a \( q \)-representative family \( \hat{S}_{u, u_1, u_2}^{\ell_1, \ell_2, q} \) in time \( 2^{O(\ell)} \cdot n^{O(1)} \).
Wrapping up the algorithm

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2. For every $\hat{P}_1^u \cup \hat{P}_2^u \in \hat{S}_{u, u_1, u_2}^{\ell_1, \ell_2, q}$, we check whether $G$ contains a $(u_1, v)$-path $P_1^v$ and a $(u_2, v)$-path $P_2^v$ of this shape:
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Overall running time: $2^O(\ell) \cdot n^{O(1)}$. 
1. Introduction

2. Our results

3. NP-hardness reduction

4. Polynomial-time algorithm

5. Sketch of the FPT algorithms

6. Conclusions
Main open question:

Finding a subdivision of a spindle $F$ is FPT parameterized by $|V(F)|$?
Further research

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When $G$ is an acyclic digraph, we can prove the following:

**Theorem**

*Given an acyclic digraph $G$ and integers $k, \ell$, deciding whether $G$ contains a subdivision of a $(k \times \ell)$-spindle can be solved in time $O(\ell^k \cdot n^{2k+1})$.**
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But is the problem FPT on acyclic digraphs? That is, in time $f(k, \ell) \cdot n^{O(1)}$?
Gràcies!