Finding subdivisions of spindles on digraphs

Júlio Araújo\textsuperscript{1} Victor A. Campos\textsuperscript{2} Ana Karolinna Maia\textsuperscript{2} Ignasi Sau\textsuperscript{1,3} Ana Silva\textsuperscript{1}

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\textsuperscript{1} Departamento de Matemática, UFC, Fortaleza, Brazil.
\textsuperscript{2} Departamento de Computação, UFC, Fortaleza, Brazil.
\textsuperscript{3} CNRS, LIRMM, Université de Montpellier, Montpellier, France.
Outline of the talk

1. Introduction
2. Our results
3. NP-hardness reduction
4. Polynomial-time algorithm
5. Sketch of the FPT algorithms
6. Conclusions
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A \textit{subdivision} of a digraph $F$ is a digraph obtained from $F$ by replacing each arc $(u, v)$ of $F$ by a directed $(u, v)$-path.

\begin{center}
\begin{tabular}{ccc}
$F$ & $F_1$ & $F_2$
\end{tabular}
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We are interested in the following problem:

\begin{center}
\textbf{Digraph Subdivision}
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\textbf{Instance:} Two digraphs $G$ and $F$.

\textbf{Question:} Does $G$ contain a subdivision of $F$ as a subdigraph?
Recent work on finding digraph subdivisions

This problem has been introduced by [Bang-Jensen, Havet, Maia. 2015]

Let $F$ be a fixed digraph.

$F$-Subdivision

Instance: A digraph $G$.

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Conjecture (Bang-Jensen, Havet, Maia. 2015)

For every fixed digraph $F$, $F$-Subdivision is either in $P$ or $NP$-complete.

This conjecture is wide open, and only examples of both cases are known.

When $|V(F)| = 4$, there are only 5 open cases. [Havet, Maia, Mohar. 2017]
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We focus on finding subdivisions of spindles

For positive integers \( \ell_1, \ldots, \ell_k \), a \((\ell_1, \ldots, \ell_k)\)-spindle is the digraph containing \(k\) paths \(P_1, \ldots, P_k\) from a vertex \(u\) to a vertex \(v\), such that

\[
|E(P_i)| = \ell_i \quad \text{for} \quad 1 \leq i \leq k
\]

and

\[
V(P_i) \setminus V(P_j) = \{u, v\} \quad \text{for} \quad 1 \leq i \neq j \leq k.
\]

If \(\ell_i = \ell\) for \(1 \leq i \leq k\), a \((\ell_1, \ldots, \ell_k)\)-spindle is also called a \((k \times \ell)\)-spindle.

\[ G \text{ contains a subdivision of a } (k \times 1)\text{-spindle} \iff \exists u, v \in V(G) : \text{the Maximum Flow from } u \text{ to } v \text{ is at least } k. \]

\[ G \text{ contains a subdivision of a } (1 \times \ell)\text{-spindle} \iff \text{the length of a Longest Path in } G \text{ is at least } \ell. \]
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$G$ contains a subdivision of a $(k \times 1)$-spindle $\iff \exists u, v \in V(G)$: the Maximum Flow from $u$ to $v$ is at least $k$.

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What is known about subdivisions of spindles

If the spindle is fixed, the problem is in $P$: [Bang-Jensen, Havet, Maia. 2015]
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![Diagram of a spindle]

We can guess all choices for the first $\ell_j$ vertices of each path.
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![Diagram of a spindle](image)

We can **guess all choices for the first** $\ell_i$ **vertices** of each path. Then, compute a **flow** from those endpoints to some vertex $v$.

The running time of this algorithm is $n^{O(|V(F)|)}$, where $n = |V(G)|$. 
What is known about subdivisions of spindles

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Is a running time $f(|V(F)|) \cdot n^{O(1)}$ possible, for some function $f$?

This question had been asked by [Bang-Jensen, Havet, Maia. 2015]
The area of parameterized complexity was introduced in the 90’s by Downey and Fellows.

Idea: given an NP-hard problem with input size $n$, fix one parameter $k$ of the input to see whether the problem gets more “tractable”.

**Example:** $k =$ length of a Longest Path.

Given a (NP-hard) problem with input of size $n$ and a parameter $k$, a fixed-parameter tractable (FPT) algorithm runs in time

$$f(k) \cdot n^{O(1)},$$

for some function $f$.

**Examples:** $k$-Vertex Cover, $k$-Longest Path.
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Max \((k \times \bullet)\)-Spindle Subdivision

For a fixed \(k \geq 1\), given an input digraph \(G\), find the largest \(\ell\) such that \(G\) contains a subdivision of a \((k \times \ell)\)-spindle.

Theorem

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Let \(\ell \geq 1\) be fixed. \(\text{Max \((\bullet \times \ell)\)-Spindle Subdivision}\) is in \(\text{P}\) if \(\ell \leq 3\), and \(\text{NP-hard}\) if \(\ell \geq 4\), even restricted to acyclic digraphs (DAGs).
Our results (I): optimization problems

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Our results (II): FPT algorithms for finding 2-spindles

2-spindle: spindle with exactly two paths.

[Benhocine, Wojda. 1983]
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Theorem

Given a digraph $G$ and $\ell \geq 1$, deciding whether there exist $\ell_1, \ell_2 \geq 1$ with $\ell_1 + \ell_2 = \ell$ such that $G$ contains a subdivision of a $(\ell_1, \ell_2)$-spindle is NP-hard and FPT parameterized by $\ell$, with running time $2^{O(\ell)} \cdot n^{O(1)}$. 
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Theorem

Given a digraph $G$ and $\ell_1, \ell_2$ with $\ell_2 \geq \ell_1 \geq 1$, deciding whether $G$ contains a subdivision of a $(\ell_1, \ell_2)$-spindle can be solved in time $2^{O(\ell_2)} \cdot n^{O(\ell_1)}$. When $\ell_1$ is a constant, the problem remains NP-hard.
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Both FPT algorithms are asymptotically optimal under the ETH.
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ETH: \(\exists\) algorithm solving 3-SAT on a formula with $n$ variables in time $2^{o(n)}$. 

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**Max (● × ℓ)-Spindle Subdivision**
For a fixed $\ell \geq 1$, given an input digraph $G$, find the largest $k$ such that $G$ contains a subdivision of a $(k \times ℓ)$-spindle.

**Theorem**

Let $\ell \geq 1$ be fixed. **Max (● × ℓ)-Spindle Subdivision** is in P if $\ell \leq 3$, and **NP-hard if $\ell \geq 4** , even restricted to DAGs.
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We prove the case $\ell = 4$, by reduction from **3-Dimensional Matching**:

Given three sets $A, B, C$ of the same size and a set of triples $T \subseteq A \times B \times C$, decide whether there exists a set $T' \subseteq T$ of pairwise disjoint triples with $|T'| = |A|$.
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Our reduction is strongly inspired by [Brewster, Hell, Pantel, Rizzi, Yeo. 2003]
Reduction for $\ell = 4$

Given $(A, B, C, T)$ of 3-Dimensional Matching, with $|A| = n$ and $T = m$, we construct $G$ of Max $(\bullet \times \ell)$-Spindle Subdivision as follows:
Reduction for $\ell = 4$

For $i \in [n]$, we add to $G$ three vertices $a_i, b_i, c_i$ (elements of sets $A, B, C$).
Reduction for \( \ell = 4 \)

For \( i \in [n] \), we add to \( G \) three vertices \( a_i, b_i, c_i \) (elements of sets \( A, B, C \)).

For \( T \in \mathcal{T} \), with \( T = (a_i, b_j, c_p) \), we add to \( G \) a copy of \( H \) and we identify vertex \( a \) with \( a_i \), vertex \( b \) with \( b_j \), and vertex \( c \) with \( c_p \).
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We add a new vertex $s$ (source) and a vertex $t$ (sink) that we connect to every other vertex. They will be the endpoints of the desired spindle.
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\textbf{Claim} $(A, B, C, \mathcal{T})$ is a \textsc{Yes}-instance of $3$-$\textsc{Dim. Matching}$ $\iff G$ contains a subdivision of a $(n + 2m \times 4)$-spindle.
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By construction of $G$, a $(n + 2m \times 4)$-spindle covers all $V(G)$, so it is equivalent to partitioning $G \setminus \{s, t\}$ into 2-paths (paths with 2 arcs).
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Key property: for every copy of $H$, there are exactly two ways the 2-paths can intersect $H$. This defines whether each triple $T \in \mathcal{T}$ is chosen or not.
Reduction for $\ell = 4$

For $i \in [n]$, we add to $G$ three vertices $a_i, b_i, c_i$ (elements of sets $A, B, C$).

For $T \in T$, with $T = (a_i, b_j, c_p)$, we add to $G$ a copy of $H$ and we identify vertex $a$ with $a_i$, vertex $b$ with $b_j$, and vertex $c$ with $c_p$.

Modification for $\ell > 4$: we define the digraph $G$ in the same way, except that we subdivide the arcs outgoing from $s$ exactly $\ell - 4$ times.
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- \(\ell = 1\): can be solved by a flow algorithm.
Cases that can be solved in polynomial time

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Let \(\ell \geq 1\) be fixed. **Max \((\bullet \times \ell)\)-Spindle Subdivision is in \(\text{P}\) if \(\ell \leq 3\)**, and **NP-hard if \(\ell \geq 4\)**, even restricted to DAGs.

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Cases that can be solved in polynomial time

**Max (● × ℓ)-Spindle Subdivision**

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- \( \ell = 1 \): can be solved by a flow algorithm.
- \( \ell = 2 \): guess two vertices, delete arcs between them, and then flow.
- \( \ell = 3 \): we reduce the problem to computing a maximum matching in an auxiliary undirected graph, as follows...
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![Diagram showing $N^+(s)$ and $N^-(t)$]
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Main ingredient for the case $\ell = 3$

**Proposition**

Let $G$ be a digraph and $X, Y \subseteq V(G)$. The maximum number of vertex-disjoint directed nontrivial paths from $X$ to $Y$ can be computed in polynomial time.
Idea of the proof

Given a digraph $G$ and $X, Y \subseteq V(G)$, we build an undirected graph $G'$:

- $V(G') = V(G) + \text{a copy of each vertex } v \not\in X \cup Y$.
- $E(G')$ for each $v \in X \cup Y$, add to $G'$ the edge $\{v, v'\}$.
- For each $(u, v)$, add $\{u, v\}$ if $v \in X \cup Y$, and $\{u, v'\}$ otherwise.

Claim $G$ contains $k$ vertex-disjoint directed nontrivial paths from $X$ to $Y$ $\iff$ $G'$ has a matching of size $k + |V(G) \setminus (X \cup Y)|$. 
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Claim: $G$ contains $k$ vertex-disjoint directed nontrivial paths from $X$ to $Y$ if and only if $G'$ has a matching of size $k + |V(G) \setminus (X \cup Y)|$. 
Introduction

Our results

NP-hardness reduction

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Conclusions
A pair $\mathcal{M} = (E, \mathcal{I})$, where $E$ is a ground set and $\mathcal{I}$ is a family of subsets of $E$, is a matroid if it satisfies the following three axioms:

1. $\emptyset \in \mathcal{I}$.
2. If $A' \subseteq A$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$.
3. If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called the independent sets of the matroid.
Representative sets in matroids

Two independent sets $A, B$ of $\mathcal{M}$ fit if $A \cap B = \emptyset$ and $A \cup B$ is independent.
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Let $\mathcal{A}$ be a family of sets of size $p$ in a matroid $M$. A subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is said to $q$-represent $\mathcal{A}$, denoted $\mathcal{A}' \subseteq_q \mathcal{A}$, if for every set $B$ of size $q$ such that there is an $A \in \mathcal{A}$ that fits $B$, there is an $A' \in \mathcal{A}'$ that also fits $B$. 

We consider the uniform matroid with ground set $V(G)$ and rank $\ell + q$, with $0 \leq q \leq 2\ell$. 

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\{A_3\} \subseteq 2\text{rep} \{A_1, A_2, A_3\}
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Finding a 2-spindle of large total size

If a subdigraph $S$ of $G$ is a subdivision of a $(\ell_1, \ell_2)$-spindle, with $\min\{\ell_1, \ell_2\} \geq 1$ and $\ell_1 + \ell_2 = \ell$, we say that $S$ is a good spindle.
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**Idea** We will $q$-represent the “first part” of the desired spindle (paths $P_u^1$ and $P_u^2$), for every $u, u_1, u_2 \in V(G)$, $\ell_1, \ell_2 \leq \ell$, and $0 \leq q \leq 2\ell$. 

![Diagram of a 2-spindle](image-url)
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![Diagram of a 2-spindle with paths $P^1_u$ and $P^2_u$, representing the desired spindle with $q$.

Using the recent techniques of [Fomin, Lokshtanov, Panolan, Saurabh. 2016], $|\hat{S}_{\ell_1, \ell_2, q}^{u, u_1, u_2}| = 2^{O(\ell)}$ and can be computed in time $2^{O(\ell)} \cdot n^{O(1)}$.}
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For every $u, u_1, u_2 \in V(G)$, $\ell_1, \ell_2 \leq \ell$, and $0 \leq q \leq 2\ell$, we compute a $q$-representative family $\hat{S}_{u,u_1,u_2}^{\ell_1,\ell_2,q}$ in time $2^{O(\ell)} \cdot n^{O(1)}$. 
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![Diagram](attachment:image.png)
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Overall running time: $2^{O(\ell)} \cdot n^{O(1)}$. 
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Given an acyclic digraph $G$ and integers $k, \ell$, deciding whether $G$ contains a subdivision of a $(k \times \ell)$-spindle can be solved in time $O(\ell^k \cdot n^{2k+1})$. 
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But is the problem FPT on acyclic digraphs? That is, in time $f(k, \ell) \cdot n^{O(1)}$?
Gràcies!