Compound Logics for Modification Problems

Fedor V. Fomin, Petr A. Golovach, **Ignasi Sau**,
Giannos Stamoulis, and Dimitrios M. Thilikos

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Thanks Dimitrios for many of the slides!!

Plan of the talk

1. Motivation for algorithmic meta-theorems based on logic
2. Definition of the new logic(s) and our results
3. Necessity of the ingredients of the logic
4. Sketch of some ideas of the proofs
5. Further research

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Input:	A graph G and an integer k ("amount of modification").
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Want: algorithms in time $\boxed{f(\varphi) \cdot n^{\mathcal{O}(1)}}$, where $n = |V(G)|$.

Algorithmic Meta-Theorems (AMTs)

For some logic L *and some class* $\mathcal{C}$ *of combinatorial structures, every algorithmic problem* Π *that is expressible in* L*, there is an* efficient *algorithm solving* Π *for inputs that belong in* $\mathcal{C}$*.*

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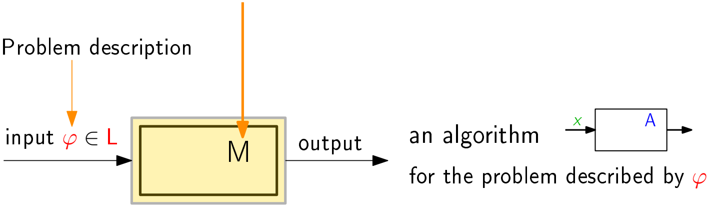

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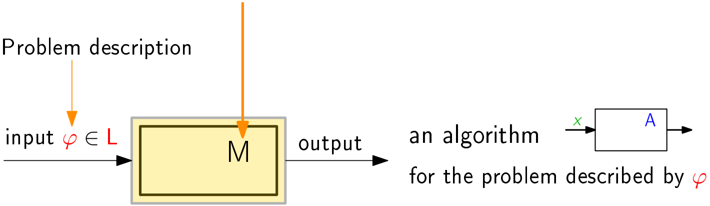
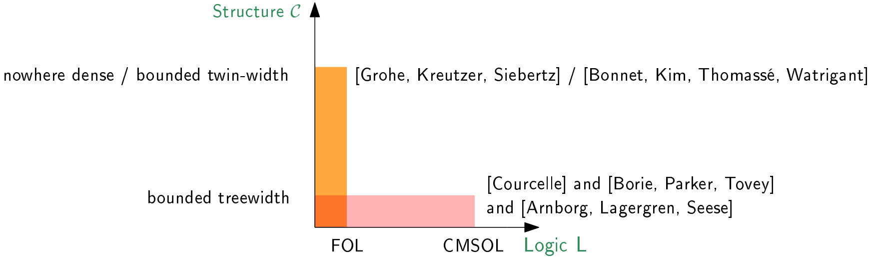


Two main logics for φ:

- FOL: First Order Logic
  - quantification on vertices or edges
- CMSOL: Counting Monadic Second Order Logic
  - quantification on sets of vertices or edges

Famous AMTs for model-checking in time FPT



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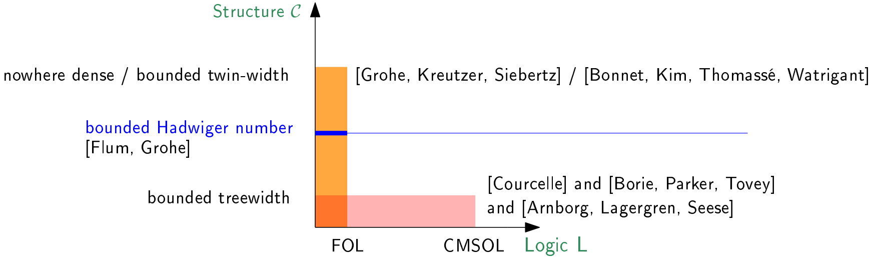

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Hadwiger number: $\mathbf{hw}(G)=$ max clique-minor of the graph G

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VERTEX DELETION TO PLANARITY
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Modulator: $X = \{x_1, \ldots, x_k\}$

Target property: minor-exclusion of $\mathcal{H} = \{K_5, K_{3,3}\}$

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Topological minor exclusion:

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Given G and k, is there an $X \subseteq V(G)^{\leq k}$ such that $G \setminus X$ is planar+more?

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- Extensions to general minor-closed target classes $\mathcal{G}$?

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- More general modification operations do not seem to be captured...

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- Can we define successive target properties?
- Hierarchical clustering?
- Multi-level modification?
- Consider different modification scenarios?
- We may demand target conditions to be satisfied by the connected components (or even the blocks) of $G \setminus X$ (CMSOL-demand).
- Multiway Cut or Multicut to some target property $\mathcal{G}$.
- We may demand vertex/edge removals with prescribed adjacencies.
- ...

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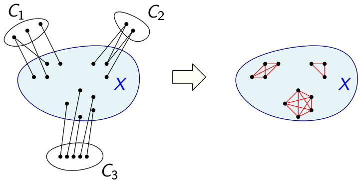

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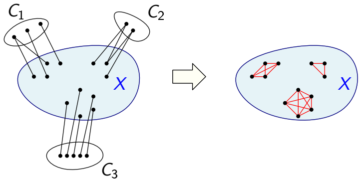

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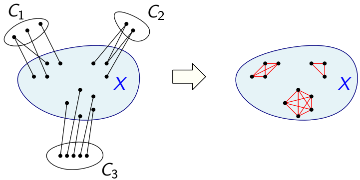

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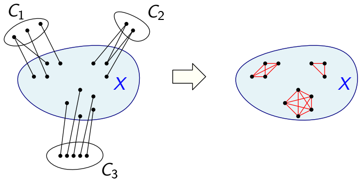

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- **p=bridge-depth**: $\mathcal{G}$-bridge-depth:
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- or just $G[X] \models \beta_k$ for some $\beta_k \in \mathsf{CMSOL}^{\mathsf{tw}}$?

  - $\mathsf{CMSOL}^{\mathsf{tw}}[\mathtt{E}, \mathtt{X}]$ (on annotated graphs):
    every $\beta \in \mathsf{CMSOL}[\mathtt{E}, \mathtt{X}]$ for which there exists some c_β such that the torsos of all the models of β have treewidth at most c_β.

Is there **one** meta-theorem that deals with all these cases?

We define a compound logic for modification problems

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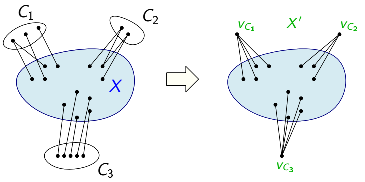

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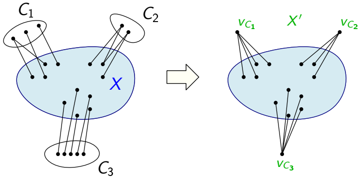

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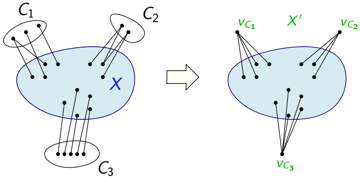

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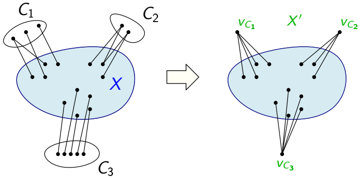

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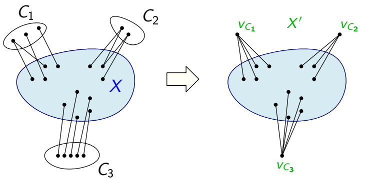

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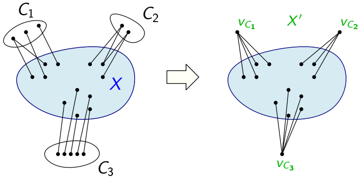

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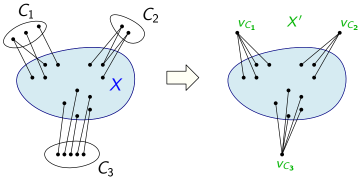

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Theorem (our result, in a less simple form)

For every $\beta \in \mathsf{CMSOL}^{\mathsf{tw}}$ and every $\gamma \in \Theta_0^{(\mathsf{c})}$, there is an algorithm deciding $\mathrm{Mod}(\beta \triangleright \gamma)$ in quadratic time.

- for $\varphi \in \mathsf{CMSOL}$, define $\varphi^{(\mathsf{c})}$: $G \models \varphi^{(\mathsf{c})}$ if $\forall C \in \mathsf{cc}(G), C \models \varphi$.
- for $\mathsf{L} \subseteq \mathsf{CMSOL}$, define $\mathsf{L}^{(\mathsf{c})} = \mathsf{L} \cup \{\varphi^{(\mathsf{c})} \mid \varphi \in \mathsf{L}\}$.

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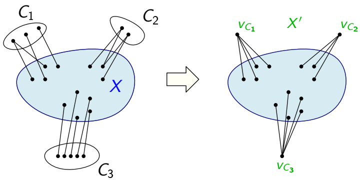

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- **MB**(L): all monotone Boolean combinations of sentences in L.

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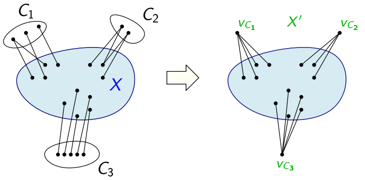

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- This automatically implies algorithms in all aforementioned directions, beyond the applicability of the theorems of Courcelle and Grohe and Flum.

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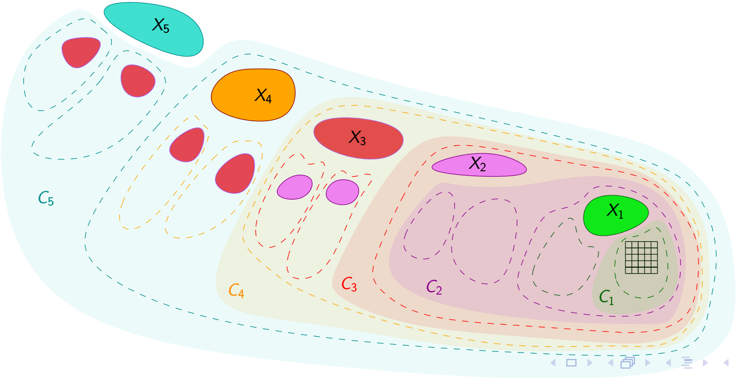

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Our results are **constructive**:

Theorem

There is a **Meta-Algorithm** M *that,*
with input a sentence $\theta \in \Theta$ and an upper bound c_θ on $\mathbf{hw}(\mathrm{Mod}(\theta))$,
returns as output the algorithm $\mathbf{A}_\theta$.

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For every $\tilde{\theta} \in \tilde{\Theta}$, there is an algorithm deciding $\mathrm{Mod}(\tilde{\theta})$ in quadratic time on graphs of fixed Hadwiger number.

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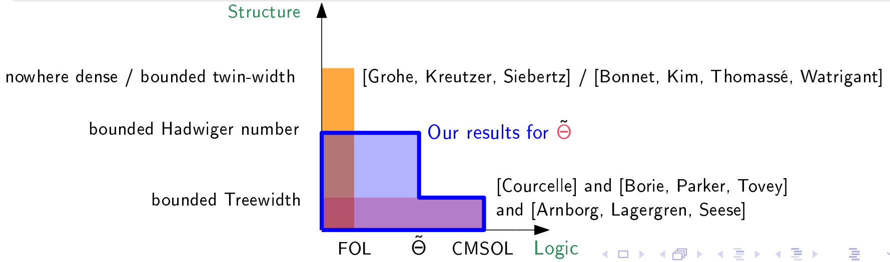

Generalization to extensions of FOL

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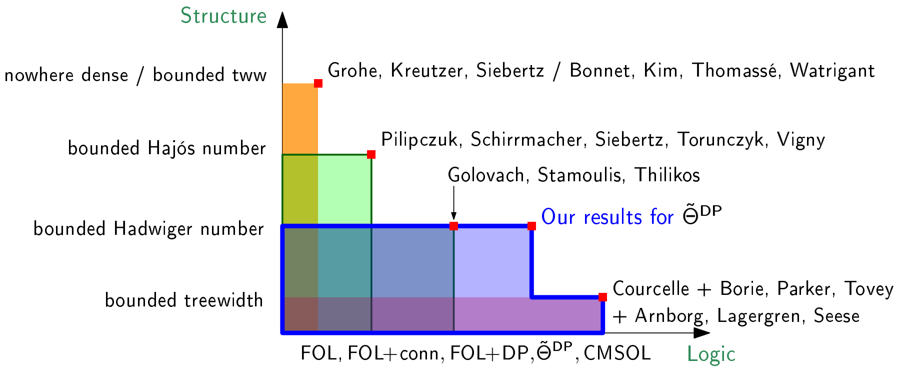
The current meta-algorithmic landscape



Missing: FOL + DP, FPT model-checking up to bounded Hajós number.

[Schirrmacher, Siebertz, Stamoulis, Thilikos, Vigny, arXiv 2023]

Necessity of the ingredients of our logic

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2. Why the target sentence $\sigma \in \mathsf{FOL}$ (or extensions)?
3. Why the target sentence μ expresses minor-exclusion?

skip

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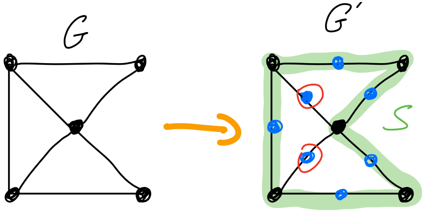

- G Hamiltonian $\Leftrightarrow$ G' has a vertex set S such that $G'[S]$ is a cycle and $G' \setminus S$ is edgeless.

- $\mathsf{tw}(G'[S]) = 2$ but $\mathsf{tw}(\mathsf{torso}(G',S)) = \mathsf{tw}(G)$ unbounded.

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Expressing whether a graph G contains a clique on k vertices is FOL-expressible, while k-Clique is W[1]-hard on general graphs
(again, consider a void modulator).

skip

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- The combinatorial/algorithmic results in

1. Ken-ichi Kawarabayashi, Robin Thomas, and Paul Wollan. A new proof of the flat wall theorem. *Journal of Combinatorial Theory, Series B*, 129:204–238, 2018.
2. Ignasi Sau, Giannos Stamoulis, and Dimitrios M. Thilikos. A more accurate view of the Flat Wall Theorem, 2021. arXiv:2102.06463.
3. Petr A. Golovach, Giannos Stamoulis, and Dimitrios M. Thilikos. Hitting topological minor models in planar graphs is fixed parameter tractable. In *Proc. of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms, (SODA)*, pages 931–950, 2020.
4. Julien Baste, Ignasi Sau, and Dimitrios M. Thilikos. A complexity dichotomy for hitting connected minors on bounded treewidth graphs: the chair and the banner draw the boundary. In *Proc. of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 951–970, 2020.
5. Ignasi Sau, Giannos Stamoulis, and Dimitrios M. Thilikos. k-apices of minor-closed graph classes. I. Bounding the obstructions. Transactions on Algorithms 2022.
6. Fedor V. Fomin, Petr A. Golovach, Giannos Stamoulis, and Dimitrios M. Thilikos. An algorithmic meta-theorem for graph modification to planarity and FOL. In *Proc. of the 28th Annual European Symposium on Algorithms (ESA)*, volume 173 of *LIPIcs*, pages 51:1–51:17, 2020.

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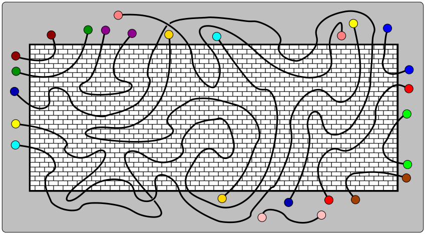
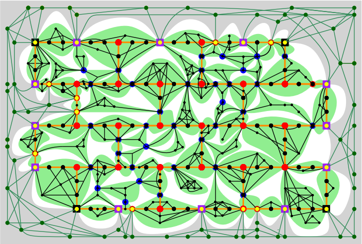
Irrelevant Vertex Technique (> 1200 citations and used in > 120 papers)

- Neil Robertson and Paul D. Seymour. Graph minors. XIII. The disjoint paths problem. *Journal of Comb. Theory, Ser. B*, 63(1):65–110, 1995.





Ultra-sketch of proof

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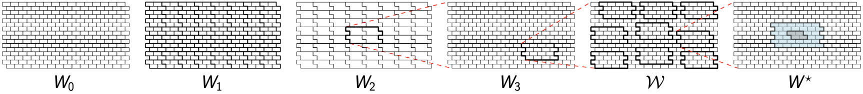

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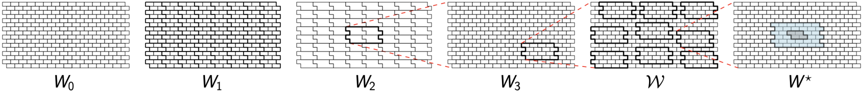
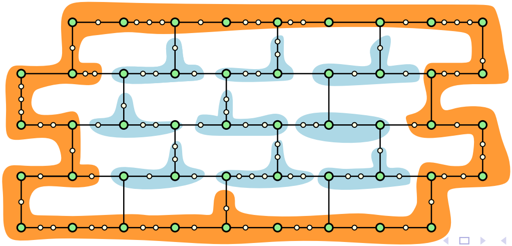


Crucial fact: the fact that the modulator sentence $\beta \in \mathsf{CMSOL}^{\mathsf{tw}}$ allows to prove that the removal of the modulator X does not destroy a flat wall too much.



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Theorem (Flat Wall Theorem. Robertson and Seymour. 1995)

There exist recursive functions $f_1 : \mathbb{N}^2 \to \mathbb{N}$ *and* $f_2 : \mathbb{N} \to \mathbb{N}$*, such that for every graph* G *and every* $q, r \in \mathbb{N}$*, one of the following holds:*

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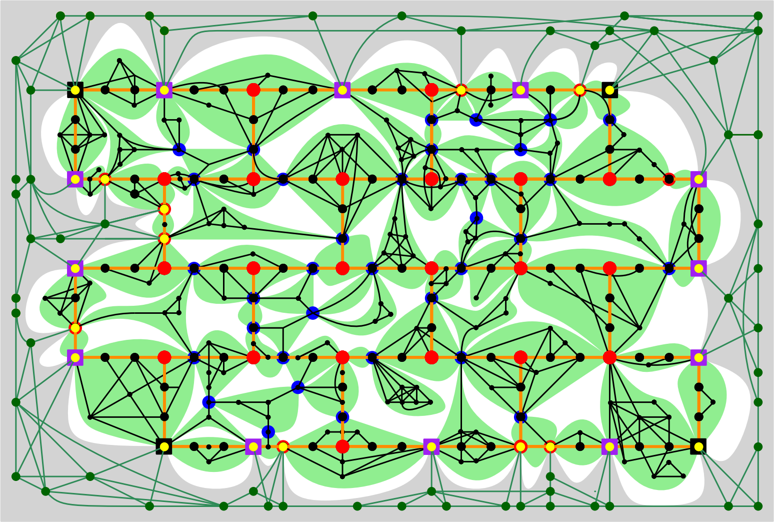
Important: possible to find one of the outputs in time $f(q, r) \cdot |V(G)|$.

How does a flat wall look like?



[Figure by Dimitrios M. Thilikos]

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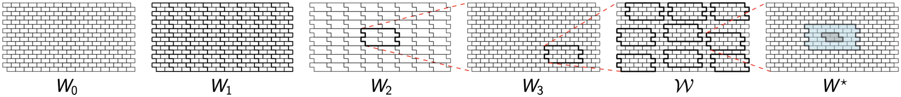
Zooming and zooming inside a flat wall



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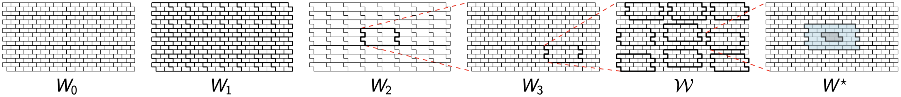

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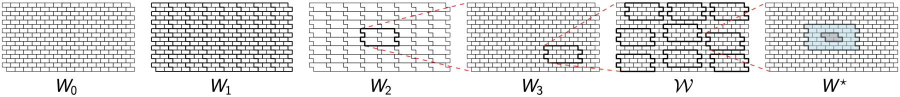

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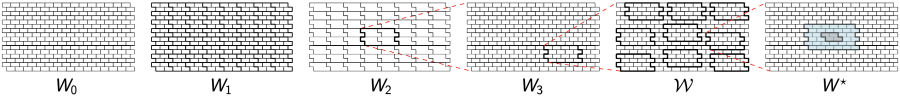

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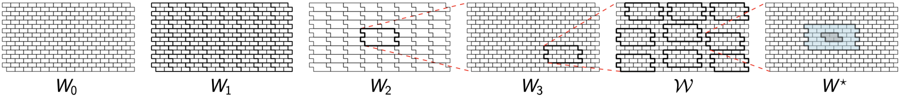

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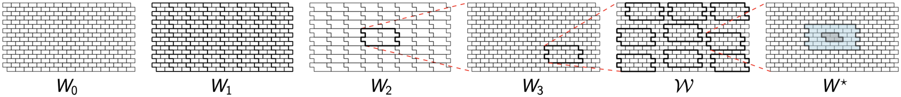

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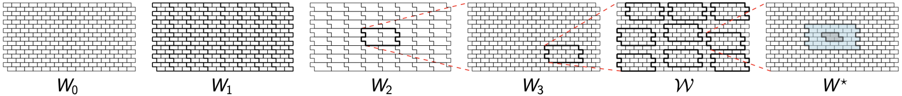
From now on, we can forget the minor-exclusion part of θ.

We keep on zooming...



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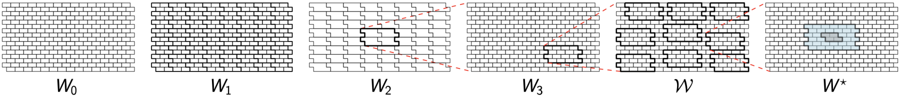

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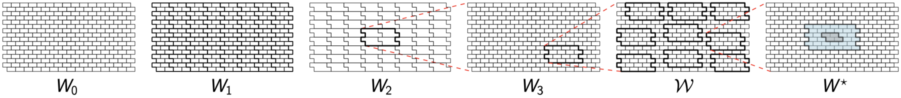

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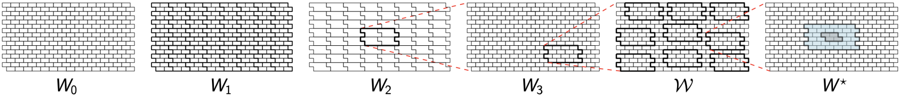

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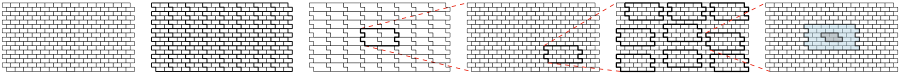

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Hardest part of the proof: prove that the central part of $W^\star$ is indeed irrelevant.

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Exploiting the bounded-treewidth property of β

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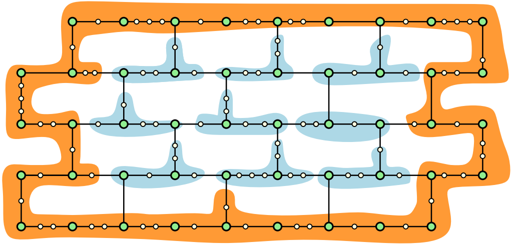

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Defining the θ-characteristic of a wall: privileged component

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This splitting gives rise to the in-signature and out-signature of a wall.

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Main new difficulty: deal with the apices corresponding to the flat wall.

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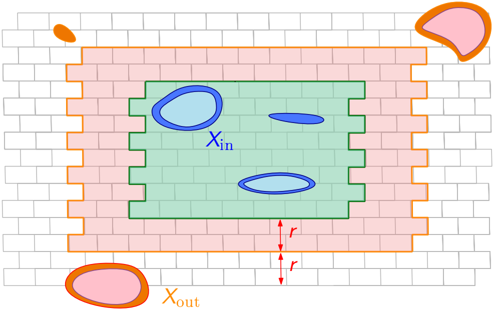

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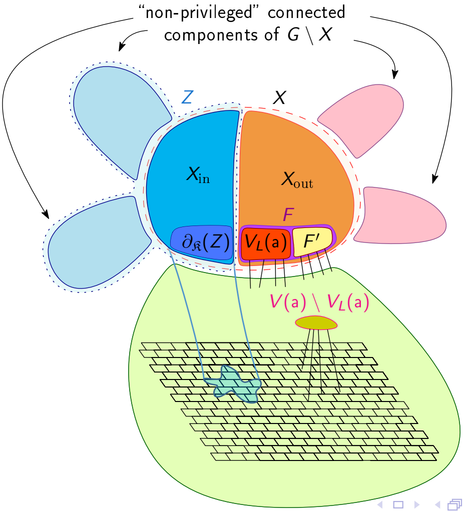

Some final remarks

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- Limitations
  - are torsos really necessary?
  - which are the optimal combinatorial assumptions on FOL+CMSOL?
- Extensions
  - irrelevant friendliness (bipartiteness)
  - other modification operations (blocks, contractions, ...)
- Open problems
  - constants hidden in $\mathcal{O}_{|\theta|}(n^2)$
  - is the Θ-hierarchy proper?
  - Is quadratic time improvable?
  - Further than minor-exclusion?