On two optimization problems:

(1) Permutation Routing on Plane Grids

(2) Find Small Subgraphs of Given Degree

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Two independent problems

(1) **Permutation Routing on Plane Grids**

Joint work with *Janez Žerovnik*

(2) **Finding Small Subgraphs of Given Minimum Degree**

Joint work with *Omid Amini* and *Saket Saurabh*

Permutation Routing on Plane Grids

Outline

- **Introduction**
  - Statement of the problem
  - Preliminaries
  - Example
- **Permutation routing algorithm for triangular grids**
  - Description
  - Correctness
  - Optimality
- Permutation routing algorithm for hexagonal grids
- (ℓ, k)-routing algorithms
- Conclusions

Permutation routing

- The **permutation routing** problem is a **packet routing** problem.
- Each processor is the **origin of at most one packet** and the **destination of at most one packet**.
- The goal is to **minimize the number of time steps** required to route all packets to their respective destinations.

Statement of the problem

- **Input:**
  - a directed graph $G = (V, E)$ (the *host* graph),
  - a subset $S \subseteq V$ of nodes,
  - and a permutation $\pi : S \rightarrow S$.
    Each node $u \in S$ wants to send a packet to $\pi(u)$.
- **Output:** Find for each pair $(u, \pi(u))$, a path form u to $\pi(u)$ in G.
- **Constraints:**
  - At each step, a packet can either move or stay at a node.
  - No arc can be crossed by two packets at the same step.
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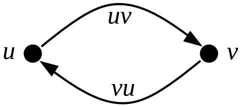
Assumptions

- We consider the **store-and-forward** and **Δ-port** model.
- **Full duplex link**: packets can be sent in the two directions of the link **simultaneously**.



- If the network is **half-duplex** →
  2 factor approximation algorithm from an optimal algorithm for the full-duplex case, by introducing *odd-even* steps.
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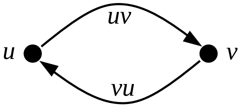

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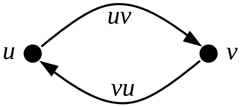

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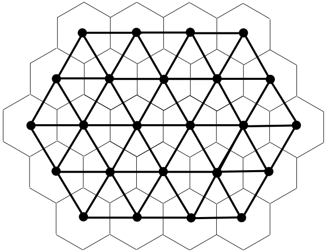
Network topologies

- There is an **ambiguity** in the notation in the literature:

$$\text{triangular grid} \leftrightarrow \text{hexagonal network},$$
$$\text{hexagonal grid} \leftrightarrow \text{honeycomb network}.$$

- Hexagonal network (△) and hexagonal tessellation (⬡):



- Hexagonal networks are finite subgraphs of the triangular grid.

Previous work

-The permutation routing problem has been studied in:

- Mobile Ad Hoc Networks
- Cube-Connected Cycle Networks
- Wireless and Radio Networks
- All-Optical Networks
- Reconfigurable Meshes...

-But, optimal algorithms:

- 2-circulant graphs, square grids.
- Triangular grids: **Two-terminal routing**
  (only one message to be sent)

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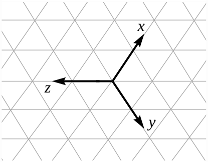
Permutation Routing on Triangular Grids

Notation and preliminary results

Nocetti, Stojmenović and Zhang [IEEE TPDS'02]:

Representation of the relative address of the nodes on a generating system ***i, j, k*** on the directions of the three axis x, y, z.



- This address **is not unique**, but we have that, being (a, b, c) and (a', b', c') the addresses of two $D - S$ pairs,

$$(a, b, c) = (a', b', c') \quad \Leftrightarrow \quad \exists \text{ an integer } d \text{ such that}$$

$$a' = a + d,$$

$$b' = b + d,$$

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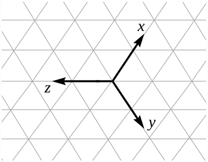

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Notation and preliminary results (2)

- A relative address $D - S = (a, b, c)$ is of the ***shortest path form*** if
  - there is a path C from S to D, C=$a\mathbf{i}$+$b\mathbf{j}$+$c\mathbf{k}$,
  - and C has the shortest length over all paths going from S to D.

Theorem (*NSZ'02*)

*An address (a, b, c) is of the **shortest path form** if and only if*

i) ***at least one component is zero*** *(that is, $abc = 0$),*

ii) *and **any two components do not have the same sign** (that is, $ab \leq 0,\ ac \leq 0,\ and\ bc \leq 0$).*

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Corollary (*NSZ'02*)

Any address has a ***unique*** *shortest path form.*

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If $D - S = (a, b, c)$, then the shortest path form is one of those:

$$(0, b-a, c-a),$$

$$(a-b, 0, c-b),$$

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and thus:

$$|D - S| = \min(|b-a| + |c-a|, |a-b| + |c-b|, |a-c| + |b-c|).$$

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- Given a packet p and its relative address (a, b, c) *in the shortest path form*,

$$\ell_p := |a| + |b| + |c|,$$

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- Trivial **lower bound**:

  Any permutation routing algorithm needs at least ℓ_{max} routing steps.

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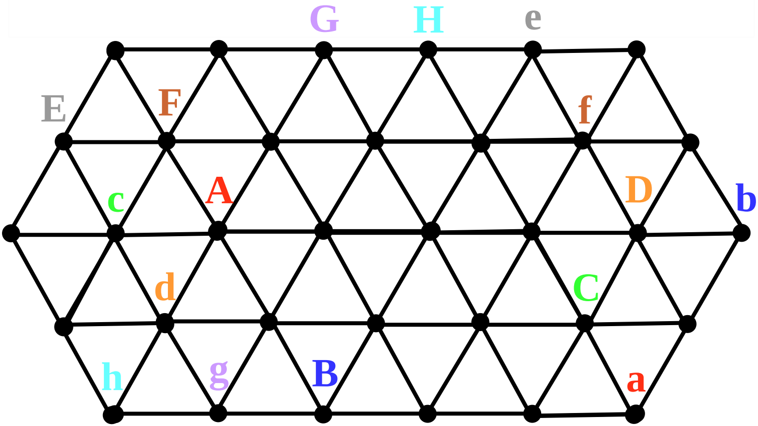
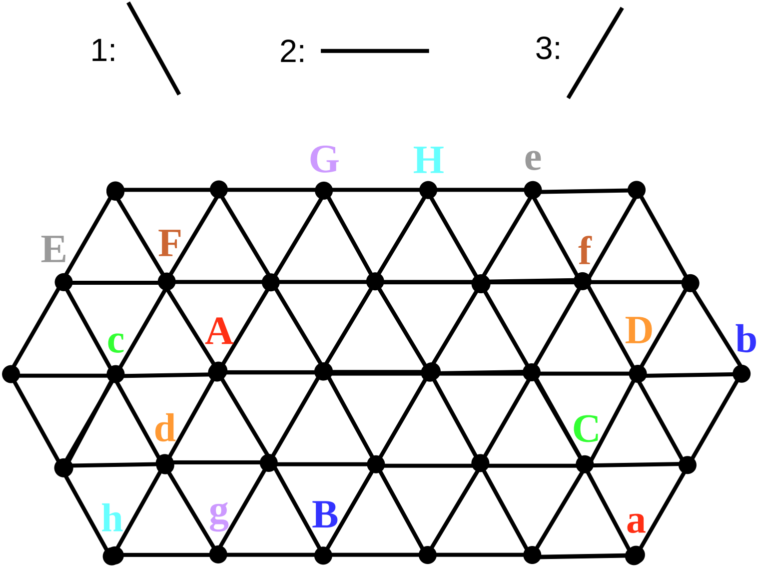
Example of an instance



A non-optimal intuitive algorithm



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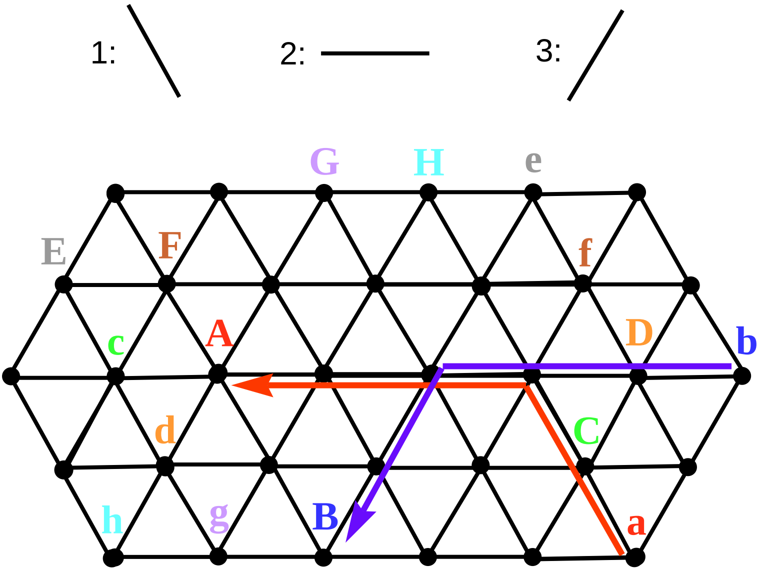

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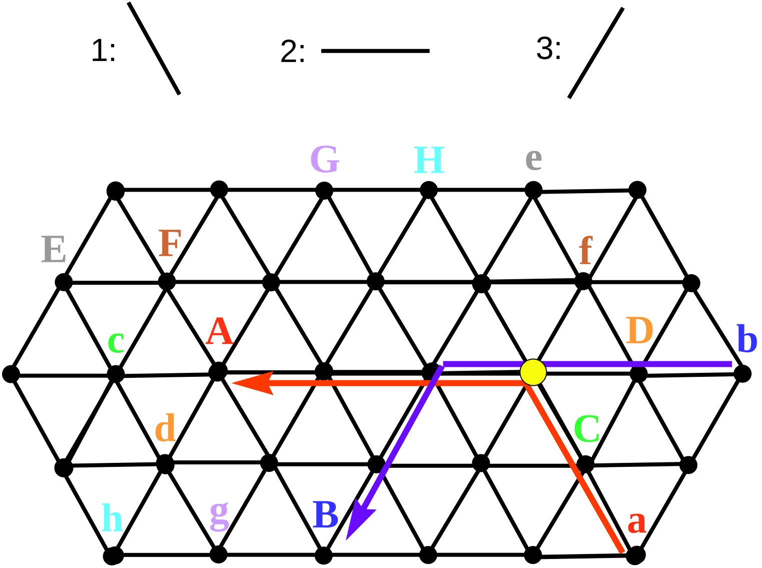

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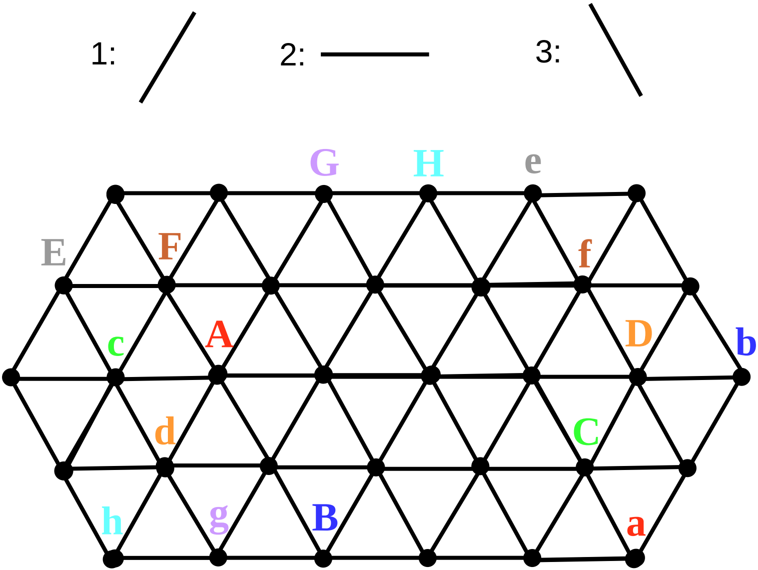

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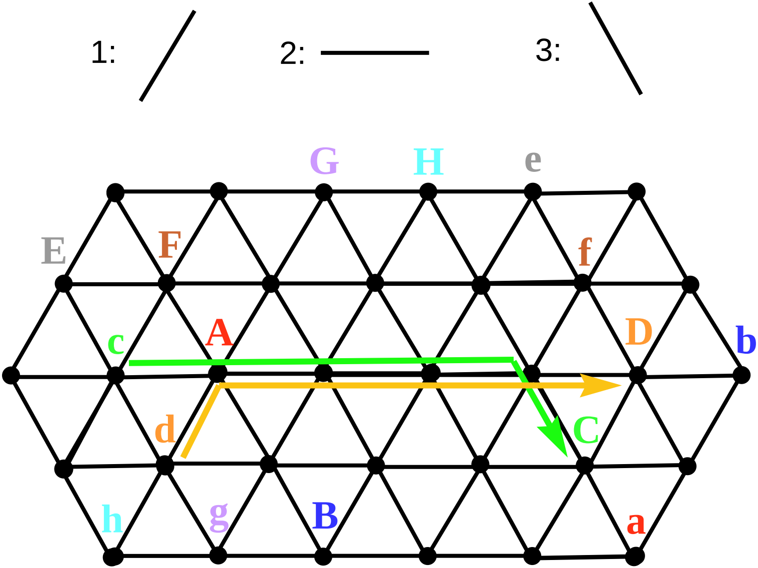

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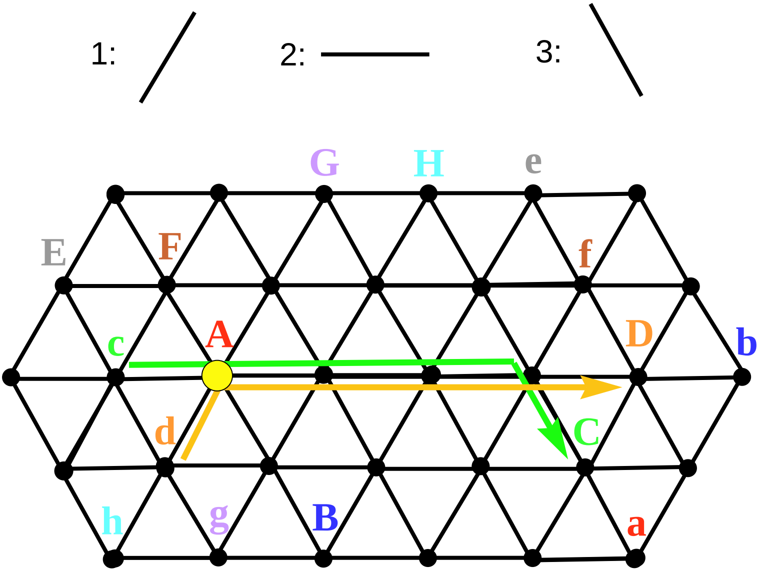

Description of Algorithm $\mathcal{A}$

At each node u of the network:

- ***Preprocessing*:** Initially, if there is a packet at u, compute the relative address $D - S$ of the message in the shortest path form, and add this information to the message.
- ***Reception phase*:** At each step, when a packet is received at u, its relative address is updated.
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  - **a)** If there are packets with **negative components**, send them **immediately** along the direction of this component.
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- Algorithm $\mathcal{A}$ defines for each packet **two directions of movement** (except if a packet has only one non-zero component)
- For instance:
  - if the packet address is of the type $(-, 0, +) \rightarrow$
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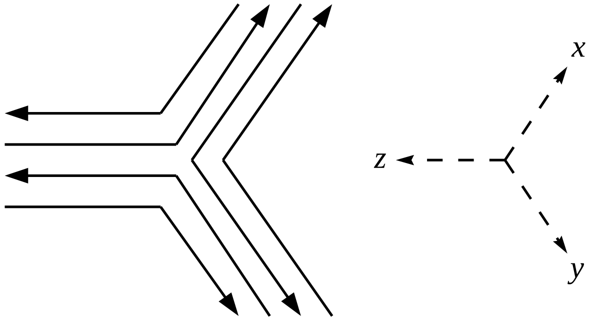
Routing the packets (2)

- In this figure all the routing rules are summarized:



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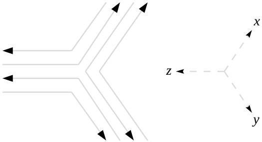
- *Key observation:*

  Packets can only **wait**, possibly, during their **last direction.**

  - this is because if two packets meet when their first direction is not finished yet, they must have the same origin node → contradiction.



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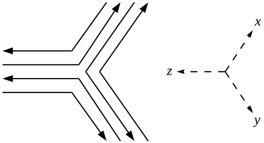

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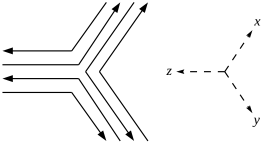

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Correctness (4)

- All the packets in in **b)** are moving along their **last direction**
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- Thus, since each node is the destination of at most one packet, in **b)** *the* **packet with maximum remaining length at each outgoing edge is unique.**

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Optimality

- Using this algorithm, at each step **all the packets with maximum remaining distance move**

  $\rightarrow$ every step the maximum remaining distance over all packets decreases by one

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Analysis

- It is a **distributed** algorithm, because each node needs only information about the packets that are its queues.
- It is an **oblivious** algorithm, since the routing of each packet depends only on the origin and destination nodes.
- It is a **translation invariant** algorithm, since only the relative address $D - S$ is necessary to route the packets.
- The only involved operations are *integer addition and comparison* among the lengths of the addresses of the packets at each node.
- Time complexity: $O(\ell_{max})$

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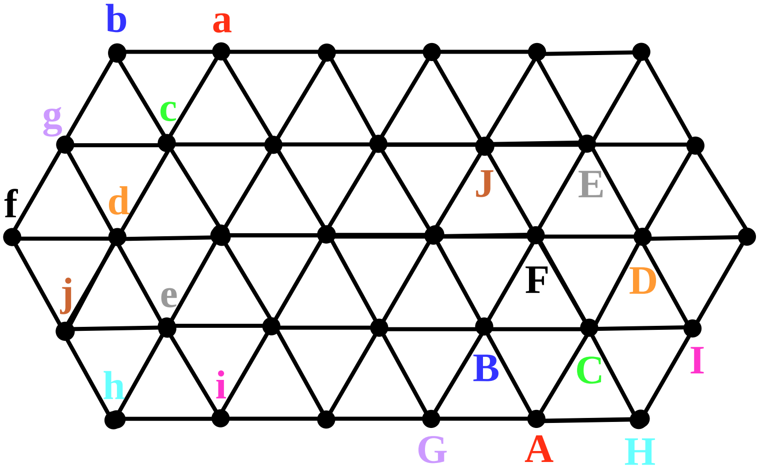

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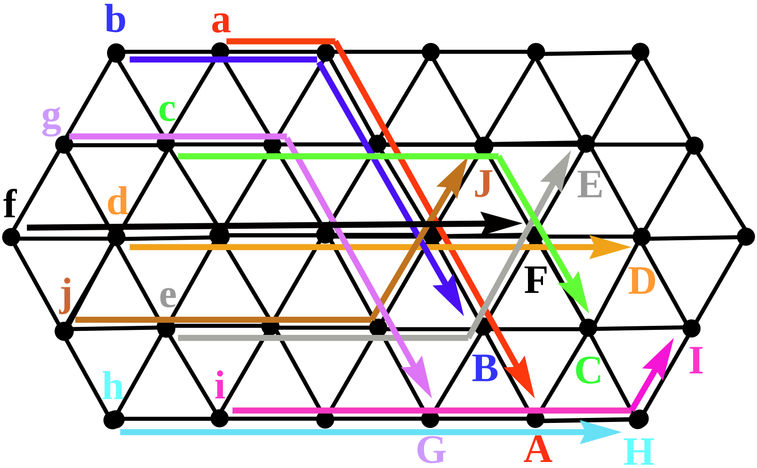
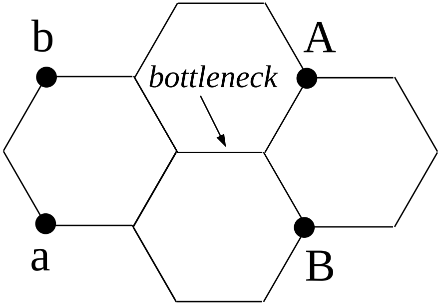


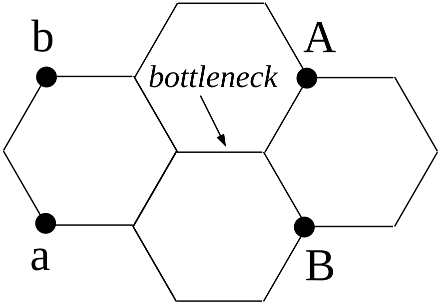
Permutation Routing on Hexagonal Grids

Counterexample



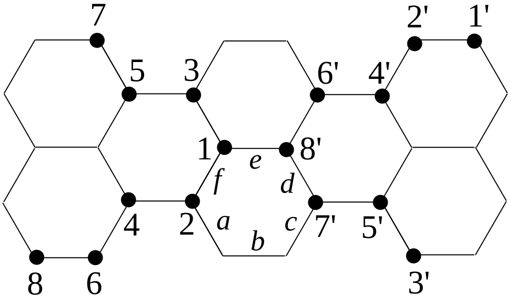
- $\ell_{max} = 4$, but it is not possible to route both packets in less than 5 steps.
- Thus, ℓ_{max} can not always be achieved!

Counterexample



- $\ell_{max} = 4$, but it is not possible to route both packets in less than 5 steps.
- Thus, ℓ_{max} can not always be achieved!

Another counterexample



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- Using edges $\{abcd\}$: 7 steps
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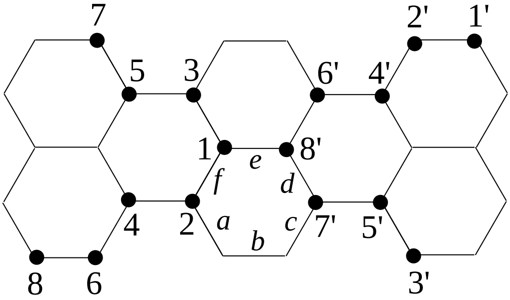

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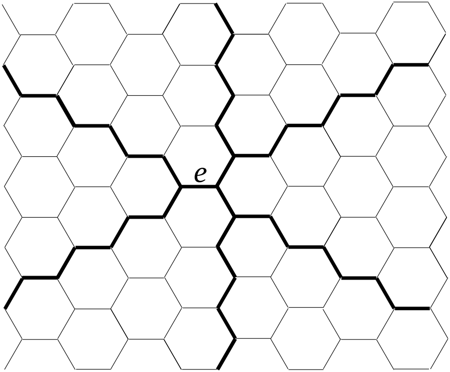
Hexagonal grid

- One can define 3 types of *zigzag chains*:



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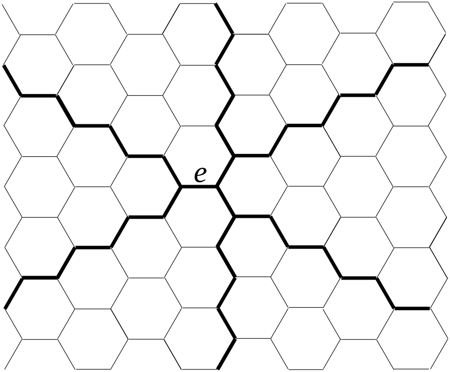

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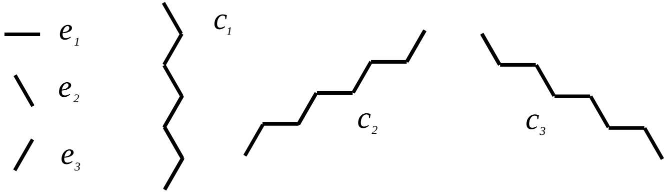
Idea

- There are 3 types of edges and 3 types of chains:



- Each edge belongs to exactly 2 different chains, and conversely each chain is made of 2 types of edges.
- Any 2 chains of different type intersect exactly on one edge.
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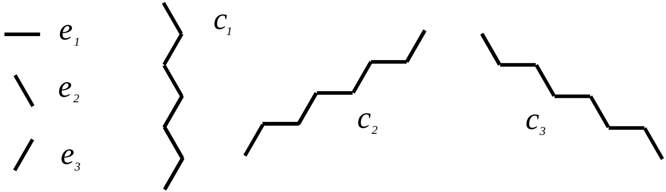

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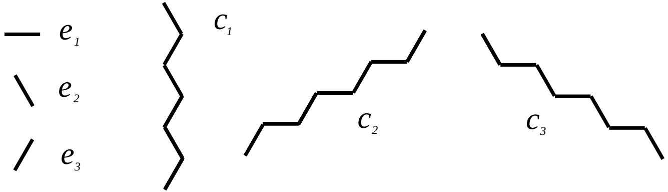

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Optimal algorithm

At each node of the network:

1) During the **first step**, move all packets along the direction of their **negative component**. If a packet's address has only a positive component, move it along this direction.

2) From now on, **change alternatively** between Phase 1 and Phase 2.

3) At each step (the same for both phases):

- **a)** If there are packets with negative components, send them immediately along the direction of this component.
- **b)** If not, for each outgoing edge order the packets according to decreasing number of remaining steps, and send the first packet of each queue.

4) At the end of the $(2\ell_{max} - 3)th$ step, move all packets along their unique non-zero component.

Running time

- Every 2 steps (one of Phase 1 and one of Phase 2) the maximum remaining distance over all packets decreases by one.
- During the first and last step all packets decrease their remaining distance by one.
- Thus, the total running time is $2 + 2(\ell_{max} - 2) = 2\ell_{max} - 2$.
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(ℓ, k)-Routing

Algorithm (in any grid)

- Each node can send at most ℓ packets and receive at most k packets
- **Idea**: represent the request set as a weighted bipartite graph H:
  - split each vertex of the original graph
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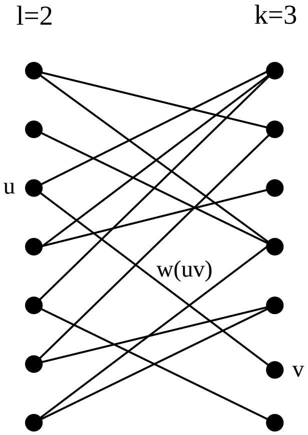

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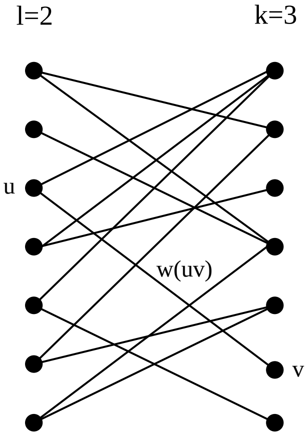

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- **Problem**: find $m := \max\{\ell, k\}$ matchings in H: $M_1, \ldots, M_m$
- Let $c(M_i) := \max\{w(e) | e \in M_i)\}$, $i = 1, \ldots, m$
- **Objective function**:

$$\min \sum_{i=1}^{m} c(M_i)$$

- **Fact**: $\min \sum_{i=1}^{m} c(M_i)$ is the running time of routing a (ℓ, k)-routing instance using this algorithm
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- Permutation routing on 3-circulant graphs is still a challenging open problem...

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Find Small Subgraphs with Given Minimum Degree

Hardness of approximation

- **Class APX (Approximable):**

  an optimization problem is in APX if can be approximated within a constant factor.

  Example: VERTEX COVER

- **Class PTAS (Polynomial-Time Approximation Scheme):**

  an optimization problem is in PTAS if can be approximated within a constant factor $1 + \varepsilon$, for *all* $\varepsilon > 0$
  (the best one can hope for an NP-complete problem).

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- We know that

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- Thus, if Π is an optimization problem:

$$\Pi \text{ is APX-hard} \quad \Rightarrow \quad \Pi \notin \text{PTAS}$$

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- **MINIMUM SUBGRAPH OF MINIMUM DEGREE** $\geq d$ **(MSMD$_d$)**:

  Let d be a natural number, and $G = (V, E)$ a given graph.

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Idea of the proof for $d = 3$

(1) We will see first that $\text{MSMD}_3 \notin \text{PTAS}$.

(2) After we will see that $\text{MSMD}_3 \notin \text{APX}$.

(1) *MSMD*$_3$ is not in *PTAS*

- Reduction from VERTEX COVER:

  Instance H of VERTEX COVER $\rightarrow$ Instance G of MSMD$_3$

- We will see that

$$\text{PTAS for } G \Rightarrow \text{PTAS for } H$$

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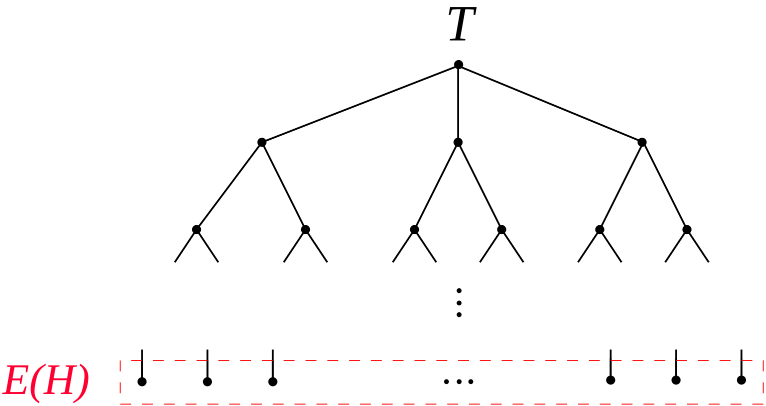
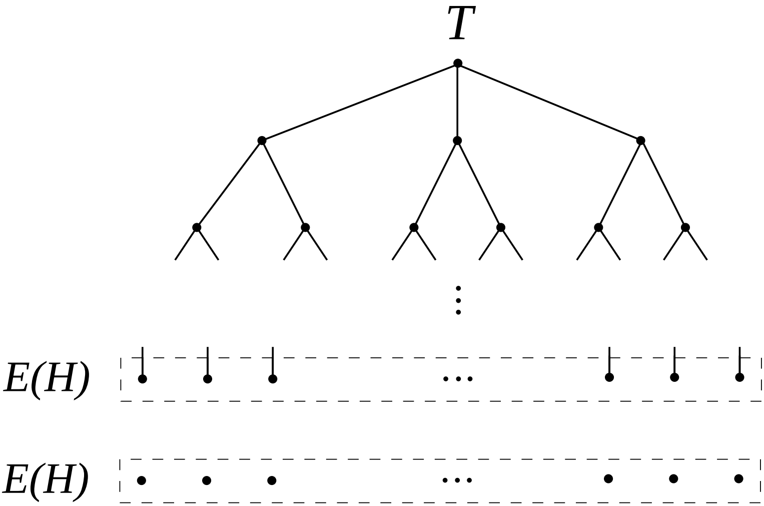
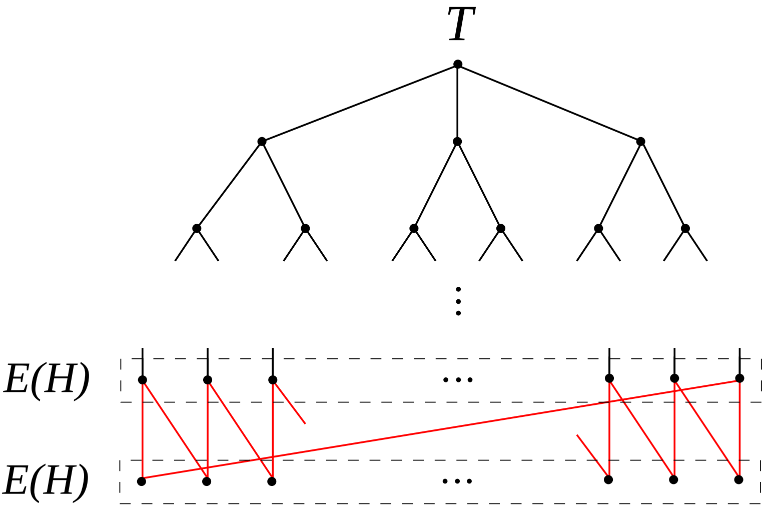
We build a complete ternary tree with $|E(H)| = 3 \cdot 2^m$ leaves:



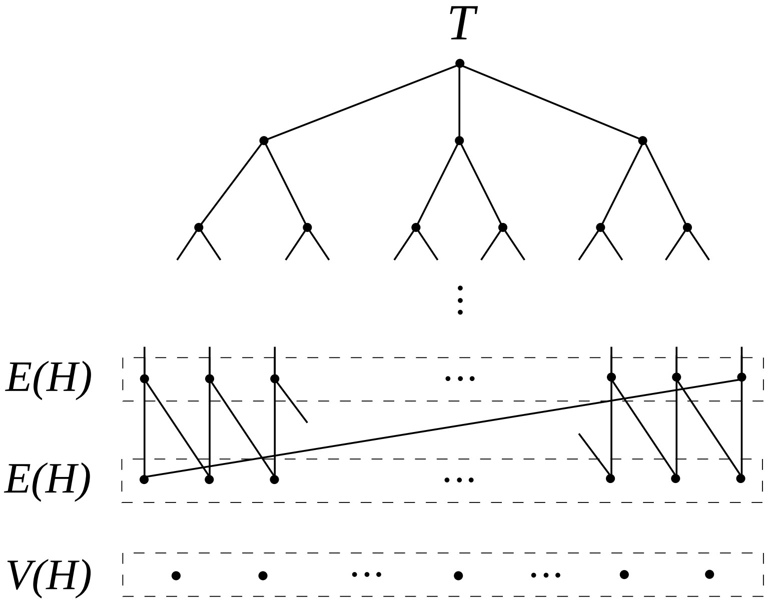
We add a copy of the set of leaves $E(H)$:



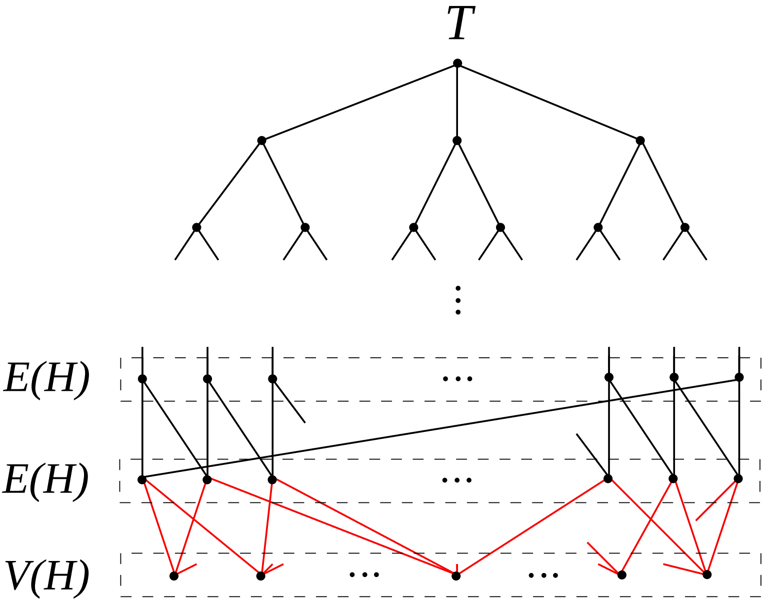
We join both sets with a Hamiltonian cycle (for technical reasons):



We add all the vertices of H:



We build the adjacency graph between $E(H)$ and $V(H)$:



(1) MSMD_3 is not in PTAS

- If we touch a vertex of $G \setminus V(H)$, we have to touch all the vertices of $G \setminus V(H)$

- Thus, MSMD_3 in G is equivalent to minimize the number of selected vertices in $V(H)$

  $\rightarrow$ this is **exactly** VERTEX COVER in H !!

- Thus,

$$OPT_{\mathrm{MSMD}_3}(G) = OPT_{\mathrm{VC}}(H) + |V(G \setminus V(H))| = OPT_{\mathrm{VC}}(H) + 9 \cdot 2^m$$

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- Let $\alpha > 1$ be the factor of innaproximability of MSMD_3
- We use a technique called **amplification of the error**:
  - We build a sequence of graphs G_k, such that MSMD_3 is hard to approximate in G_k within a factor α^k
  - This proves that the problem is not in APX

    (for any constant C, $\exists\ k > 0$ such that $\alpha^k > C$)
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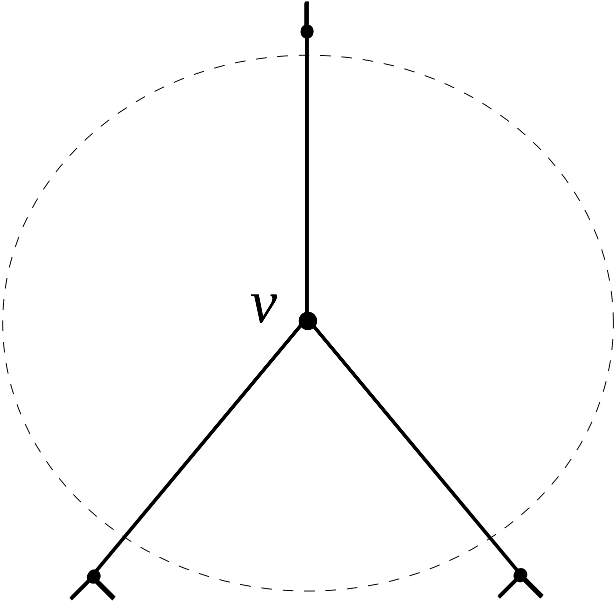
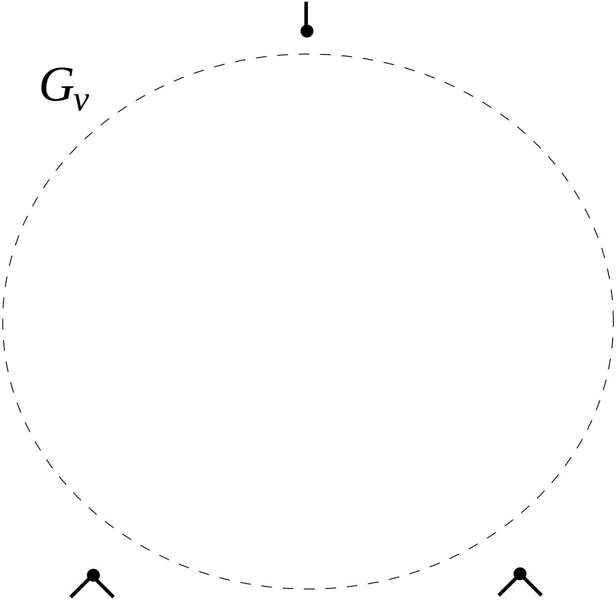
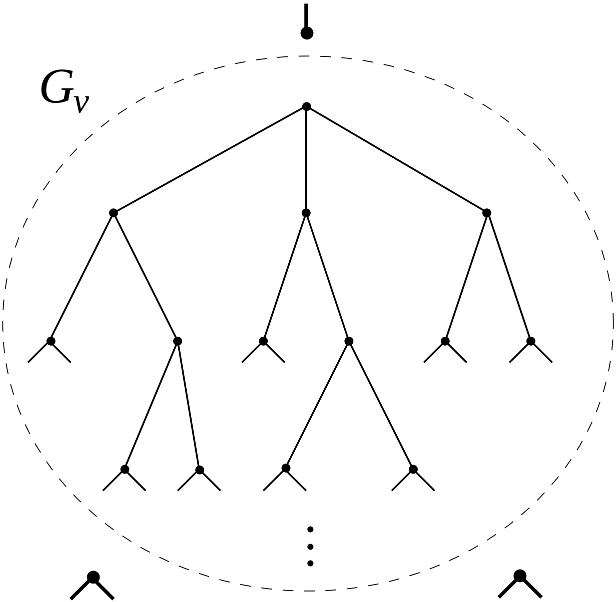
For any vertex v (note its degree by d_v):



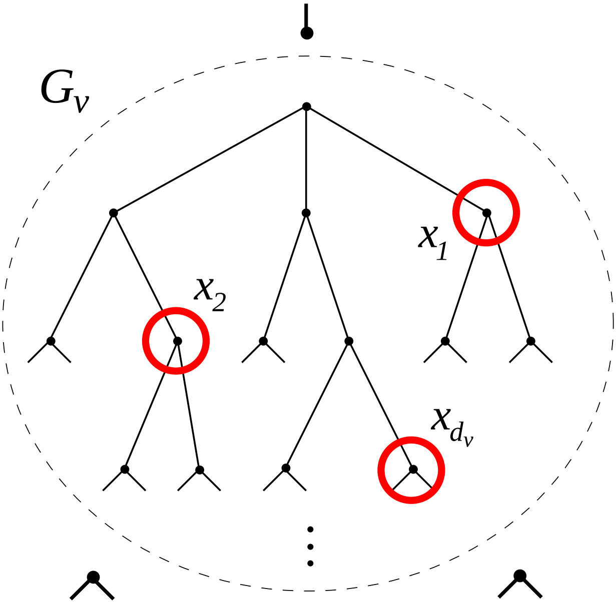
We will replace the vertex v with a graph G_v, built as follows:



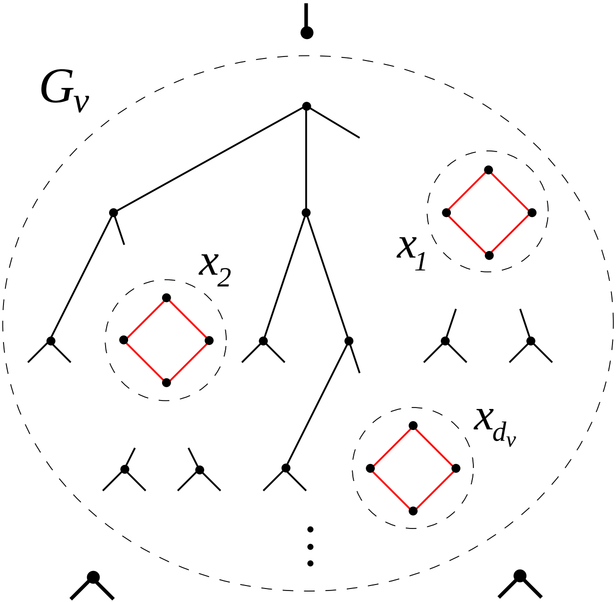
We begin by placing a copy of G (described before):



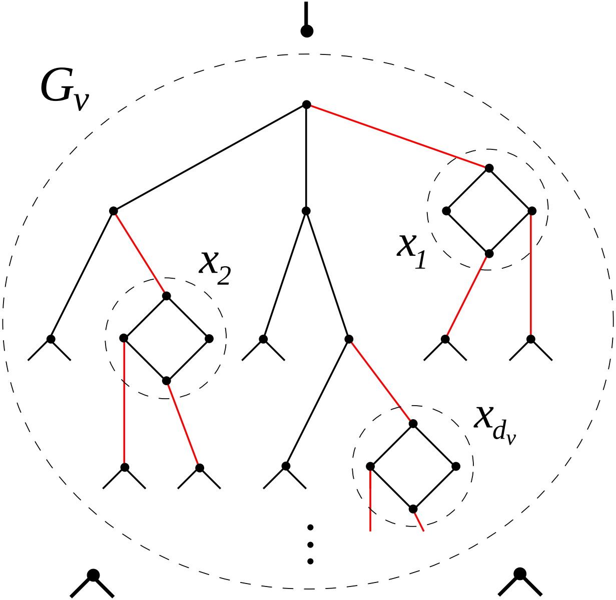
We select d_v vertices of degree 3 in $T \subset G$:



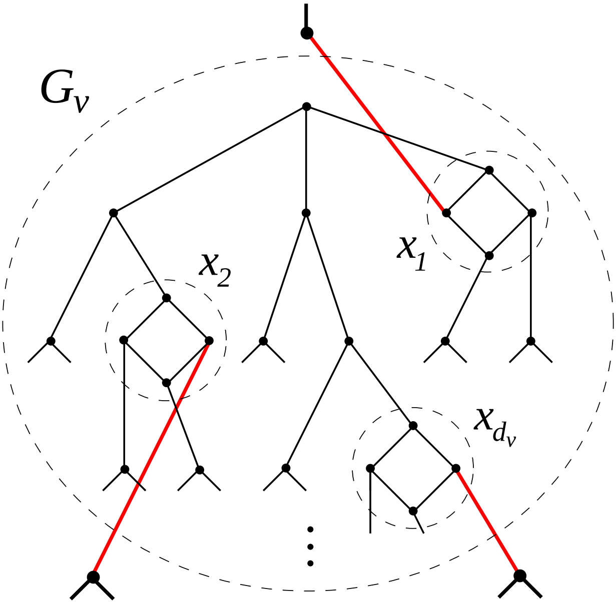
We replace each of these vertices x_i with a C_4:



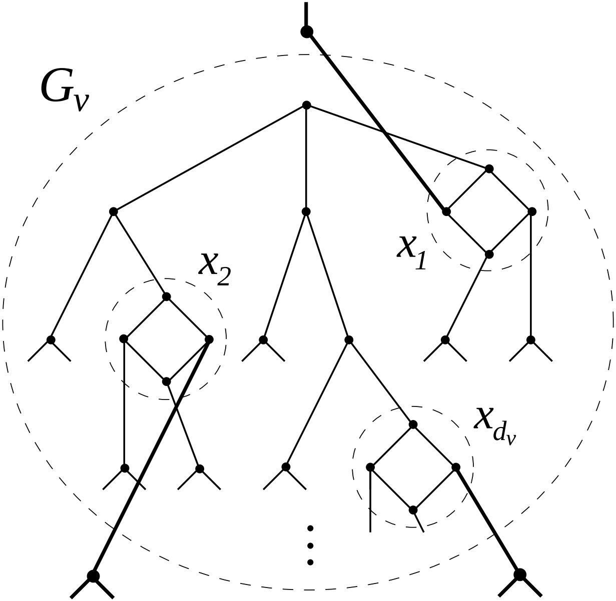
In each C_4, we join 3 of the vertices to the neighbors of x_i:



We join the d_v vertices of degree 2 to the d_v neighbors of v:



This construction for all $v \in G$ defines G_2:



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Conclusions

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- We have also proved that MSMD_d, $d \geq 3$, is W[1]-hard

  (and thus the problem is not FPT tractable in general graphs)
- We have FPT algorithms for minor free graphs

  (for instance: planar graphs, graphs of bounded local treewidth, graphs of bounded genus,...)
- We *almost* have PTAS for minor free graphs
- **Open problem**: find approximation algorithms for general graphs

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Thanks!