Edge-partitioning Regular Graphs (with Applications to Traffic Grooming)

Ignasi Sau

Postdoc at Department of Computer Science, Technion (Haifa, Israel)

Joint work with:

Xavier Muñoz

Dept. de Matemàtica Aplicada 4, Universitat Politècnica de Catalunya (Barcelona, Spain)

Zhentao Li

School of Computer Science, McGill University (Montreal, Canada)

Some of these results have been presented in:

- 34th Intern. Workshop on Graph-Theoretic Concepts in Computer Science (**WG 2008**)
- 35th Intern. Workshop on Graph-Theoretic Concepts in Computer Science (**WG 2009**)

Outline of the talk

1. Motivation: traffic grooming
2. Statement of the problem
3. The parameter $M(C, \Delta)$
4. Basic properties of $M(C, \Delta)$
5. Some results
   - Case $\Delta = 3$, $C = 4$
   - Case $\Delta \geq 4$ even
   - Case $\Delta \geq 5$ odd
   - Improved lower bound when $\Delta \equiv C \pmod{2C}$
6. Conclusions

Next section is...

1. Motivation: traffic grooming
2. Statement of the problem
3. The parameter $M(C, \Delta)$
4. Basic properties of $M(C, \Delta)$
5. Some results
6. Conclusions

Introduction

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s
- Idea:
  Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  ⟶ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)
- Objectives:
  - Better use of bandwidth
  - Reduce the equipment cost (mostly given by electronics)

Introduction

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s
- Idea:
  Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  $\longrightarrow$ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)
- Objectives:
  - Better use of bandwidth
  - Reduce the equipment cost (mostly given by electronics)

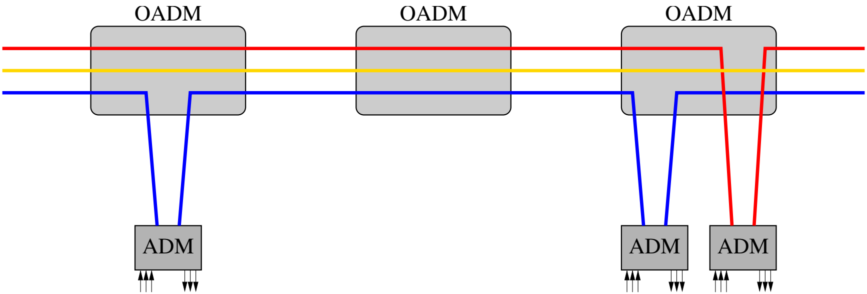
Introduction

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s
- Idea:
  Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  $\longrightarrow$ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)
- Objectives:
  - Better use of bandwidth
  - Reduce the equipment cost (mostly given by electronics)

ADM and OADM

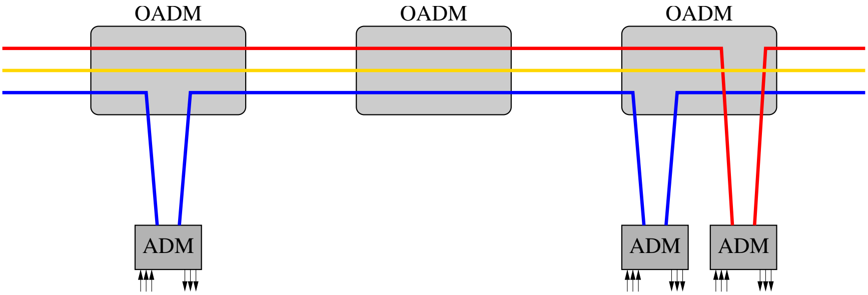
- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



⟶ we want to minimize the number of ADMs

ADM and OADM

- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



⟶ we want to minimize the number of ADMs

Definitions

- **Request** (i, j): two vertices (i, j) that want to exchange (low-speed) traffic

- **Grooming factor** C**:**

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

Example:

Capacity of one wavelength $= 2400\ Mb/s$
Capacity used by a request $= 600\ Mb/s$ $\Rightarrow\ C = 4$

- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

Definitions

- **Request** (i,j): two vertices (i,j) that want to exchange (low-speed) traffic
- **Grooming factor** C**:**

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

Example:

Capacity of one wavelength $= 2400\ Mb/s$
Capacity used by a request $= 600\ Mb/s$ $\Rightarrow\ C = 4$

- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

Definitions

- **Request** (i, j): two vertices (i, j) that want to exchange (low-speed) traffic

- **Grooming factor C:**

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

Example:

Capacity of one wavelength $= 2400\ Mb/s$
Capacity used by a request $= 600\ Mb/s$ $\quad \Rightarrow \quad C = 4$

- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

Definitions

- **Request** (i,j): two vertices (i,j) that want to exchange (low-speed) traffic
- **Grooming factor C:**

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

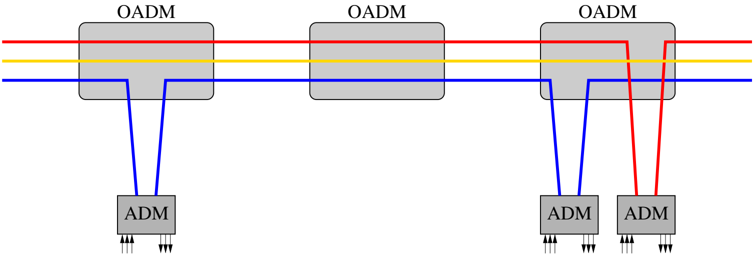
Example:

Capacity of one wavelength $= 2400\ Mb/s$
Capacity used by a request $= 600\ Mb/s$ $\Rightarrow\ C = 4$

- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

ADM and OADM

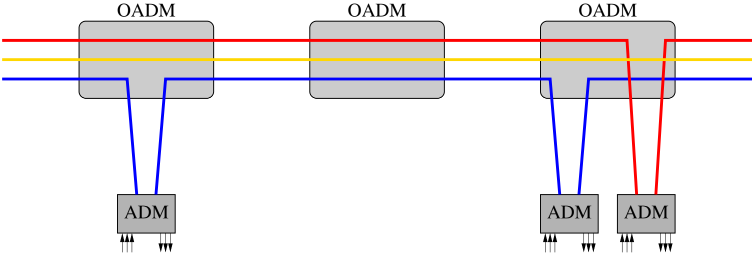
- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



- **Idea:** Use an **ADM only at the endpoints of a request** (lightpaths) in order to save as many ADMs as possible

ADM and OADM

- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



- **Idea:** Use an **ADM only at the endpoints of a request** (lightpaths) in order to save as many ADMs as possible

To fix ideas...

- Model:

Topology	$\longrightarrow$	graph G
Request set	$\longrightarrow$	graph R
Grooming factor	$\longrightarrow$	integer C
Requests in a **wavelength**	$\longrightarrow$	edges in a **subgraph** of R
ADM in a wavelength	$\longrightarrow$	vertex in a subgraph of R

- We study the case when $G = \overrightarrow{C}_n$ (**unidirectional ring**)
- We assume that the **requests** are **symmetric**

To fix ideas...

- Model:

Topology	$\longrightarrow$	graph G
Request set	$\longrightarrow$	graph R
Grooming factor	$\longrightarrow$	integer C
Requests in a **wavelength**	$\longrightarrow$	edges in a **subgraph** of R
ADM in a wavelength	$\longrightarrow$	vertex in a subgraph of R

- We study the case when $G = \overrightarrow{C}_n$ (**unidirectional ring**)
- We assume that the **requests** are **symmetric**

To fix ideas...

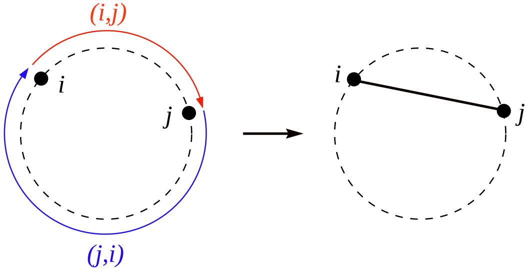
- Model:

Topology	$\longrightarrow$	graph G
Request set	$\longrightarrow$	graph R
Grooming factor	$\longrightarrow$	integer C
Requests in a **wavelength**	$\longrightarrow$	edges in a **subgraph** of R
ADM in a wavelength	$\longrightarrow$	vertex in a subgraph of R

- We study the case when $G = \overrightarrow{C}_n$ (**unidirectional ring**)
- We assume that the **requests** are **symmetric**

Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: we have both (i, j) and (j, i).

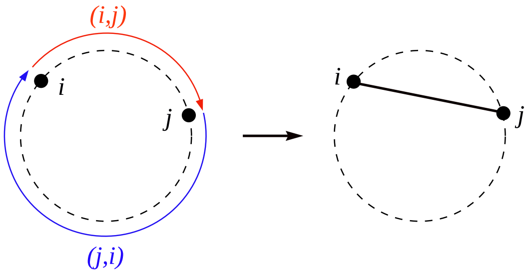


- W.l.o.g. requests (i, j) and (j, i) are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ **grooming factor** C $\Leftrightarrow$ **each subgraph has** $\leq C$ **edges.**

- C-**edge-partition** of a graph G:
  partition of $E(G)$ into subgraphs with at most C edges each.

Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: we have both (i, j) and (j, i).

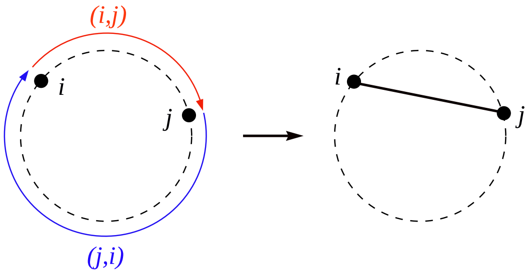


- W.l.o.g. requests (i, j) and (j, i) are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ **grooming factor C $\Leftrightarrow$ each subgraph has $\leq C$ edges.**

- **C-edge-partition** of a graph G:
  partition of $E(G)$ into subgraphs with at most C edges each.

Unidirectional Ring with Symmetric Requests

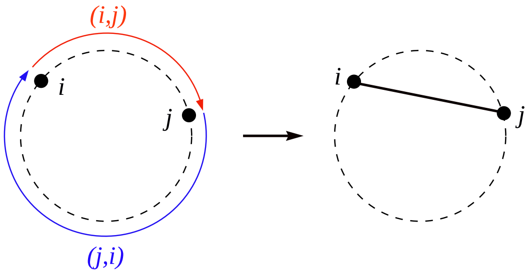
- **Symmetric requests**: we have both (i,j) and (j,i).



- W.l.o.g. requests (i,j) and (j,i) are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ **grooming factor C** $\Leftrightarrow$ **each subgraph has $\leq C$ edges.**
- ***C*-edge-partition** of a graph G:
  partition of $E(G)$ into subgraphs with at most C edges each.

Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: we have both (i,j) and (j,i).

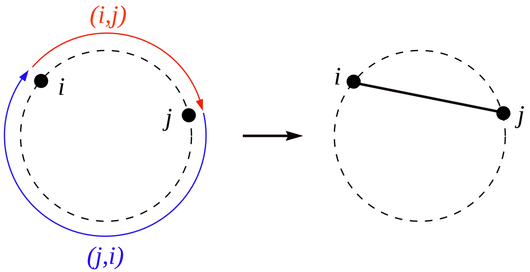


- W.l.o.g. requests (i,j) and (j,i) are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ **grooming factor $C \Leftrightarrow$ each subgraph has $\leq C$ edges.**

- *C*-**edge-partition** of a graph G:
  partition of $E(G)$ into subgraphs with at most C edges each.

Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: we have both (i, j) and (j, i).



- W.l.o.g. requests (i, j) and (j, i) are in the same subgraph
  $\rightarrow$ each pair of symmetric requests induces load 1
  $\rightarrow$ **grooming factor C $\Leftrightarrow$ each subgraph has $\leq C$ edges.**

- ***C*-edge-partition** of a graph G:
  partition of $E(G)$ into subgraphs with at most C edges each.

Statement of the "old" problem

Traffic Grooming in Unidirectional Rings

Input A cycle C_n on n nodes (network);
An *undirected* graph R on n nodes (request set);
A grooming factor C.

Output A C-edge-partition of R into subgraphs $R_1, \ldots, R_W$.

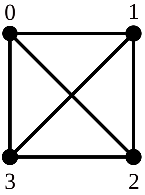
Objective Minimize $\sum_{\omega=1}^{W} |V(R_\omega)|$.

Example (unidirectional ring with symmetric requests)

$n = 4$

$R = K_4$

$C = 3$

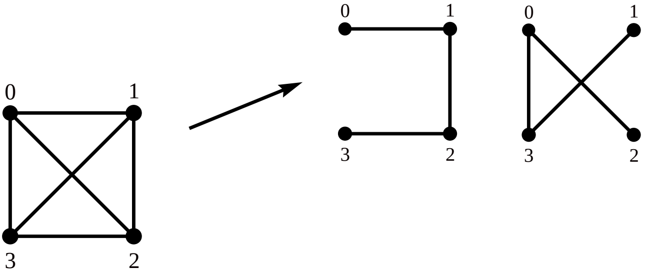


Example (unidirectional ring with symmetric requests)

$n = 4$

$R = K_4$

$C = 3$

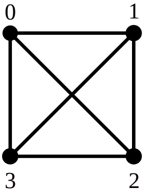


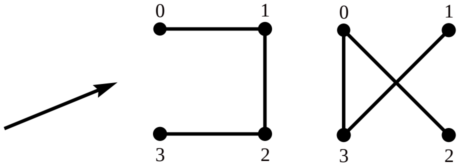
Example (unidirectional ring with symmetric requests)

$n = 4$

$R = K_4$

$C = 3$





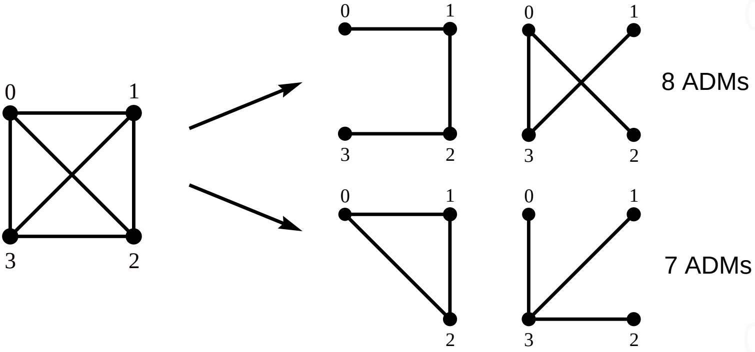
8 ADMs

Example (unidirectional ring with symmetric requests)

$n = 4$

$R = K_4$

$C = 3$



Next section is...

New model

- **Non-exhaustive** previous work (a lot!):
  - Bermond, Coudert, and Muñoz - **ONDM 2003**.
  - Bermond and Coudert - **ICC 2003**.
  - Bermond, Braud, and Coudert - **SIROCCO 2005**.
  - Bermond *et al.* - **SIDMA 2005**.
  - Flammini, Moscardelli, Shalom and Zaks - **ISAAC 2005**.
  - Flammini, Monaco, Moscardelli, Shalom and Zaks - **WG 2006**.
  - Amini, Pérennes and S. - **ISAAC 2007, TCS 2009**.
  - Bermond, Muñoz, and S. - Manusc. **2009**.
  - Bermond, Colbourn, Gionfriddo, Quattrocchi and S.- **SIDMA 2010**.

- In all of them: place ADMs at nodes for a **fixed request graph.**
  $\rightarrow$ placement of ADMs **a posteriori.**

- **New model**: place the ADMs at nodes such that the network can support **any request graph with maximum degree at most Δ.**
  $\rightarrow$ placement of ADMs **a priori.**

New model

- **Non-exhaustive** previous work (a lot!):
  - Bermond, Coudert, and Muñoz - **ONDM 2003**.
  - Bermond and Coudert - **ICC 2003**.
  - Bermond, Braud, and Coudert - **SIROCCO 2005**.
  - Bermond *et al*. - **SIDMA 2005**.
  - Flammini, Moscardelli, Shalom and Zaks - **ISAAC 2005**.
  - Flammini, Monaco, Moscardelli, Shalom and Zaks - **WG 2006**.
  - Amini, Pérennes and S. - **ISAAC 2007, TCS 2009**.
  - Bermond, Muñoz, and S. - Manusc. **2009**.
  - Bermond, Colbourn, Gionfriddo, Quattrocchi and S.- **SIDMA 2010**.

- In all of them: place ADMs at nodes for a **fixed request graph**.
  → placement of ADMs **a posteriori**.

- **New model**: place the ADMs at nodes such that the network can support **any request graph with maximum degree at most Δ**.
  → placement of ADMs **a priori**.

New model

- **Non-exhaustive** previous work (a lot!):
  - Bermond, Coudert, and Muñoz - **ONDM 2003**.
  - Bermond and Coudert - **ICC 2003**.
  - Bermond, Braud, and Coudert - **SIROCCO 2005**.
  - Bermond *et al.* - **SIDMA 2005**.
  - Flammini, Moscardelli, Shalom and Zaks - **ISAAC 2005**.
  - Flammini, Monaco, Moscardelli, Shalom and Zaks - **WG 2006**.
  - Amini, Pérennes and S. - **ISAAC 2007, TCS 2009**.
  - Bermond, Muñoz, and S. - Manusc. **2009**.
  - Bermond, Colbourn, Gionfriddo, Quattrocchi and S.- **SIDMA 2010**.

- In all of them: place ADMs at nodes for a **fixed request graph.**
  $\rightarrow$ placement of ADMs **a posteriori**.

- **New model**: place the ADMs at nodes such that the network can support **any request graph with maximum degree at most Δ.**
  $\rightarrow$ placement of ADMs **a priori**.

New model

- **Non-exhaustive** previous work (a lot!):
  - Bermond, Coudert, and Muñoz - **ONDM 2003**.
  - Bermond and Coudert - **ICC 2003**.
  - Bermond, Braud, and Coudert - **SIROCCO 2005**.
  - Bermond *et al.* - **SIDMA 2005**.
  - Flammini, Moscardelli, Shalom and Zaks - **ISAAC 2005**.
  - Flammini, Monaco, Moscardelli, Shalom and Zaks - **WG 2006**.
  - Amini, Pérennes and S. - **ISAAC 2007, TCS 2009**.
  - Bermond, Muñoz, and S. - Manusc. **2009**.
  - Bermond, Colbourn, Gionfriddo, Quattrocchi and S.- **SIDMA 2010**.

- In all of them: place ADMs at nodes for a **fixed request graph.**
  $\rightarrow$ placement of ADMs **a posteriori**.

- **New model**: place the ADMs at nodes such that the network can support **any request graph with maximum degree at most Δ.**
  $\rightarrow$ placement of ADMs **a priori**.

Statement of the "new" problem

Traffic Grooming in Unidirectional Rings with Bounded-Degree Request Graph

Input An integer n (size of the ring);
An integer C (grooming factor);
An integer Δ (maximum degree).

Output An assignment of $A(v)$ ADMs to each $v \in V(C_n)$, in such a way that **for any graph** R on n nodes with **maximum degree at most** Δ, there exists a C-edge-partition of R into subgraphs $R_1, \ldots, R_W$ s.t. each $v \in V(C_n)$ is in at most $A(v)$ subgraphs.

Objective Minimize $\sum_{v \in V(C_n)} A(v)$,
and the optimum is denoted $A(n, C, \Delta)$.

Next section is...

$M(C, \Delta)$

Definition

Let $M(C, \Delta)$ be the smallest number M such that, for all $n \geq 1$, the inequality $A(n, C, \Delta) \leq M \cdot n$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** v, except for a subset whose size is **independent of** n.
- $M(C, \Delta)$ is always an **integer**.
- Equivalently:

  $M(C, \Delta)$ is the smallest integer M such that the edges of **any** graph with maximum degree at most Δ can be C-edge-partitioned in such a way that each vertex appears in at most M subgraphs.

- In the sequel we focus on determining $M(C, \Delta)$.

$M(C, \Delta)$

Definition

Let $M(C, \Delta)$ be the smallest number M such that, for all $n \geq 1$, the inequality $A(n, C, \Delta) \leq M \cdot n$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** v, except for a subset whose size is **independent of** n.
- $M(C, \Delta)$ is always an **integer**.
- Equivalently:

  $M(C, \Delta)$ is the smallest integer M such that the edges of **any** graph with maximum degree at most Δ can be C-edge-partitioned in such a way that each vertex appears in at most M subgraphs.

- In the sequel we focus on determining $M(C, \Delta)$.

$M(C,\Delta)$

Definition

Let $M(C,\Delta)$ be the smallest number M such that, for all $n \geq 1$, the inequality $A(n,C,\Delta) \leq M \cdot n$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** v, except for a subset whose size is **independent of** n.
- $M(C,\Delta)$ is always an **integer**.
- Equivalently:

  $M(C,\Delta)$ is the smallest integer M such that the edges of **any** graph with maximum degree at most Δ can be C-edge-partitioned in such a way that each vertex appears in at most M subgraphs.

- In the sequel we focus on determining $M(C,\Delta)$.

$M(C, \Delta)$

Definition

Let $M(C, \Delta)$ be the smallest number M such that, for all $n \geq 1$, the inequality $A(n, C, \Delta) \leq M \cdot n$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** v, except for a subset whose size is **independent of** n.
- $M(C, \Delta)$ is always an **integer**.
- Equivalently:

  $M(C, \Delta)$ is the smallest integer M such that the edges of **any** graph with maximum degree at most Δ can be C-edge-partitioned in such a way that each vertex appears in at most M subgraphs.

- In the sequel we focus on determining $M(C, \Delta)$.

$M(C, \Delta)$

Definition

Let $M(C, \Delta)$ be the smallest number M such that, for all $n \geq 1$, the inequality $A(n, C, \Delta) \leq M \cdot n$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** v, except for a subset whose size is **independent of** n.
- $M(C, \Delta)$ is always an **integer**.
- Equivalently:

  $M(C, \Delta)$ is the smallest integer M such that the edges of **any** graph with maximum degree at most Δ can be C-edge-partitioned in such a way that each vertex appears in at most M subgraphs.

- In the sequel we focus on determining $M(C, \Delta)$.

More formally... (we can forget traffic grooming)

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most Δ.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of C-edge-partitions of G.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- If the graphs are restricted to belong to a subclass $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding positive integer is denoted by $M(C, \Delta, \mathcal{C})$.

More formally... (we can forget traffic grooming)

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most Δ.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of C-edge-partitions of G.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- If the graphs are restricted to belong to a subclass $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding positive integer is denoted by $M(C, \Delta, \mathcal{C})$.

More formally... (we can forget traffic grooming)

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most Δ.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of C-edge-partitions of G.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- If the graphs are restricted to belong to a subclass $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding positive integer is denoted by $M(C, \Delta, \mathcal{C})$.

More formally... (we can forget traffic grooming)

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most Δ.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of C-edge-partitions of G.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- If the graphs are restricted to belong to a subclass $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding positive integer is denoted by $M(C, \Delta, \mathcal{C})$.

More formally... (we can forget traffic grooming)

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most Δ.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of C-edge-partitions of G.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- If the graphs are restricted to belong to a subclass $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding positive integer is denoted by $M(C, \Delta, \mathcal{C})$.

Next section is...

Basic properties of $M(C, \Delta)$

- As any graph in $\mathcal{G}_\Delta$ is a subgraph of a Δ-regular graph, $M(C, \Delta) = M(C, \Delta, \mathcal{C})$, where $\mathcal{C}$ is the class of Δ-regular graphs.
- From now on, we only deal with **Δ-regular graphs**.

- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.

- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Basic properties of $M(C, \Delta)$

- As any graph in $\mathcal{G}_\Delta$ is a subgraph of a Δ-regular graph, $M(C, \Delta) = M(C, \Delta, \mathcal{C})$, where $\mathcal{C}$ is the class of Δ-regular graphs.
- From now on, we only deal with **Δ-regular graphs**.
- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.
- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Basic properties of $M(C, \Delta)$

- As any graph in $\mathcal{G}_\Delta$ is a subgraph of a Δ-regular graph, $M(C, \Delta) = M(C, \Delta, \mathcal{C})$, where $\mathcal{C}$ is the class of Δ-regular graphs.
- From now on, we only deal with **Δ-regular graphs**.

- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.

- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ for all $C, \Delta \geq 1$.

Basic properties of $M(C, \Delta)$

- As any graph in $\mathcal{G}_\Delta$ is a subgraph of a Δ-regular graph, $M(C, \Delta) = M(C, \Delta, \mathcal{C})$, where $\mathcal{C}$ is the class of Δ-regular graphs.
- From now on, we only deal with **Δ-regular graphs**.
- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.
- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Basic properties of $M(C, \Delta)$

- As any graph in $\mathcal{G}_\Delta$ is a subgraph of a Δ-regular graph, $M(C, \Delta) = M(C, \Delta, \mathcal{C})$, where $\mathcal{C}$ is the class of Δ-regular graphs.
- From now on, we only deal with **Δ-regular graphs**.
- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.
- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Basic properties of $M(C, \Delta)$

- As any graph in $\mathcal{G}_\Delta$ is a subgraph of a Δ-regular graph, $M(C, \Delta) = M(C, \Delta, \mathcal{C})$, where $\mathcal{C}$ is the class of Δ-regular graphs.
- From now on, we only deal with **Δ-regular graphs**.
- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.
- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Basic properties of $M(C,\Delta)$

- As any graph in $\mathcal{G}_\Delta$ is a subgraph of a Δ-regular graph, $M(C,\Delta) = M(C,\Delta,\mathcal{C})$, where $\mathcal{C}$ is the class of Δ-regular graphs.
- From now on, we only deal with **Δ-regular graphs**.
- $C \geq C' \Rightarrow M(C,\Delta) \leq M(C',\Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C,\Delta) \geq M(C,\Delta')$ for all $C \geq 1$.
- **Upper bound:** $M(C,\Delta) \leq M(1,\Delta) = \Delta$.

Proposition (Lower Bound)

$M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$ *for all* $C,\Delta \geq 1$.

Lower bound

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C + 1$.
  (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta \cdot |V(G)|}{2} \cdot \frac{C+1}{C} = \frac{C+1}{C} \frac{\Delta}{2} \cdot |V(G)|$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C} \frac{\Delta}{2}$ subgraphs.

Lower bound

Proposition (Lower Bound)

$M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C+1$.
  (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta\cdot|V(G)|}{2}\cdot\frac{C+1}{C} = \frac{C+1}{C}\frac{\Delta}{2}\cdot|V(G)|$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C}\frac{\Delta}{2}$ subgraphs.

Lower bound

Proposition (Lower Bound)

$M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C+1$.
  (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta \cdot |V(G)|}{2} \cdot \frac{C+1}{C} = \frac{C+1}{C}\frac{\Delta}{2}|V(G)|$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C}\frac{\Delta}{2}$ subgraphs.

Lower bound

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C + 1$.
  (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta \cdot |V(G)|}{2} \cdot \frac{C+1}{C} = \frac{C+1}{C} \frac{\Delta}{2} \cdot |V(G)|.$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C} \frac{\Delta}{2}$ subgraphs.

Lower bound

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C+1$. (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta \cdot |V(G)|}{2} \cdot \frac{C+1}{C} = \frac{C+1}{C} \frac{\Delta}{2} \cdot |V(G)|.$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C} \frac{\Delta}{2}$ subgraphs.

Lower bound

Proposition (Lower Bound)

$M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C+1$.
  (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta \cdot |V(G)|}{2} \cdot \frac{C+1}{C} = \frac{C+1}{C}\frac{\Delta}{2} \cdot |V(G)|.$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C}\frac{\Delta}{2}$ subgraphs.

Lower bound

Proposition (Lower Bound)

$M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C+1$.
  (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta \cdot |V(G)|}{2} \cdot \frac{C+1}{C} = \frac{C+1}{C}\frac{\Delta}{2} \cdot |V(G)|.$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C}\frac{\Delta}{2}$ subgraphs.

□

Lower bound

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Sketch of the proof.

- Let G be a Δ-regular graph with girth at least $C + 1$.
  (Such a graph G exists by [Erdős and Sachs, 1963].)
- Then, the subgraphs involved in any C-edge-partition of G are trees.
- The total number of vertices of any C-edge-partition is at least

$$\frac{\Delta \cdot |V(G)|}{2} \cdot \frac{C+1}{C} = \frac{C+1}{C} \frac{\Delta}{2} \cdot |V(G)|.$$

- Therefore, there is a vertex which appears in at least $\frac{C+1}{C} \frac{\Delta}{2}$ subgraphs.

$\square$

Basic properties of $M(C, \Delta)$ (II)

- $\Delta = 1$: $M(C, 1) = 1$ for all C (trivial).
- $\Delta = 2$: $M(C, 2) = 2$ for all C (not difficult).
- $\Delta = 3$: Cubic graphs. First “interesting” case:
  - If $C \leq 3$, then $M(C, 3) = 3$.
  - If $C \geq 5$, then $M(C, 3) = 2$ (using the **linear arboricity**).
  - If $C = 4$: $M(3, 4) = 2$ or 3 ???

Basic properties of $M(C, \Delta)$ (II)

- $\Delta = 1$: $M(C, 1) = 1$ for all C (trivial).
- $\Delta = 2$: $M(C, 2) = 2$ for all C (not difficult).
- $\Delta = 3$: Cubic graphs. First “interesting” case:
  - If $C \leq 3$, then $M(C, 3) = 3$.
  - If $C \geq 5$, then $M(C, 3) = 2$ (using the **linear arboricity**).
  - If $C = 4$: $M(3, 4) = 2$ or 3 ???

Basic properties of $M(C, \Delta)$ (II)

- $\Delta = 1$: $M(C, 1) = 1$ for all C (trivial).
- $\Delta = 2$: $M(C, 2) = 2$ for all C (not difficult).
- $\Delta = 3$: Cubic graphs. First "interesting" case:
  - If $C \leq 3$, then $M(C, 3) = 3$.
  - If $C \geq 5$, then $M(C, 3) = 2$ (using the **linear arboricity**).
  - If $C = 4$: $M(3, 4) = 2$ or 3 ???

Basic properties of $M(C, \Delta)$ (II)

- $\Delta = 1$: $M(C, 1) = 1$ for all C (trivial).
- $\Delta = 2$: $M(C, 2) = 2$ for all C (not difficult).
- $\Delta = 3$: Cubic graphs. First "interesting" case:
  - If $C \leq 3$, then $M(C, 3) = 3$.
  - If $C \geq 5$, then $M(C, 3) = 2$ (using the **linear arboricity**).
  - If $C = 4$: $M(3, 4) = 2$ or 3 ???

Basic properties of $M(C, \Delta)$ (II)

- $\Delta = 1$: $M(C, 1) = 1$ for all C (trivial).
- $\Delta = 2$: $M(C, 2) = 2$ for all C (not difficult).
- $\Delta = 3$: Cubic graphs. First "interesting" case:
  - If $C \leq 3$, then $M(C, 3) = 3$.
  - If $C \geq 5$, then $M(C, 3) = 2$ (using the **linear arboricity**).
  - If $C = 4$: $M(3, 4) = 2$ or 3 ???

Next section is...

Next subsection is...

Case $\Delta = 3$, $C = 4$

Proposition

$M(4,3) = 2$.

Idea of the proof.

(in fact, we prove a slightly stronger result)

- Let G be a minimal counterexample (i.e., $|V(G)|$ is minimum).
- If G has no bridges, then it can be "easily" proved.
- If G has a bridge e, then the property is true for U and V.
- Finally, we merge "carefully" the partitions of U and V to obtain a partition of G $\Rightarrow$ contradiction.

Case $\Delta = 3$, $C = 4$

Proposition

$M(4, 3) = 2$.

Idea of the proof.

(in fact, we prove a slightly stronger result)

- Let G be a minimal counterexample (i.e., $|V(G)|$ is minimum).
- If G has no bridges, then it can be "easily" proved.
- If G has a bridge e, then the property is true for U and V.
- Finally, we merge "carefully" the partitions of U and V to obtain a partition of G $\Rightarrow$ contradiction.

Case $\Delta = 3$, $C = 4$

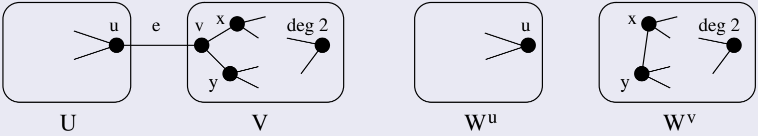
Proposition

$M(4,3) = 2$.

Idea of the proof.

(in fact, we prove a slightly stronger result)

- Let G be a minimal counterexample (i.e., $|V(G)|$ is minimum).
- If G has no bridges, then it can be "easily" proved.
- If G has a bridge e, then the property is true for U and V.



(a) (b)

- Finally, we merge "carefully" the partitions of U and V to obtain a partition of G $\Rightarrow$ contradiction.

Case $\Delta = 3$, $C = 4$

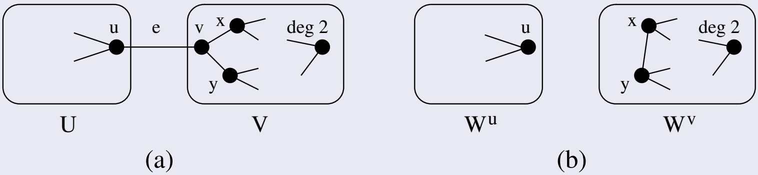
Proposition

$M(4, 3) = 2$.

Idea of the proof.

(in fact, we prove a slightly stronger result)

- Let G be a minimal counterexample (i.e., $|V(G)|$ is minimum).
- If G has no bridges, then it can be "easily" proved.
- If G has a bridge e, then the property is true for U and V.



- Finally, we merge "carefully" the partitions of U and V to obtain a partition of G $\Rightarrow$ contradiction.

Next subsection is...

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- We have already seen the lower bound.
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2}\left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

□

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$.

Proof.

- We have already seen the lower bound.
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2}\left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil.$$

□

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- We have already seen the lower bound.
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2} \left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil .$$

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- We have already seen the lower bound.
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2}\left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- We have already seen the lower bound.
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2} \left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

□

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- We have already seen the lower bound.
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2} \left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil .$$

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- We have already seen the lower bound.
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2} \left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

□

Next subsection is...

Case $\Delta \geq 5$ odd

Proposition

Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Sketch of proof.

- Since Δ is odd, $|V(G)|$ is even. Add a perfect matching M to G to obtain a $(\Delta+1)$-regular multigraph G'. Orient the edges of G' in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta+1)/2$ out-edges E_v^+.
- Remove M and partition E_v^+ into stars with C edges.
- Number of occurrences of each vertex $v \in V(G)$:
    - If an edge of M is in E_v^+, then: $\left\lceil \frac{\Delta-1}{2C} \right\rceil + \Delta - \frac{\Delta-1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.
    - Otherwise, if no edge of M is in E_v^+, then: $\left\lceil \frac{\Delta+1}{2C} \right\rceil + \Delta - \frac{\Delta+1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{1-C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Case $\Delta \geq 5$ odd

Proposition

Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Sketch of proof.

- Since Δ is odd, $|V(G)|$ is even. Add a perfect matching M to G to obtain a $(\Delta+1)$-regular multigraph G'. Orient the edges of G' in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta+1)/2$ out-edges E_v^+.
- Remove M and partition E_v^+ into stars with C edges.
- Number of occurrences of each vertex $v \in V(G)$:
  - If an edge of M is in E_v^+, then: $\left\lceil \frac{\Delta-1}{2C} \right\rceil + \Delta - \frac{\Delta-1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.
  - Otherwise, if no edge of M is in E_v^+, then:

$$\left\lceil \frac{\Delta+1}{2C} \right\rceil + \Delta - \frac{\Delta+1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{1-C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil.$$

Case $\Delta \geq 5$ odd

Proposition

Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C,\Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Sketch of proof.

- Since Δ is odd, $|V(G)|$ is even. Add a perfect matching M to G to obtain a $(\Delta+1)$-regular multigraph G'. Orient the edges of G' in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta+1)/2$ out-edges E_v^+.
- Remove M and partition E_v^+ into stars with C edges.
- Number of occurrences of each vertex $v \in V(G)$:
  - If an edge of M is in E_v^+, then: $\left\lceil \frac{\Delta-1}{2C} \right\rceil + \Delta - \frac{\Delta-1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.
  - Otherwise, if no edge of M is in E_v^+, then:

    $\left\lceil \frac{\Delta+1}{2C} \right\rceil + \Delta - \frac{\Delta+1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{1-C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Case $\Delta \geq 5$ odd

Proposition

Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Sketch of proof.

- Since Δ is odd, $|V(G)|$ is even. Add a perfect matching M to G to obtain a $(\Delta+1)$-regular multigraph G'. Orient the edges of G' in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta+1)/2$ out-edges E_v^+.
- Remove M and partition E_v^+ into stars with C edges.
- Number of occurrences of each vertex $v \in V(G)$:
  - If an edge of M is in E_v^+, then: $\left\lceil \frac{\Delta-1}{2C} \right\rceil + \Delta - \frac{\Delta-1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.
  - Otherwise, if no edge of M is in E_v^+, then: $\left\lceil \frac{\Delta+1}{2C} \right\rceil + \Delta - \frac{\Delta+1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{1-C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

□

Case $\Delta \geq 5$ odd

Proposition

Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Sketch of proof.

- Since Δ is odd, $|V(G)|$ is even. Add a perfect matching M to G to obtain a $(\Delta+1)$-regular multigraph G'. Orient the edges of G' in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta+1)/2$ out-edges E_v^+.
- Remove M and partition E_v^+ into stars with C edges.
- Number of occurrences of each vertex $v \in V(G)$:
  - If an edge of M is in E_v^+, then: $\left\lceil \frac{\Delta-1}{2C} \right\rceil + \Delta - \frac{\Delta-1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.
  - Otherwise, if no edge of M is in E_v^+, then:
    $\left\lceil \frac{\Delta+1}{2C} \right\rceil + \Delta - \frac{\Delta+1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{1-C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

$\square$

Case $\Delta \geq 5$ odd (II)

Corollary

Let $\Delta \geq 5$ be odd. If $\Delta \pmod{2C} = 1$ or $\Delta \pmod{2C} \geq C+1$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$.

Corollary (Case $C = 2$)

For any $\Delta \geq 5$ odd, $M(2,\Delta) = \left\lceil \frac{3\Delta}{4} \right\rceil$.

Proposition

Let $\Delta \geq 5$ be odd and let $\mathcal{C}$ be the class of Δ-regular graphs than contain a perfect matching. Then $M(C,\Delta,\mathcal{C}) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$.

Case $\Delta \geq 5$ odd (II)

Corollary

Let $\Delta \geq 5$ be odd. If $\Delta \pmod{2C} = 1$ or $\Delta \pmod{2C} \geq C+1$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Corollary (Case $C = 2$)

For any $\Delta \geq 5$ odd, $M(2,\Delta) = \left\lceil \frac{3\Delta}{4} \right\rceil$.

Proposition

Let $\Delta \geq 5$ be odd and let $\mathcal{C}$ be the class of Δ-regular graphs than contain a perfect matching. Then $M(C,\Delta,\mathcal{C}) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Case $\Delta \geq 5$ odd (II)

Corollary

Let $\Delta \geq 5$ be odd. If $\Delta \pmod{2C} = 1$ or $\Delta \pmod{2C} \geq C + 1$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Corollary (Case $C = 2$)

For any $\Delta \geq 5$ odd, $M(2, \Delta) = \left\lceil \frac{3\Delta}{4} \right\rceil$.

Proposition

Let $\Delta \geq 5$ be odd and let $\mathcal{C}$ be the class of Δ-regular graphs than contain a perfect matching. Then $M(C, \Delta, \mathcal{C}) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Next subsection is...

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

For any $\Delta \geq 5$ odd, $M(3, \Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C, \Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

For any $\Delta \geq 5$ odd, $M(3,\Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C,\Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C,\Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

For any $\Delta \geq 5$ odd, $M(3,\Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C,\Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

For any $\Delta \geq 5$ odd, $M(3,\Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C,\Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

For any $\Delta \geq 5$ odd, $M(3,\Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C,\Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

For any $\Delta \geq 5$ odd, $M(3, \Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C, \Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

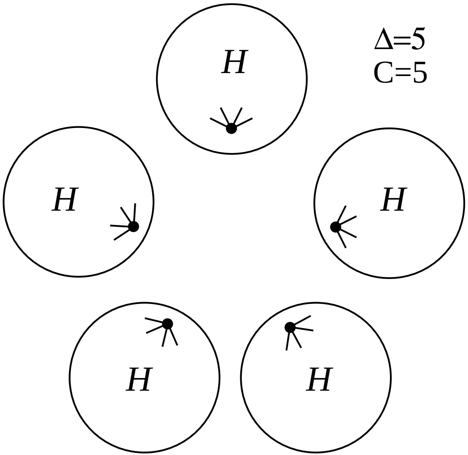
For any $\Delta \geq 5$ odd, $M(3, \Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C, \Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

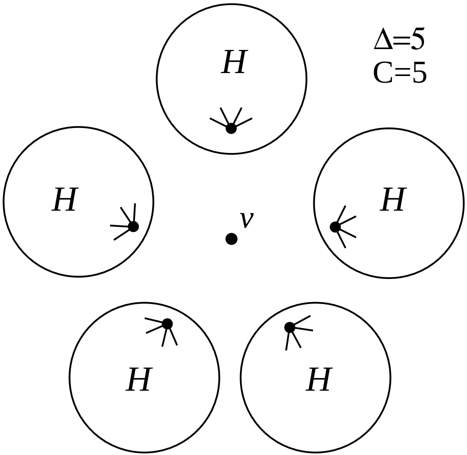
Idea of the proof

- Make Δ copies of H



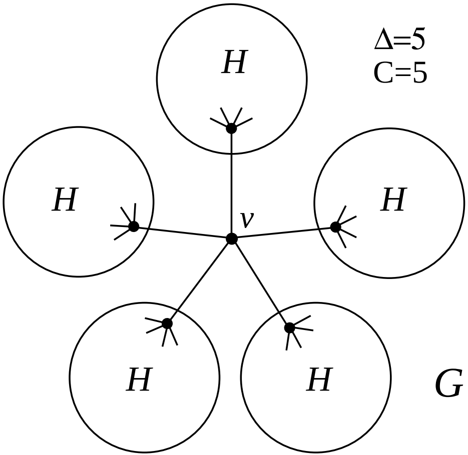
Idea of the proof

- Make Δ copies of H and add a cut-vertex v joined to all vertices of degree $\Delta - 1$ to make our Δ-regular graph G.



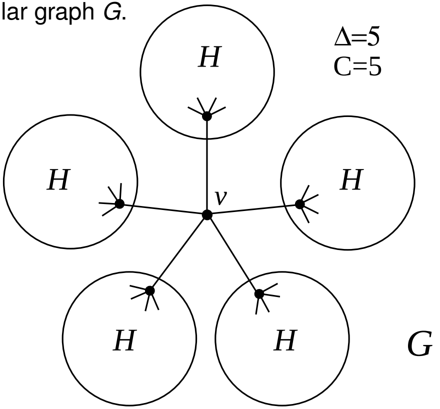
Idea of the proof

- Make Δ copies of H and add a cut-vertex v joined to all vertices of degree $\Delta - 1$ to make our Δ-regular graph G.



Idea of the proof

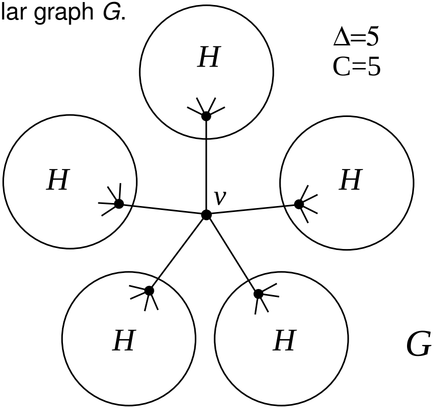
- Make Δ copies of H and add a cut-vertex v joined to all vertices of degree $\Delta - 1$ to make our Δ-regular graph G.



- Now suppose for the sake of contradiction that there is a C-edge-partition $\mathcal{B}$ of G where each vertex appears in at most LB(C, Δ) subgraphs.
- Since the girth of G is greater than C, all the subgraphs in $\mathcal{B}$ are trees.

Idea of the proof

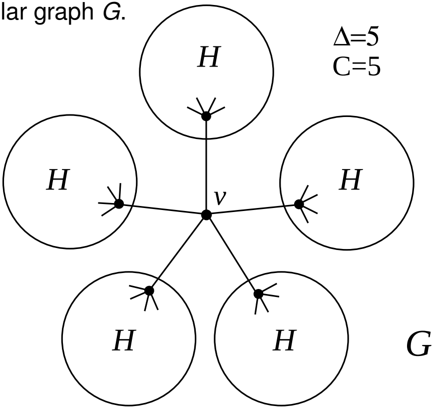
- Make Δ copies of H and add a cut-vertex v joined to all vertices of degree $\Delta - 1$ to make our Δ-regular graph G.



- Now suppose for the sake of contradiction that there is a C-edge-partition $\mathcal{B}$ of G where each vertex appears in at most $\mathrm{LB}(C, \Delta)$ subgraphs.
- Since the girth of G is greater than C, all the subgraphs in $\mathcal{B}$ are trees.

Idea of the proof

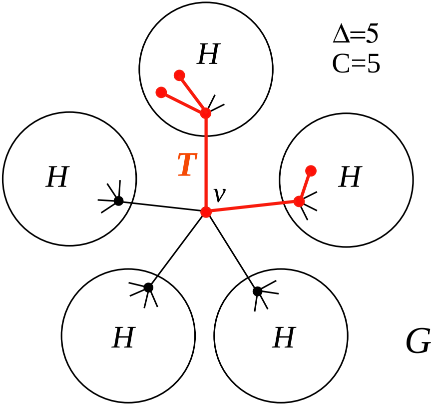
- Make Δ copies of H and add a cut-vertex v joined to all vertices of degree $\Delta - 1$ to make our Δ-regular graph G.



- Now suppose for the sake of contradiction that there is a C-edge-partition $\mathcal{B}$ of G where each vertex appears in at most $\mathrm{LB}(C, \Delta)$ subgraphs.
- Since the girth of G is greater than C, all the subgraphs in $\mathcal{B}$ are trees.

Idea of the proof

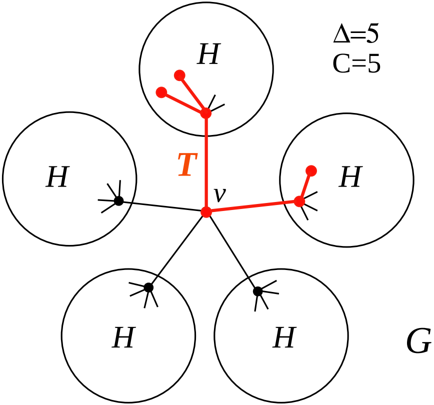
- Since $LB(C, \Delta) < \Delta$, v must have degree at least 2 in some subgraph $T \in \mathcal{B}$.



- Since $|E(T)| \leq C$, the tree T contains at most $\lfloor \frac{C-2}{2} \rfloor = \frac{C-3}{2}$ edges of a copy of H intersecting T.
- Now we only work in this copy H. Let $\alpha = |E(T \cap H)| \leq \frac{C-3}{2}$ ($\alpha = \frac{5-3}{2} = 1$ in the example).

Idea of the proof

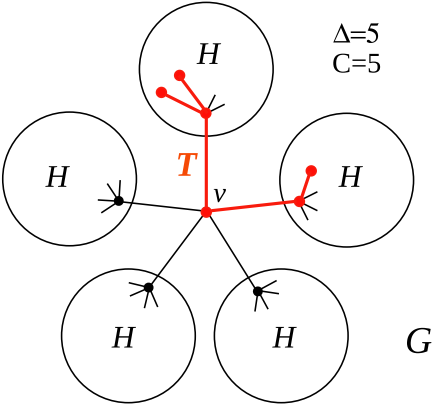
- Since $LB(C, \Delta) < \Delta$, v must have degree at least 2 in some subgraph $T \in \mathcal{B}$.



- Since $|E(T)| \leq C$, the tree T contains at most $\left\lfloor \frac{C-2}{2} \right\rfloor = \frac{C-3}{2}$ edges of a copy of H intersecting T.
- Now we only work in this copy H. Let $\alpha = |E(T \cap H)| \leq \frac{C-3}{2}$ $(\alpha = \frac{5-3}{2} = 1$ in the example).

Idea of the proof

- Since $LB(C, \Delta) < \Delta$, v must have degree at least 2 in some subgraph $T \in \mathcal{B}$.

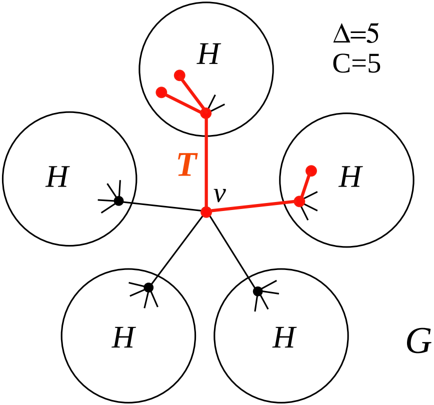


- Since $|E(T)| \leq C$, the tree T contains at most $\left\lfloor \frac{C-2}{2} \right\rfloor = \frac{C-3}{2}$ edges of a copy of H intersecting T.
- Now we only work in this copy H. Let $\alpha = |E(T \cap H)| \leq \frac{C-3}{2}$

($\alpha = \frac{5-3}{2} = 1$ in the example).

Idea of the proof

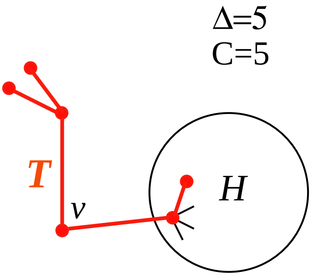
- Since $LB(C, \Delta) < \Delta$, v must have degree at least 2 in some subgraph $T \in \mathcal{B}$.



- Since $|E(T)| \leq C$, the tree T contains at most $\lfloor \frac{C-2}{2} \rfloor = \frac{C-3}{2}$ edges of a copy of H intersecting T.
- Now we only work in this copy H. Let $\alpha = |E(T \cap H)| \leq \frac{C-3}{2}$ ($\alpha = \frac{5-3}{2} = 1$ in the example).

Idea of the proof

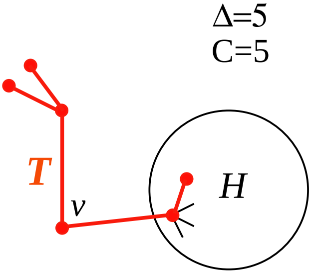
- Let $\mathcal{B}' = \{B \cap H\}_{B \in (\mathcal{B} - \{T\})}$, with the empty subgraphs removed. That is, $\mathcal{B}'$ contains the subgraphs in $\mathcal{B}$ that partition the edges in H that are not in T.



- Let $n = |V(H)|$, which is odd as in H there is one vertex of degree $\Delta - 1$ and all the others have degree Δ.

Idea of the proof

- Let $\mathcal{B}' = \{B \cap H\}_{B \in (\mathcal{B} - \{T\})}$, with the empty subgraphs removed. That is, $\mathcal{B}'$ contains the subgraphs in $\mathcal{B}$ that partition the edges in H that are not in T.



- Let $n = |V(H)|$, which is odd as in H there is one vertex of degree $\Delta - 1$ and all the others have degree Δ.

Idea of the proof

- Therefore, the total number of edges of the trees in $\mathcal{B}'$ is
$$\sum_{T\in\mathcal{B}'} |E(T)| \;=\; |E(H)| - \alpha \;=\; \frac{n\Delta - 1}{2} - \alpha \;=\; \frac{nkC - 1}{2} - \alpha. \qquad (1)$$

- As $\alpha \leq \frac{C-3}{2}$, from (1) we get
$$\sum_{T\in\mathcal{B}'} |E(T)| \;\geq\; \frac{nkC - 1}{2} - \frac{C-3}{2} \;=\; \left(\frac{nk-1}{2}\right) \cdot C + 1. \qquad (2)$$

- As each tree in $\mathcal{B}'$ has at most C edges, from (2) we get that
$$|\mathcal{B}'| \;\geq\; \left\lceil \frac{nk-1}{2} + \frac{1}{C} \right\rceil \;=\; \frac{nk-1}{2} + \left\lceil \frac{1}{C} \right\rceil \;=\; \frac{nk-1}{2} + 1. \qquad (3)$$

- Clearly, $\sum_{T\in\mathcal{B}'} |V(T)| = \sum_{T\in\mathcal{B}'} |E(T)| + |\mathcal{B}'|$, and $|V(T\cap H)| = \alpha + 1$.

Idea of the proof

- Therefore, the total number of edges of the trees in $\mathcal{B}'$ is

$$\sum_{T \in \mathcal{B}'} |E(T)| \; = \; |E(H)| - \alpha \; = \; \frac{n\Delta - 1}{2} - \alpha \; = \; \frac{nkC - 1}{2} - \alpha. \qquad (1)$$

- As $\alpha \leq \frac{C-3}{2}$, from (1) we get

$$\sum_{T \in \mathcal{B}'} |E(T)| \; \geq \; \frac{nkC - 1}{2} - \frac{C - 3}{2} \; = \; \left(\frac{nk - 1}{2}\right) \cdot C + 1. \qquad (2)$$

- As each tree in $\mathcal{B}'$ has at most C edges, from (2) we get that

$$|\mathcal{B}'| \; \geq \; \left\lceil \frac{nk - 1}{2} + \frac{1}{C} \right\rceil \; = \; \frac{nk - 1}{2} + \left\lceil \frac{1}{C} \right\rceil \; = \; \frac{nk - 1}{2} + 1. \qquad (3)$$

- Clearly, $\sum_{T \in \mathcal{B}'} |V(T)| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'|$, and $|V(T \cap H)| = \alpha + 1$.

Idea of the proof

- Therefore, the total number of edges of the trees in $\mathcal{B}'$ is

$$\sum_{T \in \mathcal{B}'} |E(T)| \;=\; |E(H)| - \alpha \;=\; \frac{n\Delta - 1}{2} - \alpha \;=\; \frac{nkC - 1}{2} - \alpha. \qquad (1)$$

- As $\alpha \leq \frac{C-3}{2}$, from (1) we get

$$\sum_{T \in \mathcal{B}'} |E(T)| \;\geq\; \frac{nkC - 1}{2} - \frac{C-3}{2} \;=\; \left(\frac{nk-1}{2}\right) \cdot C + 1. \qquad (2)$$

- As each tree in $\mathcal{B}'$ has at most C edges, from (2) we get that

$$|\mathcal{B}'| \;\geq\; \left\lceil \frac{nk-1}{2} + \frac{1}{C} \right\rceil \;=\; \frac{nk-1}{2} + \left\lceil \frac{1}{C} \right\rceil \;=\; \frac{nk-1}{2} + 1. \qquad (3)$$

- Clearly, $\sum_{T \in \mathcal{B}'} |V(T)| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'|$, and $|V(T \cap H)| = \alpha + 1$.

Idea of the proof

- Therefore, the total number of edges of the trees in $\mathcal{B}'$ is

$$\sum_{T\in\mathcal{B}'} |E(T)| \;=\; |E(H)| - \alpha \;=\; \frac{n\Delta - 1}{2} - \alpha \;=\; \frac{nkC - 1}{2} - \alpha. \qquad (1)$$

- As $\alpha \leq \frac{C-3}{2}$, from (1) we get

$$\sum_{T\in\mathcal{B}'} |E(T)| \;\geq\; \frac{nkC - 1}{2} - \frac{C-3}{2} \;=\; \left(\frac{nk-1}{2}\right) \cdot C + 1. \qquad (2)$$

- As each tree in $\mathcal{B}'$ has at most C edges, from (2) we get that

$$|\mathcal{B}'| \;\geq\; \left\lceil \frac{nk-1}{2} + \frac{1}{C} \right\rceil \;=\; \frac{nk-1}{2} + \left\lceil \frac{1}{C} \right\rceil \;=\; \frac{nk-1}{2} + 1. \qquad (3)$$

- Clearly, $\sum_{T\in\mathcal{B}'} |V(T)| = \sum_{T\in\mathcal{B}'} |E(T)| + |\mathcal{B}'|$, and $|V(T \cap H)| = \alpha + 1$.

Idea of the proof

- Therefore, using (1) and (3), we get that the total number of occurrences of the vertices of H in some tree of $\mathcal{B}$ is

$$\begin{aligned}
\sum_{v \in V(H)} |\{T \in \mathcal{B} : v \in T\}| &= \sum_{T \in \mathcal{B}'} |V(T)| + |V(T \cap H)| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'| + \alpha + 1 \\
&= \frac{nkC - 1}{2} - \alpha + |\mathcal{B}'| + \alpha + 1 \geq \frac{nkC - 1}{2} + \frac{nk - 1}{2} + 2 \\
&= nk \cdot \frac{C + 1}{2} + 1 = n \cdot \mathrm{LB}(C, \Delta) + 1 ,
\end{aligned}$$

- which implies that at least one vertex of H appears in at least $\mathrm{LB}(C, \Delta) + 1$ subgraphs, which is a contradiction to $\mathcal{B}$ being a C-edge-partition of G in which each vertex appears in at most $\mathrm{LB}(C, \Delta)$ subgraphs.

Idea of the proof

- Therefore, using (1) and (3), we get that the total number of occurrences of the vertices of H in some tree of $\mathcal{B}$ is

$$\begin{aligned}
\sum_{v \in V(H)} |\{T \in \mathcal{B} : v \in T\}| &= \sum_{T \in \mathcal{B}'} |V(T)| + |V(T \cap H)| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'| + \alpha + 1 \\
&= \frac{nkC - 1}{2} - \alpha + |\mathcal{B}'| + \alpha + 1 \geq \frac{nkC - 1}{2} + \frac{nk - 1}{2} + 2 \\
&= nk \cdot \frac{C + 1}{2} + 1 = n \cdot \mathrm{LB}(C, \Delta) + 1 \,,
\end{aligned}$$

- which implies that at least one vertex of H appears in at least $\mathrm{LB}(C, \Delta) + 1$ subgraphs, which is a contradiction to $\mathcal{B}$ being a C-edge-partition of G in which each vertex appears in at most $\mathrm{LB}(C, \Delta)$ subgraphs.

Next section is...

Summary of results: values of $M(C,\Delta)$

$C \mid \Delta$	1	2	3	4	5	6	7	...	Δ even	Δ odd
1	1	2	3	4	5	6	7	...	Δ	Δ
2	1	2	3	3	4	5	6	...	$\frac{3\Delta}{4}$	$\frac{3\Delta}{4}$
3	1	2	3 (2)	3	4	5 (4)	5	...	$\frac{2\Delta}{3}$	$\frac{2\Delta+1}{3}$ $\left(\frac{2\Delta}{3}\right)$
4	1	2	2	3	4	4	5	...	$\frac{5\Delta}{8}$	$\geq \frac{5\Delta}{8}$ $(=)$
5	1	2	2	3	4 (3)	4	5	...	$\frac{3\Delta}{5}$	$\geq \frac{3\Delta}{5}$ $(=)$
6	1	2	2	3	≥ 3 $(=)$	4	5	...	$\frac{7\Delta}{12}$	$\geq \frac{7\Delta}{12}$ $(=)$
7	1	2	2	3	≥ 3 $(=)$	4	5 (4)	...	$\frac{4\Delta}{7}$	$\geq \frac{4\Delta}{7}$ $(=)$
8	1	2	2	3	≥ 3 $(=)$	4	≥ 4 $(=)$	...	$\frac{9\Delta}{16}$	$\geq \frac{9\Delta}{16}$ $(=)$
9	1	2	2	3	≥ 3 $(=)$	4	≥ 4 $(=)$	...	$\frac{5\Delta}{9}$	$\geq \frac{5\Delta}{9}$ $(=)$
...	...	...	...	...	...	...	...	...	...	...
C	1	2	2	3	≥ 3 $(=)$	4	≥ 4 $(=)$	...	$\frac{C+1}{C}\frac{\Delta}{2}$	$\geq \frac{C+1}{C}\frac{\Delta}{2}$ $(=)$

Table: Known values of $M(C,\Delta)$. The red cases remain open.
The (blue) cases in brackets only hold if the graph has a perfect matching.
The symbol "$(=)$" means that the corresponding lower bound is attained.

Conclusions and further research

- We have studied a new model of **traffic grooming** that allows the network to support **dynamic** traffic without reconfiguring the electronic equipment at the nodes.
- We established the value of $M(C,\Delta)$ for "almost all" values of C and Δ, leaving **open** only the case where:
  - $\Delta \geq 5$ is odd;
  - $C \geq 4$;
  - $3 \leq \Delta \pmod{2C} \leq C-1$; and
  - the request graph does not contain a perfect matching.
- For these open cases, we provided upper bounds that differ from the optimal value by at most one.
- **Further Research**:
  - Determine $M(C,\Delta)$ for the remaining cases:

$$\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil \text{ or } \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1 \text{ ??}$$

  - Other classes of request graphs that **make sense** from the telecommunications point of view?

Conclusions and further research

- We have studied a new model of **traffic grooming** that allows the network to support **dynamic** traffic without reconfiguring the electronic equipment at the nodes.
- We established the value of $M(C, \Delta)$ for "almost all" values of C and Δ, leaving **open** only the case where:
  - $\Delta \geq 5$ is odd;
  - $C \geq 4$;
  - $3 \leq \Delta \pmod{2C} \leq C - 1$; and
  - the request graph does not contain a perfect matching.
- For these open cases, we provided upper bounds that differ from the optimal value by at most one.
- **Further Research**:
  - Determine $M(C, \Delta)$ for the remaining cases:

$$\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil \text{ or } \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1 \text{ ??}$$

  - Other classes of request graphs that **make sense** from the telecommunications point of view?

Conclusions and further research

- We have studied a new model of **traffic grooming** that allows the network to support **dynamic** traffic without reconfiguring the electronic equipment at the nodes.
- We established the value of $M(C,\Delta)$ for "almost all" values of C and Δ, leaving **open** only the case where:
  - $\Delta \geq 5$ is odd;
  - $C \geq 4$;
  - $3 \leq \Delta \pmod{2C} \leq C-1$; and
  - the request graph does not contain a perfect matching.
- For these open cases, we provided upper bounds that differ from the optimal value by at most one.
- **Further Research**:
  - Determine $M(C,\Delta)$ for the remaining cases:

    $$\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil \text{ or } \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1 \text{ ??}$$
  - Other classes of request graphs that **make sense** from the telecommunications point of view?

Conclusions and further research

- We have studied a new model of **traffic grooming** that allows the network to support **dynamic** traffic without reconfiguring the electronic equipment at the nodes.
- We established the value of $M(C, \Delta)$ for "almost all" values of C and Δ, leaving **open** only the case where:
  - $\Delta \geq 5$ is odd;
  - $C \geq 4$;
  - $3 \leq \Delta \pmod{2C} \leq C - 1$; and
  - the request graph does not contain a perfect matching.
- For these open cases, we provided upper bounds that differ from the optimal value by at most one.
- **Further Research**:
  - Determine $M(C, \Delta)$ for the remaining cases:

    $$\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil \text{ or } \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1 \text{ ??}$$
  - Other classes of request graphs that **make sense** from the telecommunications point of view?

Gràcies!