Edge-partitioning Regular Graphs
(with Applications to Traffic Grooming)

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Some of these results have been presented in:
- 34th Intern. Workshop on Graph-Theoretic Concepts in Computer Science (WG 2008)
- 35th Intern. Workshop on Graph-Theoretic Concepts in Computer Science (WG 2009)
Outline of the talk

1. Motivation: traffic grooming
2. Statement of the problem
3. The parameter $M(C, \Delta)$
4. Basic properties of $M(C, \Delta)$
5. Some results
   - Case $\Delta = 3, C = 4$
   - Case $\Delta \geq 4$ even
   - Case $\Delta \geq 5$ odd
   - Improved lower bound when $\Delta \equiv C \pmod{2C}$
6. Conclusions
1 Motivation: traffic grooming

2 Statement of the problem

3 The parameter $M(C, \Delta)$

4 Basic properties of $M(C, \Delta)$

5 Some results

6 Conclusions
WDM (Wavelength Division Multiplexing) networks
- 1 wavelength (or frequency) = up to 40 Gb/s
- 1 fiber = hundreds of wavelengths = Tb/s

Idea:
Traffic grooming consists in packing low-speed traffic flows into higher speed streams

→ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)

Objectives:
- Better use of bandwidth
- Reduce the equipment cost (mostly given by electronics)
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**ADM and OADM**

- **OADM** (Optical Add/Drop Multiplexer) = insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer) = insert/extract an OC/STM (electric low-speed signal) to/from a wavelength

→ we want to minimize the number of ADMs
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**Definitions**

- **Request** \((i, j)\): two vertices \((i, j)\) that want to exchange (low-speed) traffic

- **Grooming factor** \(C\):

  \[
  C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}
  \]

  **Example:**
  
  Capacity of one wavelength = 2400 \(Mb/s\)
  Capacity used by a request = 600 \(Mb/s\)  \(\Rightarrow\) \(C = 4\)

- **Load** of an arc in a wavelength: number of requests using this arc in this wavelength \((\leq C)\)
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To fix ideas...

Model:

- Topology $\rightarrow$ graph $G$
- Request set $\rightarrow$ graph $R$
- Grooming factor $\rightarrow$ integer $C$
- Requests in a wavelength $\rightarrow$ edges in a subgraph of $R$
- ADM in a wavelength $\rightarrow$ vertex in a subgraph of $R$

- We study the case when $G = \vec{C}_n$ (unidirectional ring)
- We assume that the requests are symmetric
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**Symmetric requests**: we have both \((i, j)\) and \((j, i)\).

- W.l.o.g. requests \((i, j)\) and \((j, i)\) are in the same subgraph
  \(\rightarrow\) each pair of symmetric requests induces load 1
  \(\rightarrow\) **grooming factor** \(C \iff\) each subgraph has \(\leq C\) edges.

- **C-edge-partition** of a graph \(G\):
  partition of \(E(G)\) into subgraphs with at most \(C\) edges each.
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\(C\text{-edge-partition}\) of a graph \(G\):
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Statement of the “old” problem

Traffic Grooming in Unidirectional Rings

Input
A cycle $C_n$ on $n$ nodes (network);
An undirected graph $R$ on $n$ nodes (request set);
A grooming factor $C$.

Output
A $C$-edge-partition of $R$ into subgraphs $R_1, \ldots, R_W$.

Objective
Minimize $\sum_{\omega=1}^{W} |V(R_\omega)|$. 
Example (unidirectional ring with symmetric requests)

\[ n = 4 \]
\[ R = K_4 \]
\[ C = 3 \]
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7 ADMs
Motivation: traffic grooming

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Some results

Conclusions
New model

- **Non-exhaustive** previous work (a lot!):
  - Flammini, Moscardelli, Shalom and Zaks - ISAAC 2005.
  - Flammini, Monaco, Moscardelli, Shalom and Zaks - WG 2006.
  - Bermond, Colbourn, Gionfriddo, Quattrocchi and S.- SIDMA 2010.

- In all of them: place ADMs at nodes for a **fixed request graph**. → placement of ADMs *a posteriori*.

- **New model**: place the ADMs at nodes such that the network can support **any** request graph with maximum degree at most $\Delta$. → placement of ADMs *a priori*.
New model

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  - Bermond *et al.* - **SIDMA 2005**.
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  - Amini, Pérennes and S. - **ISAAC 2007, TCS 2009**.
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**Statement of the “new” problem**

**Traffic Grooming in Unidirectional Rings with Bounded-Degree Request Graph**

**Input**
- An integer $n$ (size of the ring);
- An integer $C$ (grooming factor);
- An integer $\Delta$ (maximum degree).

**Output**
An assignment of $A(v)$ ADMs to each $v \in V(C_n)$, in such a way that for any graph $R$ on $n$ nodes with maximum degree at most $\Delta$, there exists a $C$-edge-partition of $R$ into subgraphs $R_1, \ldots, R_W$ s.t. each $v \in V(C_n)$ is in at most $A(v)$ subgraphs.

**Objective**
Minimize $\sum_{v \in V(C_n)} A(v)$, and the optimum is denoted $A(n, C, \Delta)$. 
Motivation: traffic grooming

Statement of the problem

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Basic properties of $M(C, \Delta)$

Some results

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Definition

Let $M(C, \Delta)$ be the smallest number $M$ such that, for all $n \geq 1$, the inequality $A(n, C, \Delta) \leq M \cdot n$ holds.

- Due to symmetry, it can be seen that $A(\nu)$ is the same for all nodes $\nu$, except for a subset whose size is independent of $n$.

- $M(C, \Delta)$ is always an integer.

- Equivalently:

  $M(C, \Delta)$ is the smallest integer $M$ such that the edges of any graph with maximum degree at most $\Delta$ can be $C$-edge-partitioned in such a way that each vertex appears in at most $M$ subgraphs.

- In the sequel we focus on determining $M(C, \Delta)$. 
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More formally... (we can forget traffic grooming)

- Let $G_{\Delta}$ be the class of (simple undirected) graphs with maximum degree at most $\Delta$.

- For $G \in G_{\Delta}$, let $P_{C}(G)$ be the set of $C$-edge-partitions of $G$.

- For $P \in P_{C}(G)$, let

$$\text{occ}(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in G_{\Delta}} \left( \min_{P \in P_{C}(G)} \text{occ}(P) \right)$$

- If the graphs are restricted to belong to a subclass $C \subseteq G_{\Delta}$, then the corresponding positive integer is denoted by $M(C, \Delta, C)$.
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Basic properties of $M(C, \Delta)$

- As any graph in $G_\Delta$ is a subgraph of a $\Delta$-regular graph, $M(C, \Delta) = M(C, \Delta, C)$, where $C$ is the class of $\Delta$-regular graphs.

- From now on, we only deal with $\Delta$-regular graphs.

- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.

- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.

- Upper bound: $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$$M(C, \Delta) \geq \left[ \frac{C+1}{C} \frac{\Delta}{2} \right] \text{ for all } C, \Delta \geq 1.$$
Basic properties of $M(C, \Delta)$

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$$M(C, \Delta) \geq \left\lfloor \frac{C + 1}{c} \frac{\Delta}{2} \right\rfloor$$ for all $C, \Delta \geq 1$. 


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Basic properties of \( M(C, \Delta) \)

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Lower bound

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Sketch of the proof.

- Let \( G \) be a \( \Delta \)-regular graph with girth at least \( C + 1 \).
  (Such a graph \( G \) exists by [Erdős and Sachs, 1963].)
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Proposition (Lower Bound)

\[ M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil \quad \text{for all } C, \Delta \geq 1. \]

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- $\Delta = 1$: $M(C, 1) = 1$ for all $C$ (trivial).

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- $\Delta = 3$: Cubic graphs. First “interesting” case:
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- Case $\Delta = 3, \ C = 4$
- Case $\Delta \geq 4$ even
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### Proposition

$M(4, 3) = 2.$

### Idea of the proof.

(in fact, we prove a slightly stronger result)

1. Let $G$ be a minimal counterexample (i.e., $|V(G)|$ is minimum).
2. If $G$ has no bridges, then it can be “easily” proved.
3. If $G$ has a bridge $e$, then the property is true for $U$ and $V$.

Finally, we merge “carefully” the partitions of $U$ and $V$ to obtain a partition of $G \Rightarrow$ contradiction.
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Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

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- **Construction:**
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**Proposition**

Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

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  - If an edge of $M$ is in $E_v^+$, then: $\left\lceil \frac{\Delta-1}{2C} \right\rceil + \Delta - \frac{\Delta-1}{2} = \left\lceil \frac{C+1}{2} \frac{\Delta}{C} + \frac{C-1}{2C} \right\rceil$.
  - Otherwise, if no edge of $M$ is in $E_v^+$, then:
    $$\left\lceil \frac{\Delta+1}{2C} \right\rceil + \Delta - \frac{\Delta+1}{2} = \left\lceil \frac{C+1}{2} \frac{\Delta}{C} + \frac{1-C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{2} \frac{\Delta}{C} + \frac{C-1}{2C} \right\rceil.$$
Case \( \Delta \geq 5 \) odd (II)

**Corollary**

Let \( \Delta \geq 5 \) be odd. If \( \Delta \pmod{2C} = 1 \) or \( \Delta \pmod{2C} \geq C + 1 \), then

\[
M(C, \Delta) = \left\lceil \frac{C + 1}{2} \frac{\Delta}{2} \right\rceil.
\]

**Corollary (Case \( C = 2 \))**

For any \( \Delta \geq 5 \) odd, \( M(2, \Delta) = \left\lceil \frac{3\Delta}{4} \right\rceil \).

**Proposition**

Let \( \Delta \geq 5 \) be odd and let \( C \) be the class of \( \Delta \)-regular graphs that contain a perfect matching. Then

\[
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Motivation: traffic grooming

Statement of the problem

The parameter $M(C, \Delta)$

Basic properties of $M(C, \Delta)$

Some results

- Case $\Delta = 3, C = 4$
- Case $\Delta \geq 4$ even
- Case $\Delta \geq 5$ odd
- Improved lower bound when $\Delta \equiv C \pmod{2C}$

Conclusions
Improved lower bound when $\Delta \equiv C \pmod{2C}$

**Theorem**

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.

**Corollary (Case $C = 3$)**

For any $\Delta \geq 5$ odd, $M(3, \Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

**Idea of the proof of the Theorem.**

- We prove that if $\Delta = kC$ with $k$ odd, then $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.
- Since both $\Delta$ and $k$ are odd, so is $C$, and therefore $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a $\Delta$-regular graph $G$ with no $C$-edge-partition where each vertex appears in at most $k \cdot \frac{C+1}{2} =: LB(C, \Delta)$ subgraphs.
- First, we construct a graph $H$ where all vertices have degree $\Delta$ except one which has degree $\Delta - 1$. Furthermore, we build $H$ so that it has girth strictly greater than $C$. Such a graph $H$ exists by [Chandran, SIAM J. Discr. Math., 2003].
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![Diagram showing the construction of a $\Delta$-regular graph G from $\Delta$ copies of H, with a cut-vertex v connected to all vertices of degree $\Delta - 1$.]
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Now suppose for the sake of contradiction that there is a $C$-edge-partition $B$ of $G$ where each vertex appears in at most $LB(C, \Delta)$ subgraphs.

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- Since $LB(C, \Delta) < \Delta$, $v$ must have degree at least 2 in some subgraph $T \in \mathcal{B}$.

- Since $|E(T)| \leq C$, the tree $T$ contains at most $\left\lfloor \frac{C-2}{2} \right\rfloor = \frac{C-3}{2}$ edges of a copy of $H$ intersecting $T$.

- Now we only work in this copy $H$. Let $\alpha = |E(T \cap H)| \leq \frac{C-3}{2}$ ($\alpha = \frac{5-3}{2} = 1$ in the example).
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Let $B' = \{ B \cap H \}_{B \in (B - \{ T \})}$, with the empty subgraphs removed. That is, $B'$ contains the subgraphs in $B$ that partition the edges in $H$ that are not in $T$.

Let $n = |V(H)|$, which is odd as in $H$ there is one vertex of degree $\Delta - 1$ and all the others have degree $\Delta$. 
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Therefore, the total number of edges of the trees in $B'$ is

$$\sum_{T \in B'} |E(T)| = |E(H)| - \alpha = \frac{n\Delta - 1}{2} - \alpha = \frac{nkC - 1}{2} - \alpha. \quad (1)$$

As $\alpha \leq \frac{C-3}{2}$, from (1) we get

$$\sum_{T \in B'} |E(T)| \geq \frac{nkC - 1}{2} - \frac{C-3}{2} = \left( \frac{nk - 1}{2} \right) \cdot C + 1. \quad (2)$$

As each tree in $B'$ has at most $C$ edges, from (2) we get that

$$|B'| \geq \left\lceil \frac{nk - 1}{2} + \frac{1}{C} \right\rceil = \frac{nk - 1}{2} + \left\lceil \frac{1}{C} \right\rceil = \frac{nk - 1}{2} + 1. \quad (3)$$

Clearly, $\sum_{T \in B'} |V(T)| = \sum_{T \in B'} |E(T)| + |B'|$, and $|V(T \cap H)| = \alpha + 1$. 
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Therefore, the total number of edges of the trees in $B'$ is

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As each tree in $B'$ has at most $C$ edges, from (2) we get that

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Therefore, using (1) and (3), we get that the total number of occurrences of the vertices of $H$ in some tree of $B$ is

$$\sum_{v \in V(H)} \left| \{ T \in B : v \in T \} \right| = \sum_{T \in B'} |V(T)| + |V(T \cap H)| = \sum_{T \in B'} |E(T)| + |B'| + \alpha + 1$$

$$= \frac{nkC - 1}{2} - \alpha + |B'| + \alpha + 1 \geq \frac{nkC - 1}{2} + \frac{nk - 1}{2} + 2$$

$$= nk \cdot \frac{C + 1}{2} + 1 = n \cdot \text{LB}(C, \Delta) + 1,$$

which implies that at least one vertex of $H$ appears in at least $\text{LB}(C, \Delta) + 1$ subgraphs, which is a contradiction to $B$ being a $C$-edge-partition of $G$ in which each vertex appears in at most $\text{LB}(C, \Delta)$ subgraphs.
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Motivation: traffic grooming

Statement of the problem

The parameter $M(C, \Delta)$

Basic properties of $M(C, \Delta)$

Some results

Conclusions
Summary of results: values of $M(C, \Delta)$

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<td>2</td>
<td>3</td>
<td>$\geq 3$ (=)</td>
<td>4</td>
<td>5 (4)</td>
<td>\ldots</td>
<td>$\frac{4\Delta}{7}$</td>
<td>$\geq \frac{4\Delta}{7}$ (=)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>$\geq 3$ (=)</td>
<td>4</td>
<td>$\geq 4$ (=)</td>
<td>\ldots</td>
<td>$\frac{9\Delta}{16}$</td>
<td>$\geq \frac{9\Delta}{16}$ (=)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>$\geq 3$ (=)</td>
<td>4</td>
<td>$\geq 4$ (=)</td>
<td>\ldots</td>
<td>$\frac{5\Delta}{9}$</td>
<td>$\geq \frac{5\Delta}{9}$ (=)</td>
<td></td>
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<td>\ldots</td>
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<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>$\geq 3$ (=)</td>
<td>4</td>
<td>$\geq 4$ (=)</td>
<td>\ldots</td>
<td>$\frac{C+1}{C} \frac{\Delta}{2}$</td>
<td>$\geq \frac{C+1}{C} \frac{\Delta}{2}$ (=)</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Known values of $M(C, \Delta)$. The red cases remain open. The (blue) cases in brackets only hold if the graph has a perfect matching. The symbol “(=)” means that the corresponding lower bound is attained.
Conclusions and further research

- We have studied a new model of traffic grooming that allows the network to support dynamic traffic without reconfiguring the electronic equipment at the nodes.

- We established the value of $M(C, \Delta)$ for “almost all” values of $C$ and $\Delta$, leaving open only the case where:
  - $\Delta \geq 5$ is odd;
  - $C \geq 4$;
  - $3 \leq \Delta \pmod{2C} \leq C - 1$; and
  - the request graph does not contain a perfect matching.

- For these open cases, we provided upper bounds that differ from the optimal value by at most one.

- **Further Research:**
  - Determine $M(C, \Delta)$ for the remaining cases:
    - $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ or $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$??
  - Other classes of request graphs that make sense from the telecommunications point of view?
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Gràcies!