Computing Distances on Graph Associahedra is Fixed-parameter Tractable

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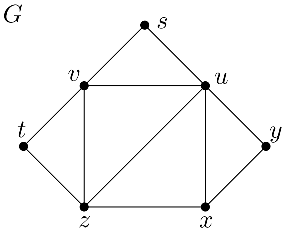
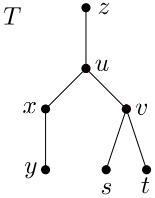
Elimination trees

Tree T obtained from a graph G by picking recursively a vertex in each connected component of the current graph:





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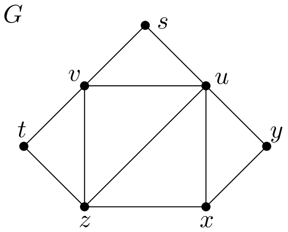

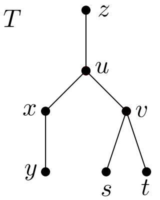

Elimination trees have been studied extensively in many contexts:

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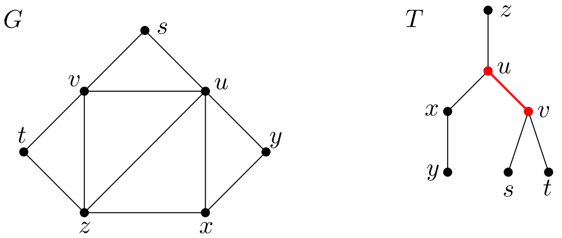

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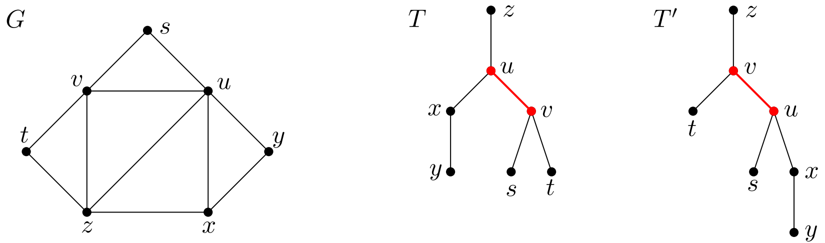

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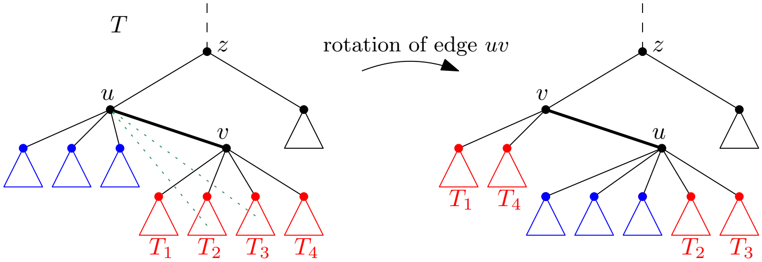

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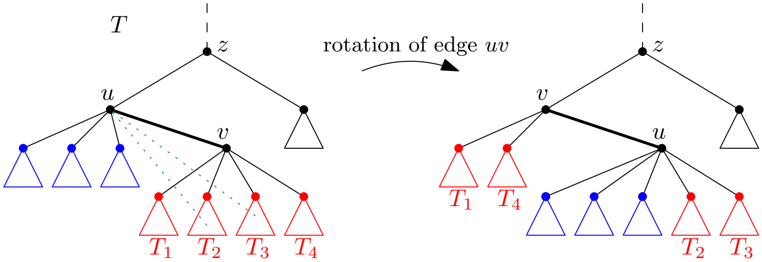

The rotation distance between two elimination trees (forests) T, T' of a graph G, denoted by $\mathrm{dist}(T, T')$, is the minimum number of rotations it takes to transform T into T'.

Graph associahedra

For any graph G, the flip graph of elimination forests of G under edge rotations is the skeleton of a polytope: graph associahedron $\mathcal{A}(G)$.

Object introduced by [Carr, Devadoss, Postnikov. 2006-2009]

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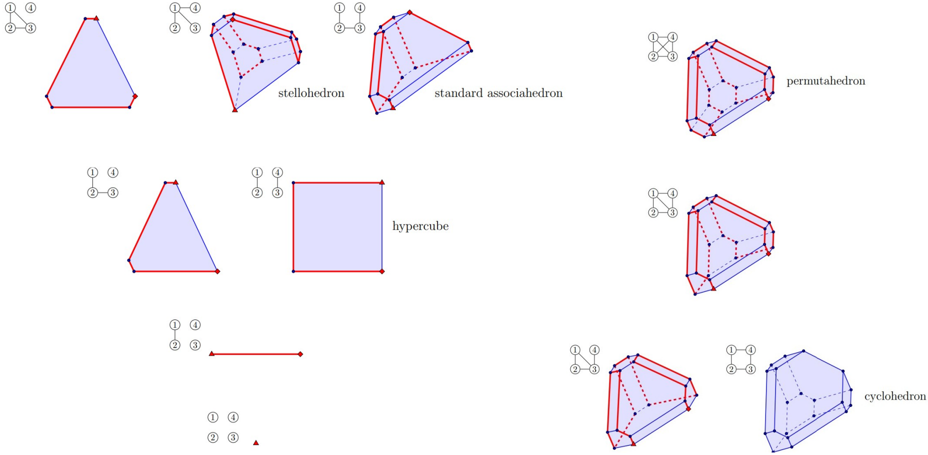
Famous particular cases of $\mathcal{A}(G)$ depending on the underlying graph G:

G	$\mathcal{A}(G)$
path	(standard) associahedron
complete graph	permutahedron
cycle	cyclohedron
star	stellohedron
matching	hypercube

Illustration of some famous examples



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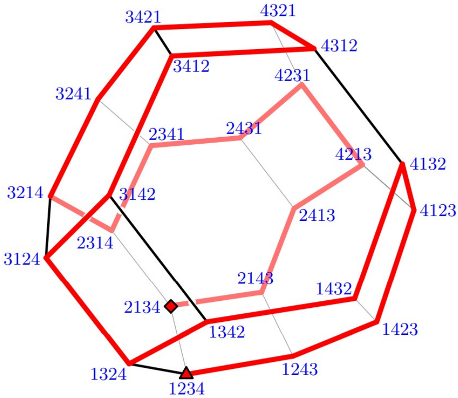
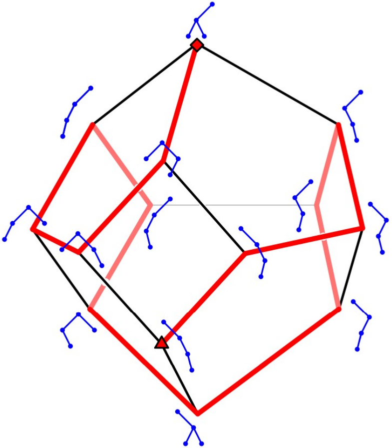
Zooming in: permutahedron and (standard) associahedron





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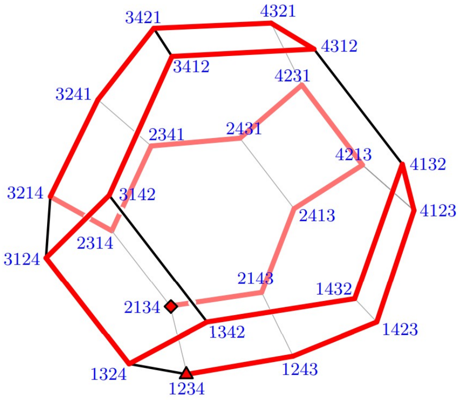

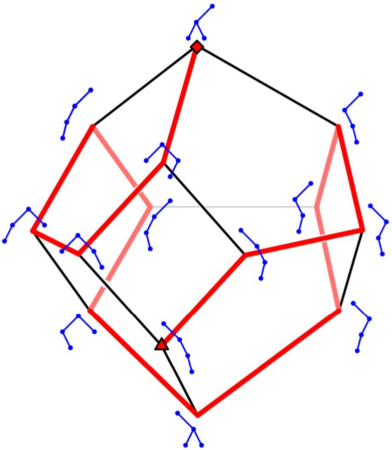

The (standard) associahedron has a rich history and literature, connecting computer science, combinatorics, algebra, and topology.

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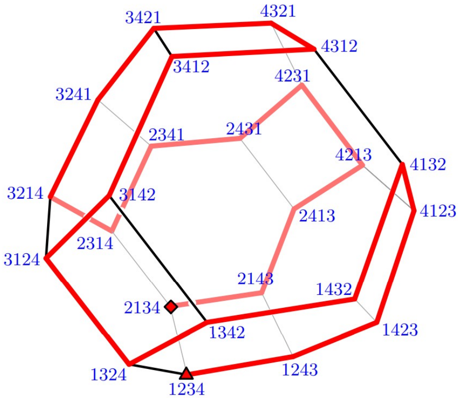

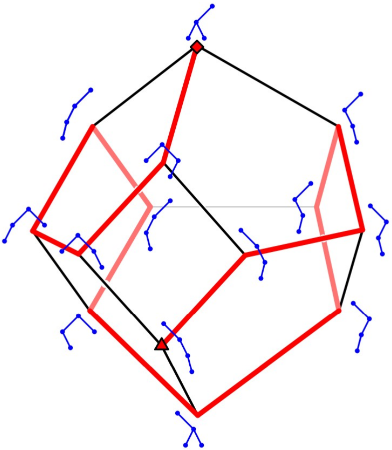

Binary trees are in bijection with many other Catalan objects:
triangulations of a convex polygon, well-formed parenthesis, Dyck paths, ...

Intensively studied: diameter of graph associahedra

Determining the diameter exactly, or upper/lower bounds, or estimates:

- If G is a path: [Sleator, Tarjan, Thurston. 1998] [Pournin. 2014]
- If G is a star: [Manneville, Pilaud. 2010]
- If G is a cycle: [Pournin. 2017]
- If G is a tree: [Manneville, Pilaud. 2010] [Cardinal, Langerman, Pérez-Lantero. 2018]
- If G is a complete bipartite or trivially perfect graph: [Cardinal, Pournin, Valencia-Pabon. 2022]
- If G is a caterpillar: [Berendsohn. 2022]
- If G has bounded treedepth or treewidth: [Cardinal, Pournin, Valencia-Pabon. 2022]

Our focus: computing distances on graph associahedra

Suppose for simplicity that the considered graph G is connected.

ROTATION DISTANCE

Instance: A graph G, two elimination trees T and T' of G, and a positive integer k.

Question: Is the rotation distance between T and T' at most k?

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This is $\boxed{\text{not}}$ the problem we solve!

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This motivates the study of the $\boxed{\text{parameterized complexity}}$ of the problem.

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- para-NP-hard problem: NP-hard for a fixed value of the parameter.

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$$\text{with} \quad f(k) = k^{k \cdot 2^{2^{\cdot^{\cdot^{\cdot^{2^{\mathcal{O}(k^2)}}}}}}},$$

where the tower of exponentials has height at most $(3k+1)4k = \mathcal{O}(k^2)$.

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Prior to our work, only the case where G is a path was known to be FPT.

[Cleary, St. John. 2009] [Lucas. 2010] [Kanj, Sedgwick, Xia. 2017] [Li, Xia. 2023]

Main ideas of the FPT algorithm

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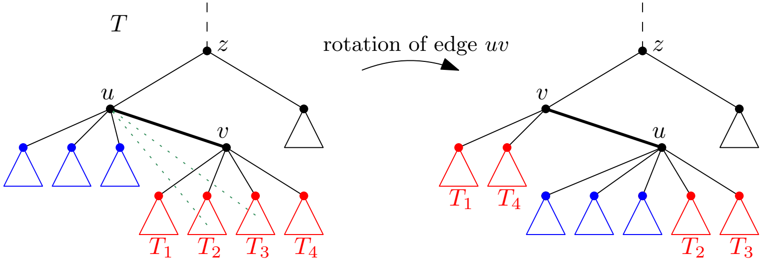
Let us see how we find such a "small" set $M \subseteq V(T)$ of marked vertices...

There are few children-bad vertices



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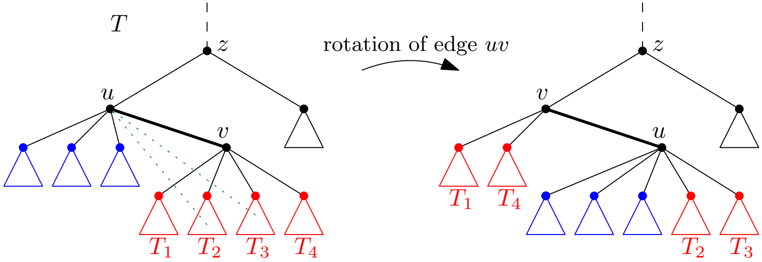

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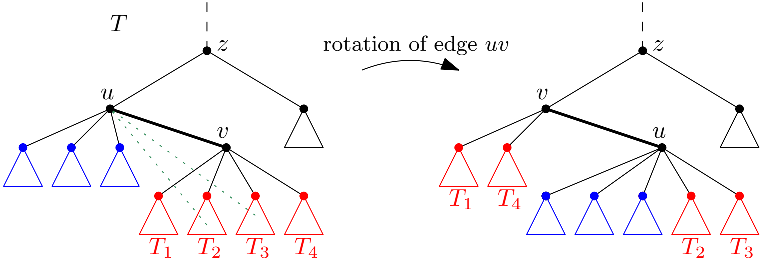

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We may assume that there are at most $3k$ (T, T')-children-bad vertices.

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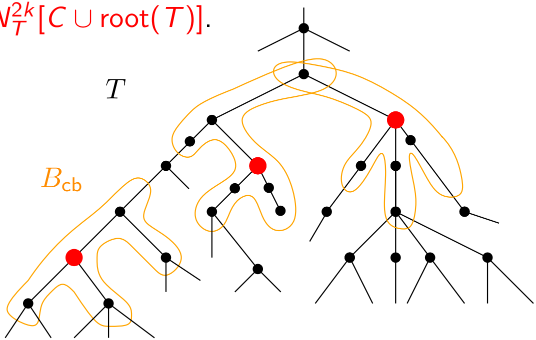

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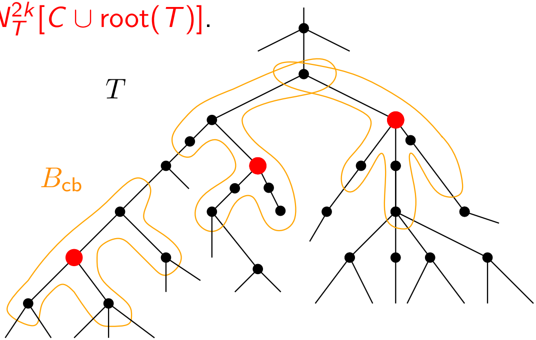

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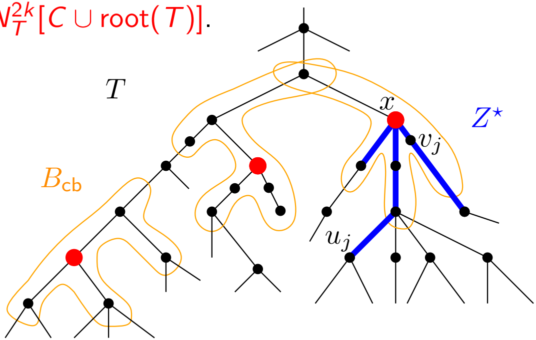

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If $\Delta(T)$ is bounded (in particular, if $\Delta(G)$ is bounded), we are done!

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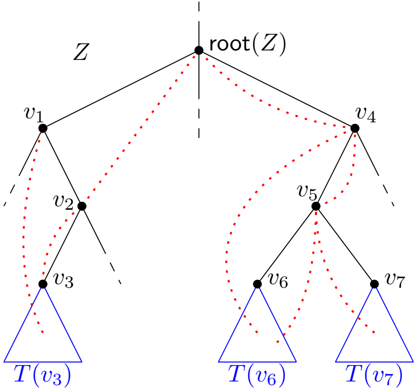

trace$(T, Z, v_1) = (1)$
trace$(T, Z, v_2) = (1, 1)$
trace$(T, Z, v_3) = (1, 1, 0)$
trace$(T, Z, v_4) = (1)$
trace$(T, Z, v_5) = (1, 0)$
trace$(T, Z, v_6) = (1, 1, 0)$
trace$(T, Z, v_7) = (1, 0, 0)$

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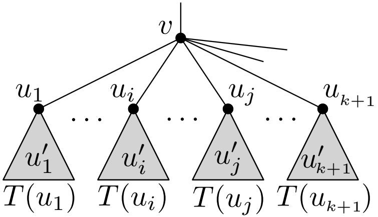

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Crucial: enough to keep track of "at least $k + 1$", not the actual number.

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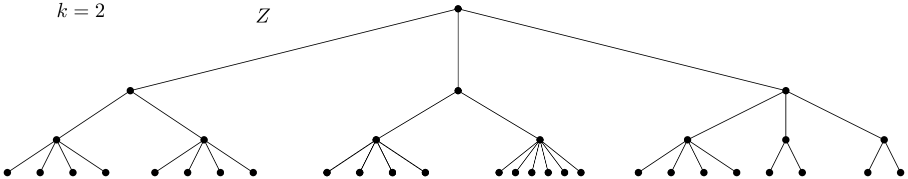

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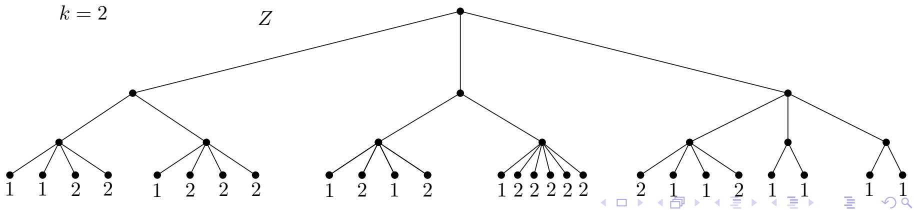

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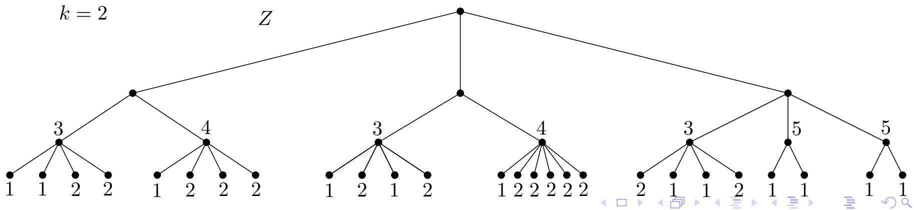

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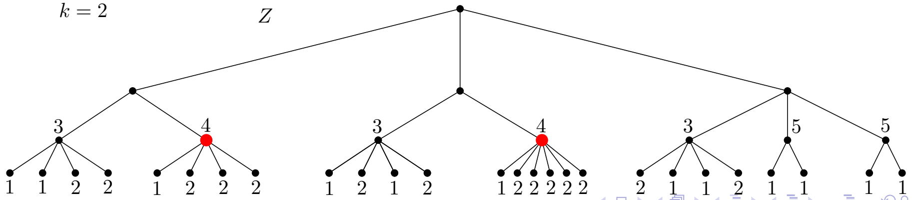

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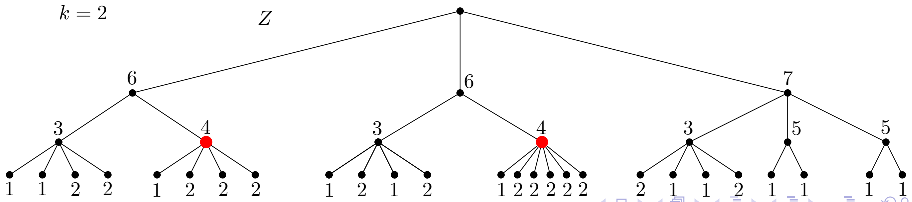

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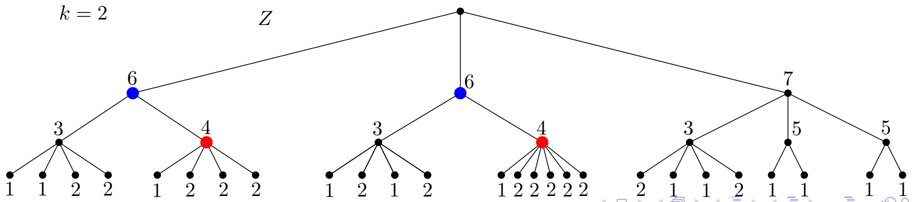

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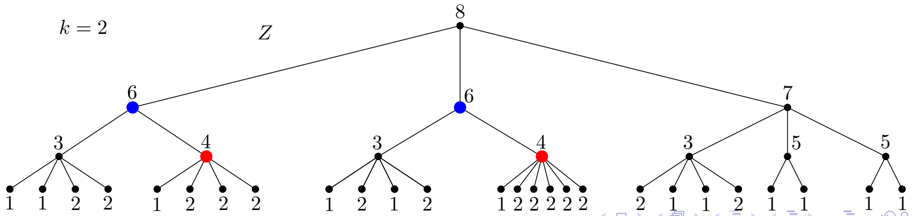

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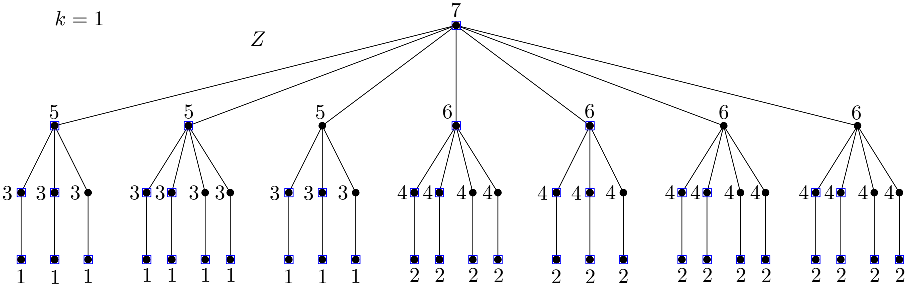

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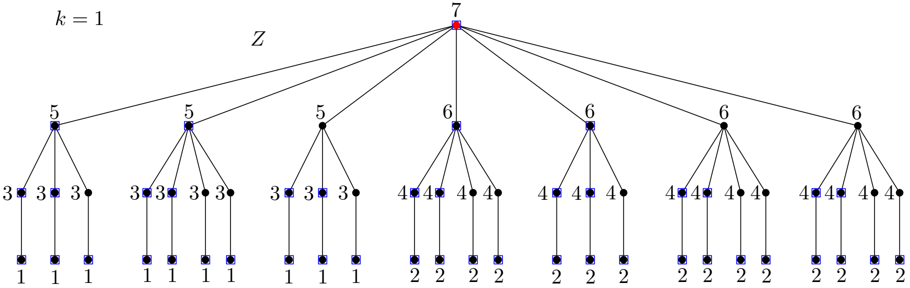

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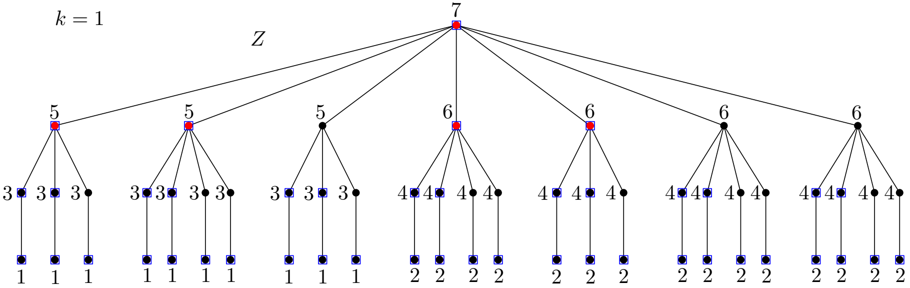

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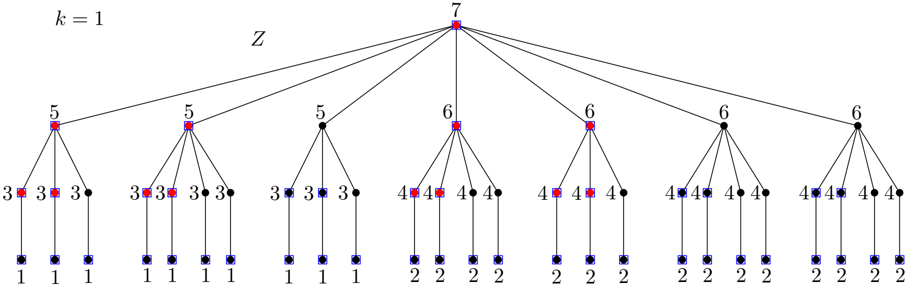

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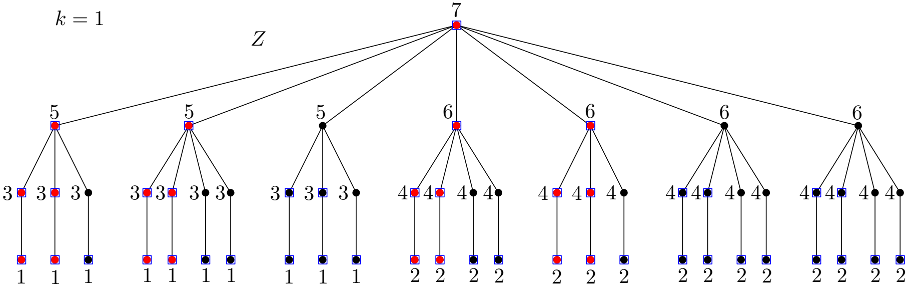

The set of marked vertices satisfies what we want

Lemma

The set $M \subseteq V(T)$ of marked vertices has size bounded by a function $h(k)$, with the same asymptotic growth as the function $g(k)$ given by the number of types. Moreover, M can be computed in time $h(k) \cdot |V(G)|$.

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- We distinguish two cases...

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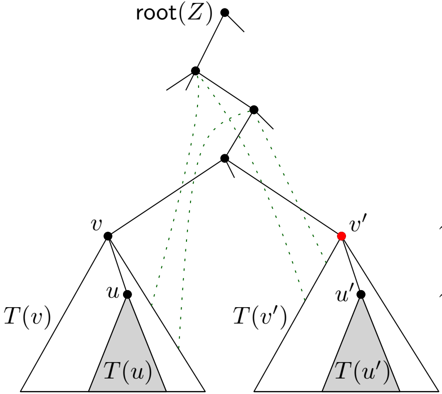

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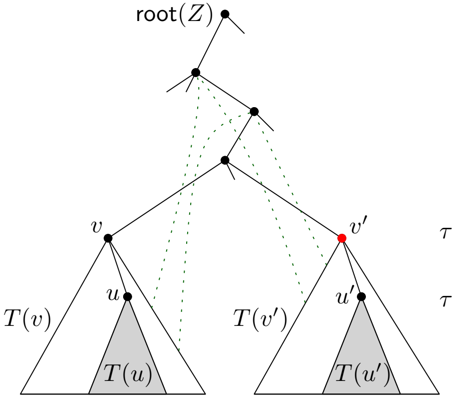

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We define σ' from σ by just replacing v with v' in all the rotations of σ involving v.

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All T-siblings v' of v with $\tau(T, Z, v) = \tau(T, Z, v')$ are non-marked.

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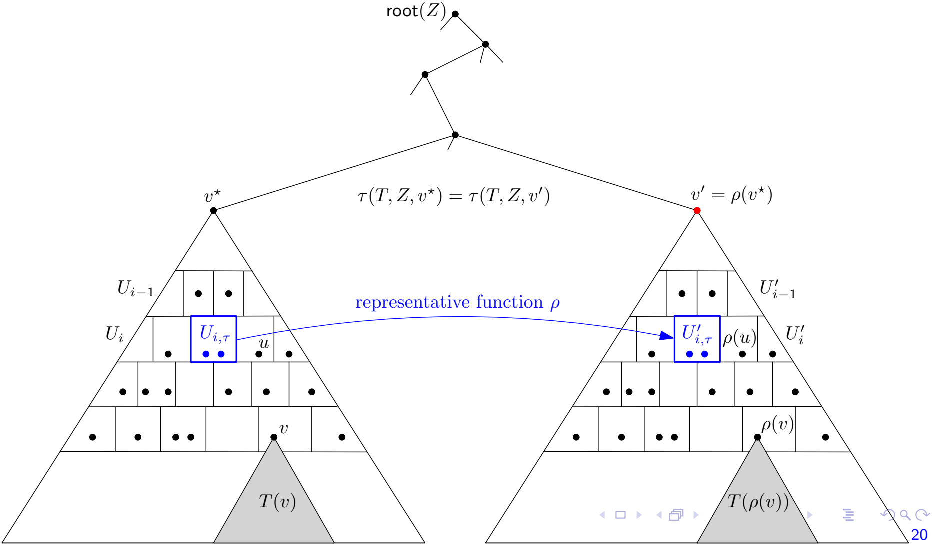
In this case, to define σ', we need to modify σ in a more global way:



New results: can we go beyond graph associahedra?

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ROTATION DISTANCE problem: distances on graph associahedra.

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Rotation Distance problem: distances on graph associahedra.

Natural generalization: distances on hypergraphic polytopes.

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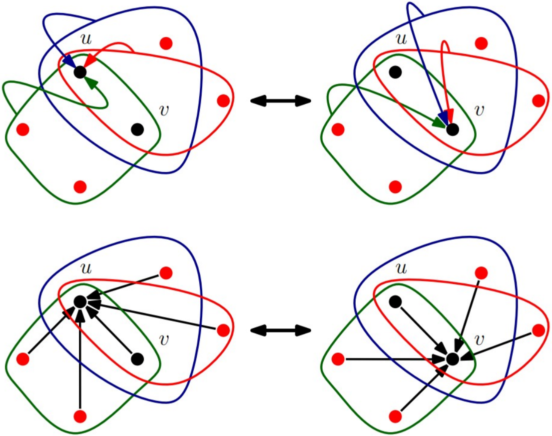
- **Vertices**: all acyclic orientations of H.
- **Edges**: if the two corresponding rotations are related by a flip.



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We present a parameterized reduction from k-Dominating Set.

Conclusions and further research

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COMBINATORIAL SHORTEST PATH ON POLYMATROIDS:

- NP-hard. [Ito, Kakimura, Kamiyama, Kobayashi, Maezawa, Nozaki, Okamoto. 2023]
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Gràcies!