Introdução sobre complexidade parametrizada

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Outline of the talk

1. Introduction
   - Parameterized complexity
   - Treewidth
2. FPT algorithms parameterized by treewidth
3. The $\mathcal{F}$-DELETION problem

Next section is...

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Crucial notion in complexity theory: NP-completeness

- Cook-Levin Theorem (1971): the SAT problem is NP-complete.
- Karp (1972): list of 21 *important* NP-complete problems.
- Nowadays, literally thousands of problems are known to be NP-hard: unless $\mathsf{P} = \mathsf{NP}$, they cannot be solved in polynomial time.

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- Nowadays, literally thousands of problems are known to be NP-hard:
  unless $\mathsf{P} = \mathsf{NP}$, they cannot be solved in polynomial time.
- But what does it mean for a problem to be NP-hard?

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Are all instances really hard to solve?

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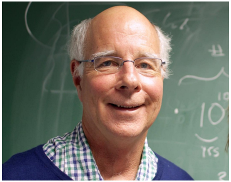
Message In many applications, not only the total size of the instance matters, but also the value of an additional parameter.

The area of parameterized complexity

Idea Measure the complexity of an algorithm in terms of the input size and an additional integer parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established area with hundreds of articles published every year in the most prestigious TCS journals and conferences.

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These three problems are NP-hard, but are they equally hard?

They behave quite differently...

1. k-Vertex Cover: solvable in time $2^k \cdot n^2$
2. k-Clique: solvable in time $k^2 \cdot n^k$
3. Vertex k-Coloring: NP-hard for every fixed $k \geq 3$

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   The problem is para-NP-hard

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Working hypothesis of parameterized complexity: **k-CLIQUE is not FPT**

(in classical complexity: 3-SAT cannot be solved in poly-time)

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Do all FPT problems admit polynomial kernels? NO!

Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)

Deciding whether a graph has a PATH *with* $\geq k$ *vertices* is FPT *but does not admit a polynomial kernel*, *unless* NP $\subseteq$ coNP/poly.

Typical approach to deal with a parameterized problem

Parameterized problem L

- k-Clique
- k-Vertex Cover
- k-Path
- Vertex k-Coloring

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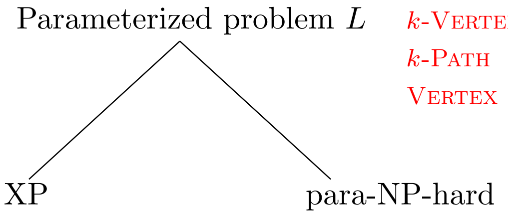

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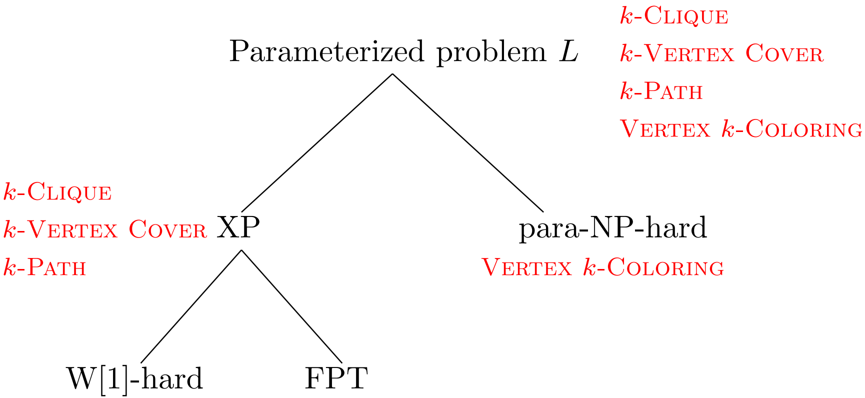

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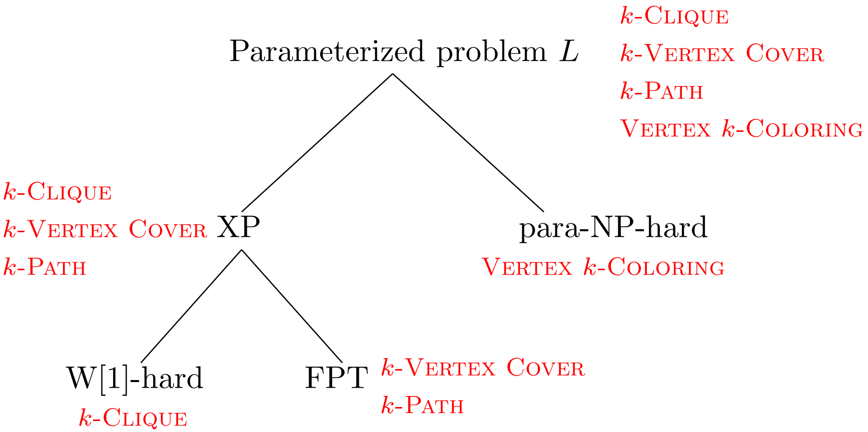

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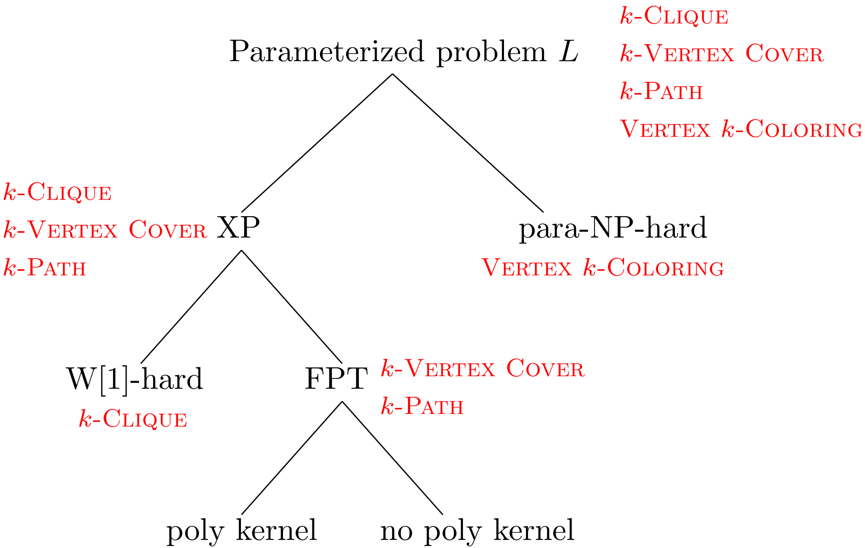

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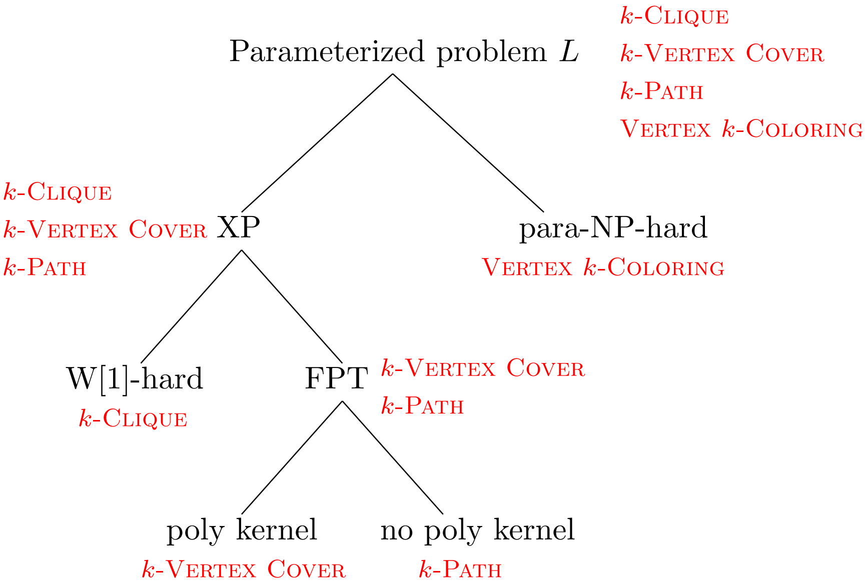

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Treewidth via k-trees

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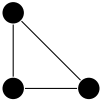

[Figure by Julien Baste]

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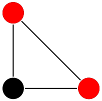

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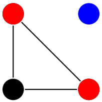

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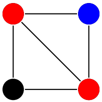

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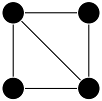

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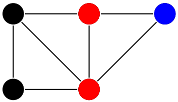

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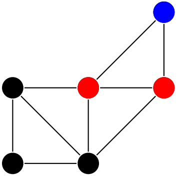

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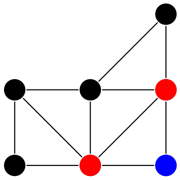

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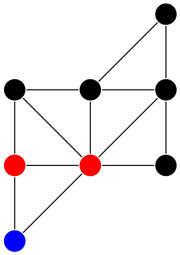

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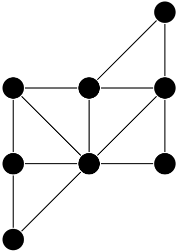

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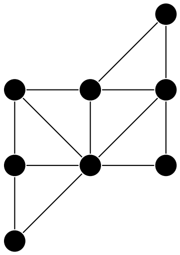

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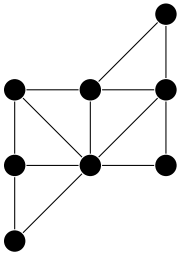

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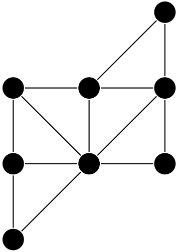

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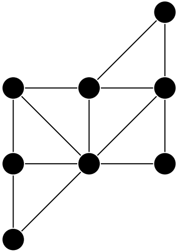

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Construction suggests the notion of tree decomposition: small separators.

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{B_t \mid t \in V(T)\})$, where
  T is a tree, and
  $B_t \subseteq V(G)\ \ \forall t \in V(T)$ (bags),

  satisfying the following:

  - $\bigcup_{t \in V(T)} B_t = V(G)$,
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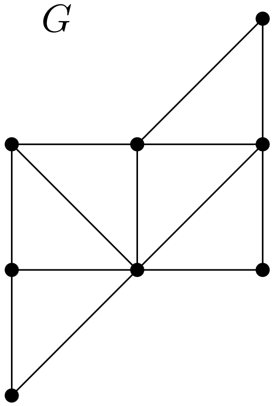
- Treewidth of a graph G: minimum width of a tree decomposition of G.



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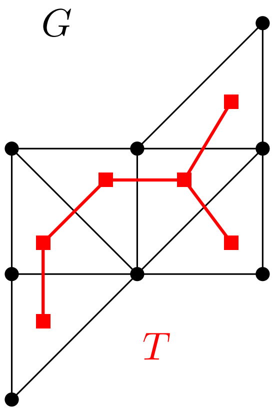

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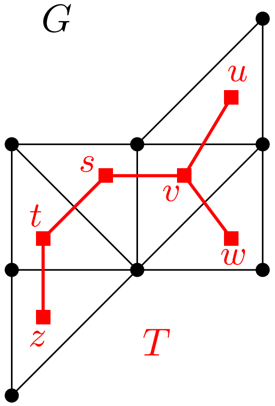

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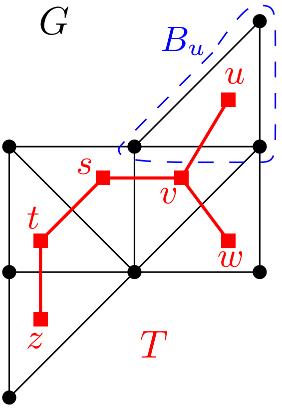

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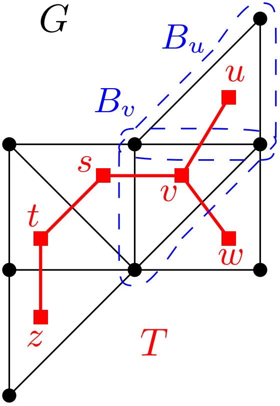

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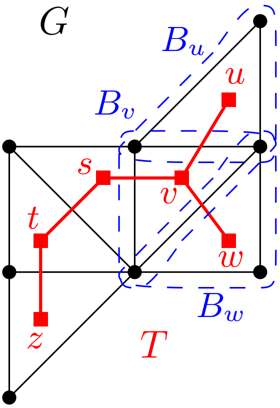

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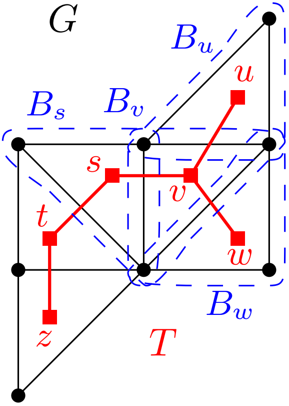

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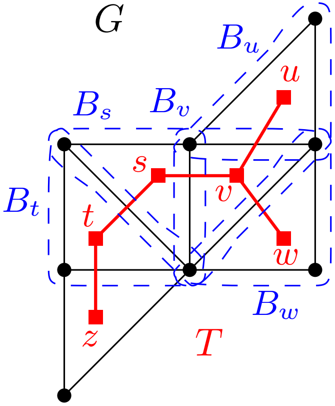

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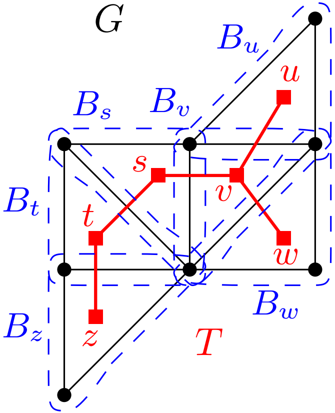

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1. Treewidth is a fundamental combinatorial tool in graph theory: key role in the Graph Minors project of Robertson and Seymour.
2. Treewidth behaves very well algorithmically, and algorithms parameterized by treewidth appear very often in FPT algorithms.
3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

Next section is...

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Graph logic that allows quantification over sets of vertices and edges.

Example: $\texttt{DomSet}(S)$: $[\ \forall v \in V(G) \setminus S, \exists u \in S : \{u, v\} \in E(G)\ ]$

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   Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

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Typical statements:

ETH $\Rightarrow$ k-Vertex Cover cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$.

ETH $\Rightarrow$ Planar k-Vertex Cover cannot in time $2^{o(\sqrt{k})} \cdot n^{O(1)}$.

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- Typically, FPT algorithms parameterized by treewidth are based on dynamic programming (DP) over a tree decomposition.
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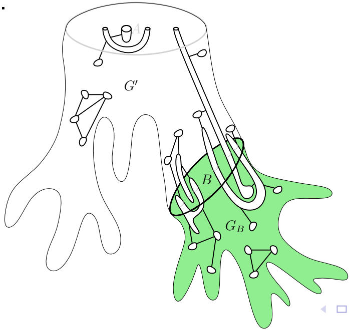
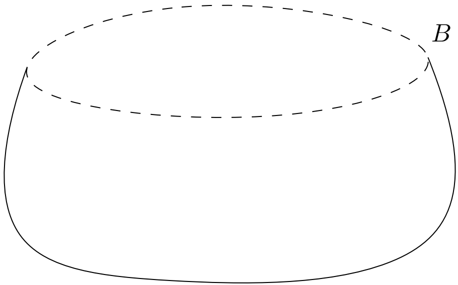
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Two behaviors for problems parameterized by treewidth

Local problems VERTEX COVER, DOMINATING SET, CLIQUE, INDEPENDENT SET, q-COLORING for fixed q.



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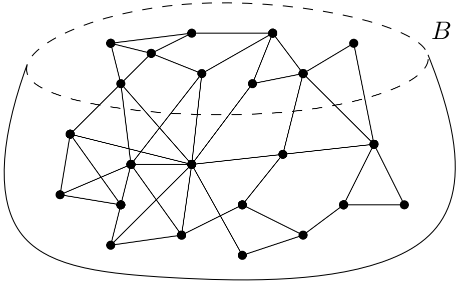

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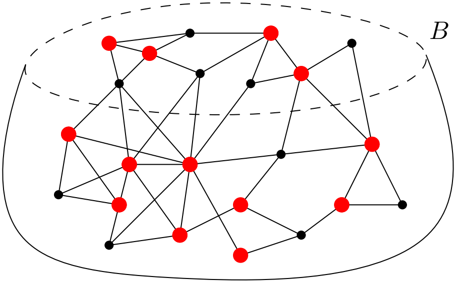

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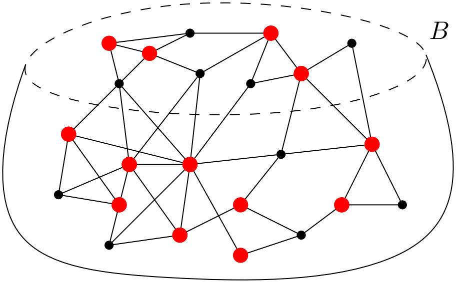

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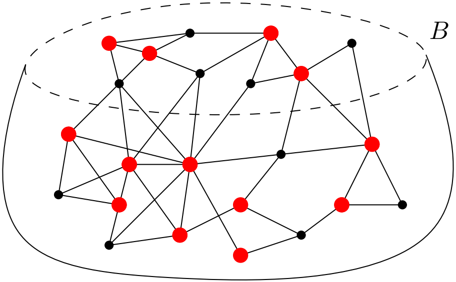

- It is sufficient to store, for each bag B, the subset of vertices of B that belong to a partial solution: 2^{tw} choices
- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

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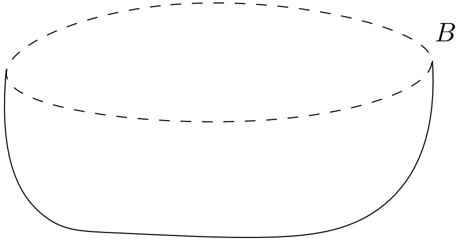
Connectivity problems seem to be more complicated...

Connectivity problems HAMILTONIAN CYCLE, LONGEST PATH, STEINER TREE, CONNECTED VERTEX COVER.



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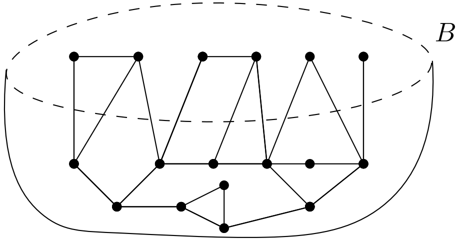

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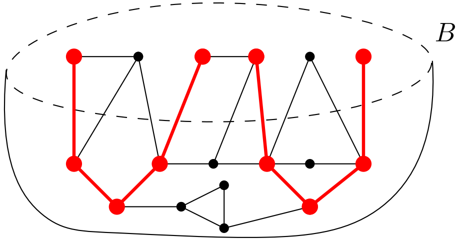

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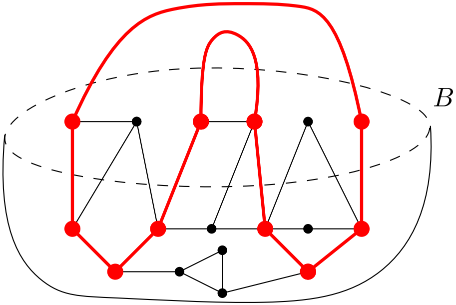

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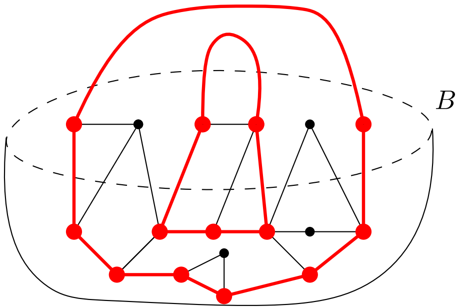

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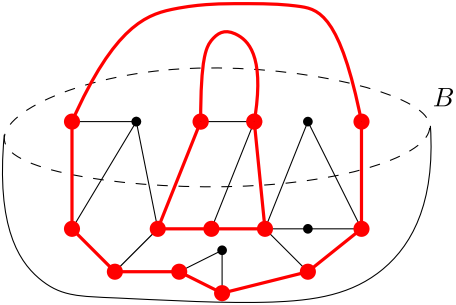

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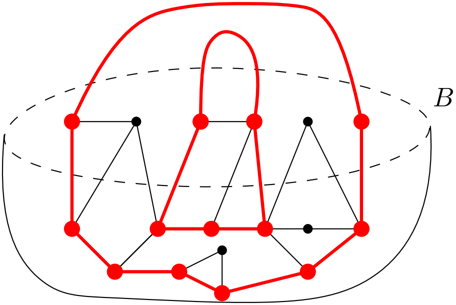

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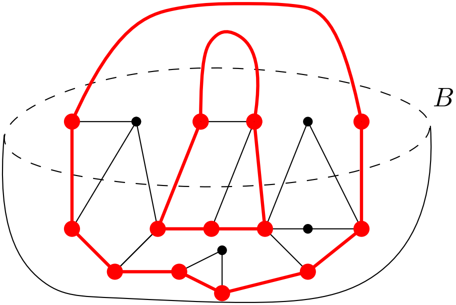

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- The "natural" DP algorithms provide only time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$.

Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

$$\boxed{2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}}$$

Vertex Cover, Dominating Set, ...

- Connectivity problems:

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Longest Path, Steiner Tree, ...

The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

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This was false!!

Cut&Count technique: [Cygan, Nederlof, Pilipczuk², van Rooij, Wojtaszczyk. 2011]

Randomized single-exponential algorithms for connectivity problems.

The revolution of single-exponential algorithms

It was believed that, except on sparse graphs (planar, surfaces), algorithms in time $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$ were optimal for connectivity problems.

This was false!!

Cut&Count technique: [Cygan, Nederlof, Pilipczuk², van Rooij, Wojtaszczyk. 2011]

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1. Relax the connectivity requirement by considering a set of cuts that contain the relevant (connected) solutions.
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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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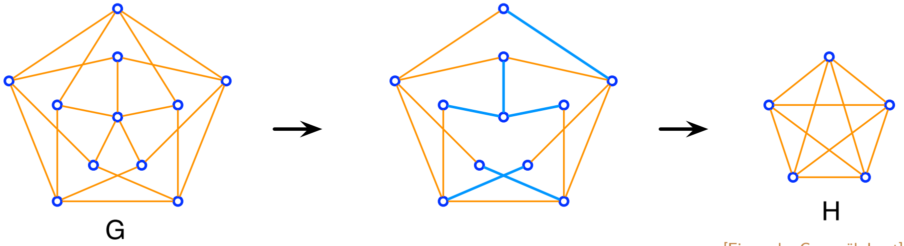
There are other examples of such problems...

Next section is...

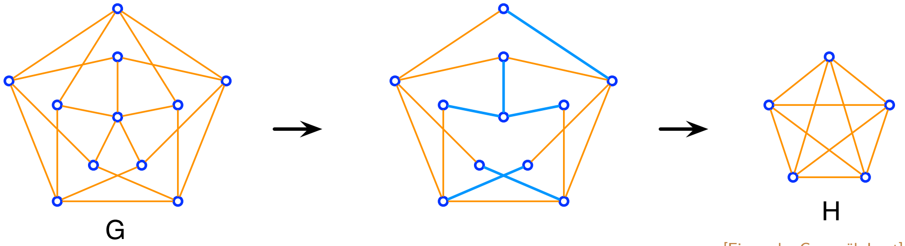
Minors and topological minors



[Figure by Gwenaël Joret]

- H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

Minors and topological minors



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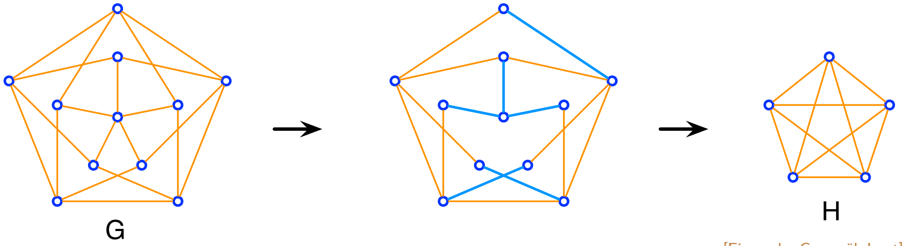

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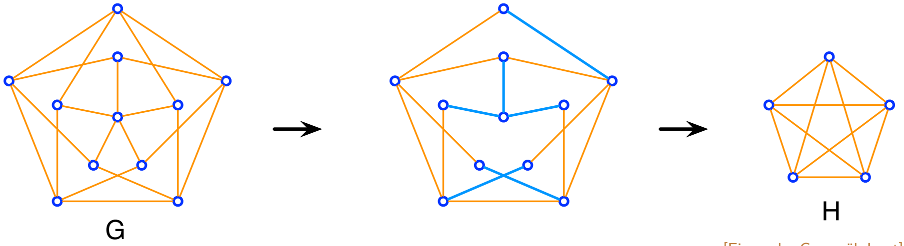

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- Therefore: H topological minor of G $\not\Leftarrow$ H minor of G

The $\mathcal{F}$-M-DELETION problem

Let $\mathcal{F}$ be a fixed finite collection of graphs.

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Both problems are NP-hard if $\mathcal{F}$ contains some edge. [Lewis, Yannakakis. 1980]

FPT by Courcelle's Theorem.

Work with Julien Baste and Dimitrios M. Thilikos (2016-)

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-Deletion/$\mathcal{F}$-TM-Deletion can be solved in time

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- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

Summary of our results: arXiv 1704.07284+1907.04442

[1] Connected collection $\mathcal{F}$: all the graphs are connected.

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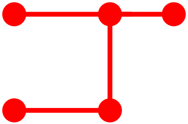
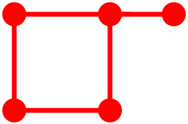
A dichotomy for hitting a connected minor





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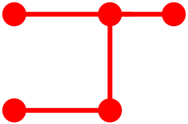

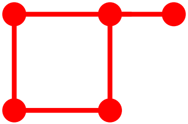

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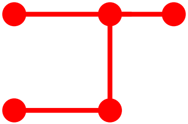

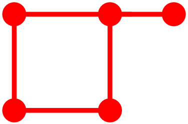

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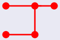
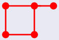
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_c$  *or* $H \preceq_c$ .

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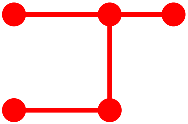

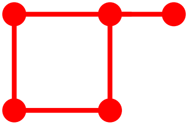

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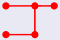
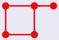
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{c}}$  *or* $H \preceq_{\mathrm{c}}$ .
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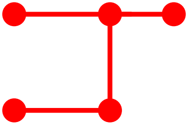

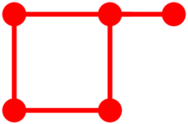

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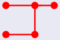
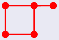
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_c$  *or* $H \preceq_c$ .
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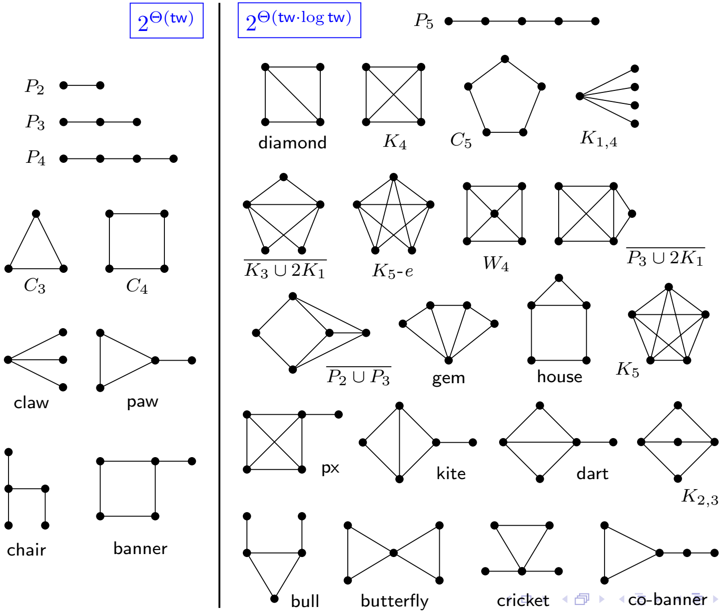
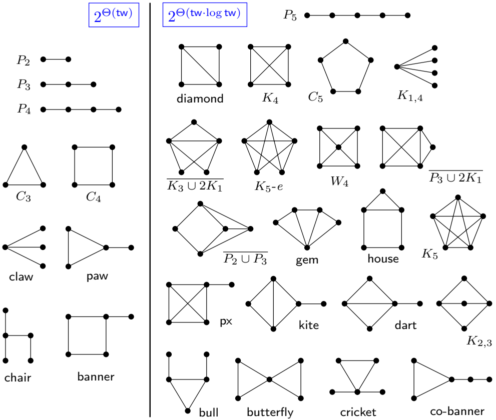
In both cases, the running time is asymptotically *optimal* *under the* ETH.

skip

Complexity of hitting a single connected minor H



A compact statement for a single connected graph



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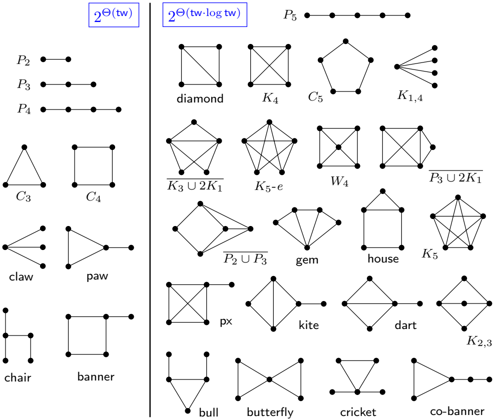

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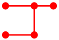
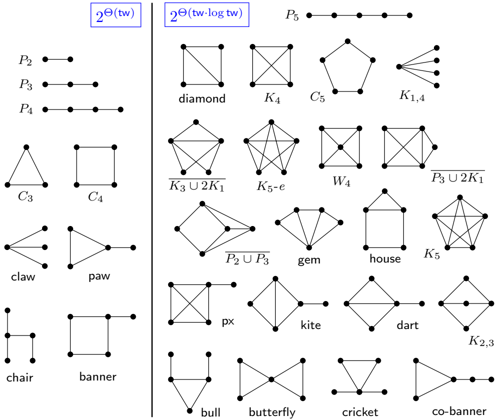
- All graphs on the left are contractions of  or 

A compact statement for a single connected graph



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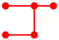
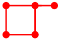
- All graphs on the left are contractions of  or 
- All graphs on the right are not contractions of  or 

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3. Lower bounds under the ETH
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     [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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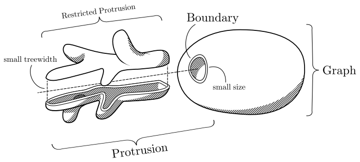

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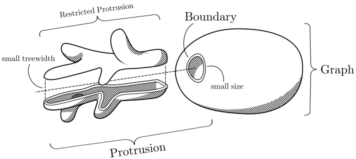

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  Extra: Bidimensionality, irrelevant vertices, protrusion decompositions...

skip

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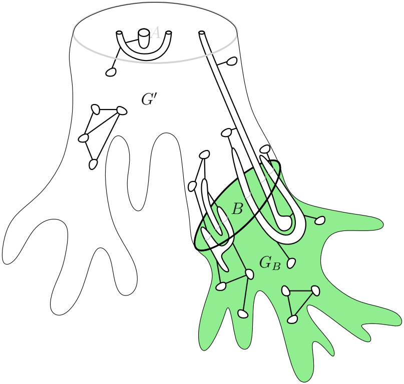

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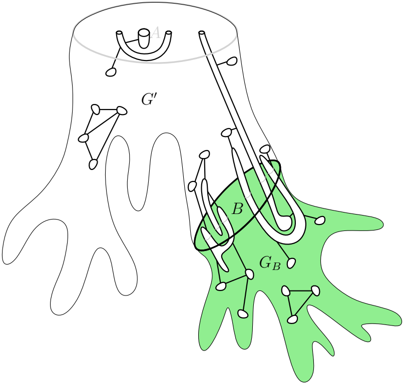

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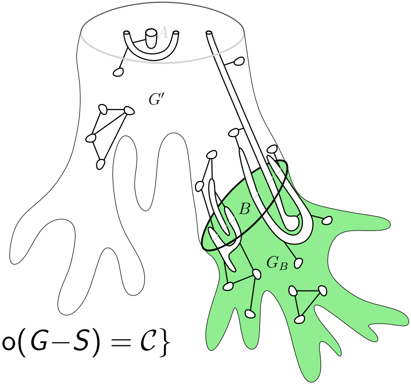

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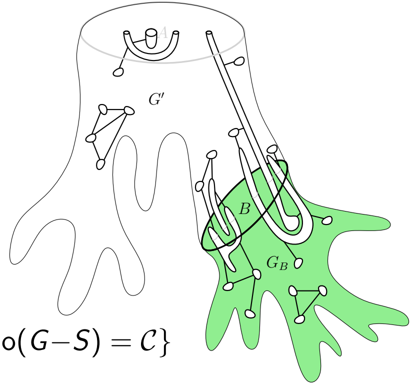

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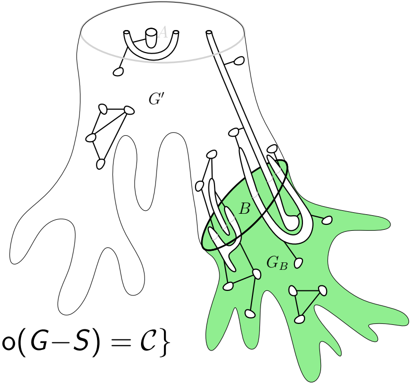

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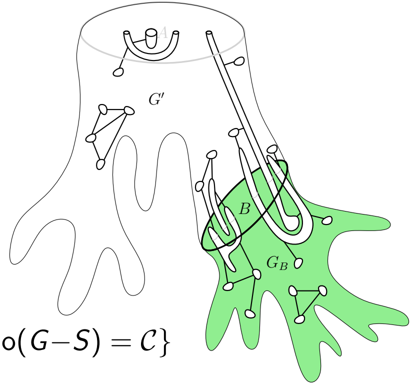

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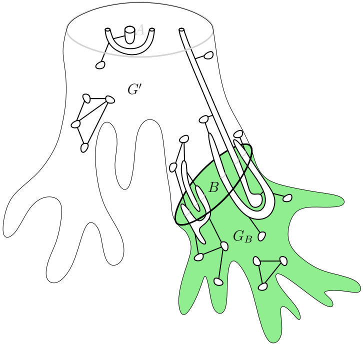

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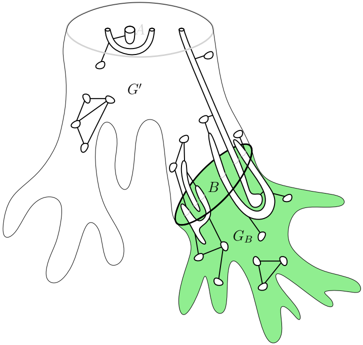

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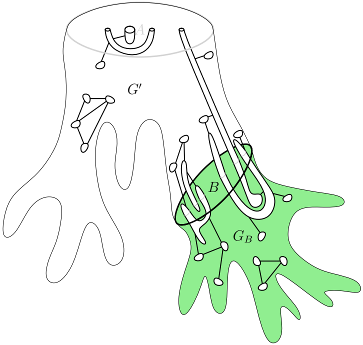

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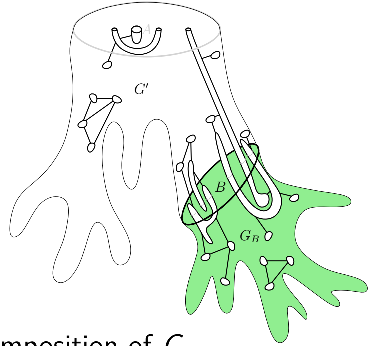

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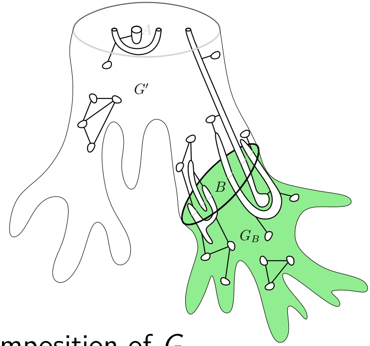

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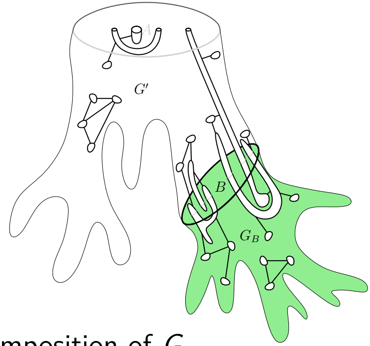

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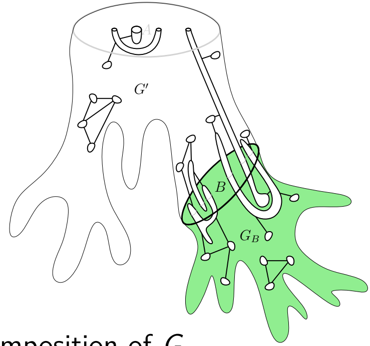

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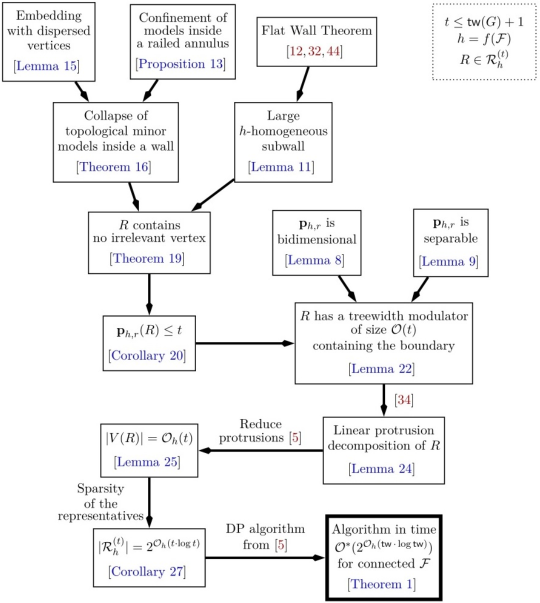

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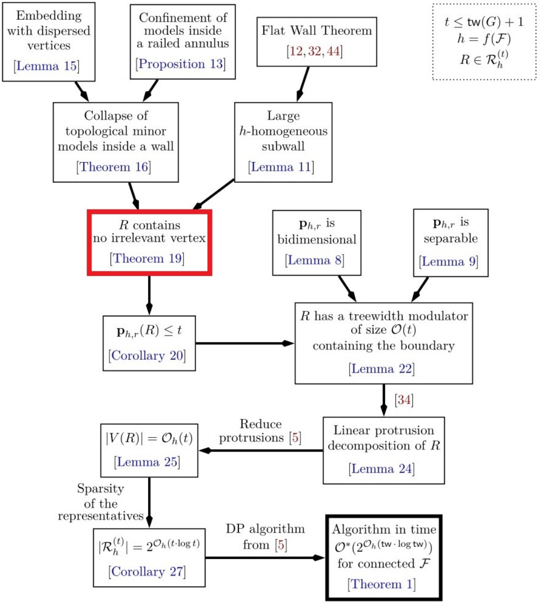

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Hard part: finding an irrelevant vertex inside a flat wall

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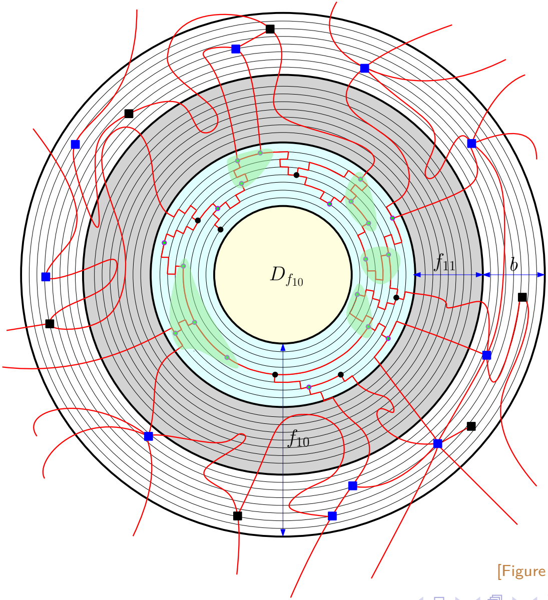

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[Figure by Dimitrios M. Thilikos]

Algorithm when the input graph G is planar

- Idea get an improved bound on $|\mathcal{R}^{(\mathcal{F},t)}|$.

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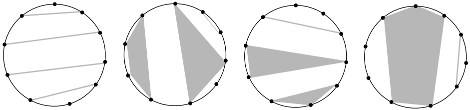

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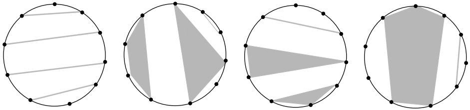

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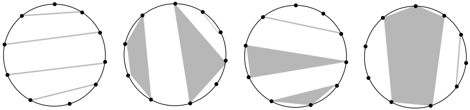

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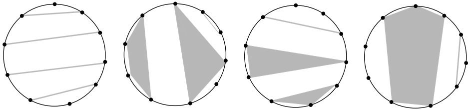

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. skip [Rué, S., Thilikos. 2014]

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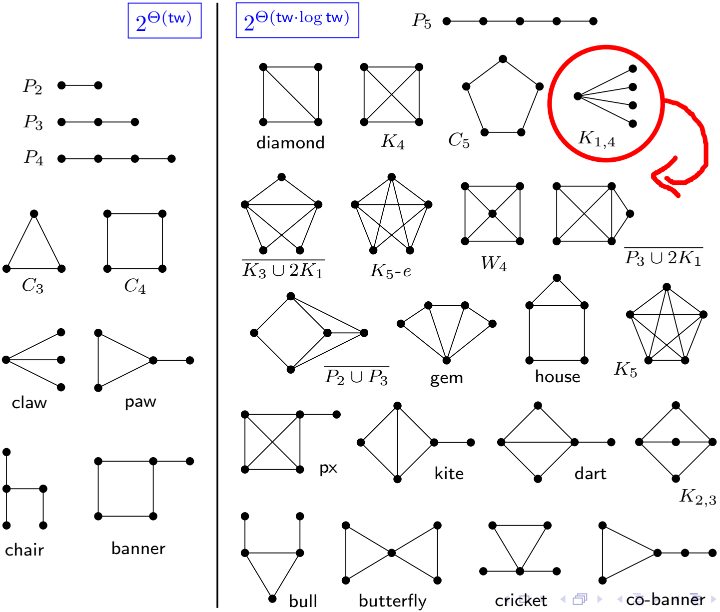
For topological minors, there is (at least) one change



Gràcies!