# Efficient algorithms parameterized by treewidth

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**Algorithms and Combinatorics seminar**
**PESC-UFRJ, Rio de Janeiro, Brazil**
December 2019

# Outline of the talk

1. Introduction
   - Parameterized complexity
   - Treewidth
2. FPT algorithms parameterized by treewidth
3. The $\mathcal{F}$-Deletion problem

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# Crucial notion in complexity theory: NP-completeness

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- Karp (1972): list of 21 *important* NP-complete problems.
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- Nowadays, literally thousands of problems are known to be NP-hard:
  unless $\mathrm{P} = \mathrm{NP}$, they cannot be solved in polynomial time.
- But what does it mean for a problem to be NP-hard?

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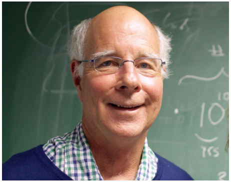
Message In many applications, not only the total size of the instance matters, but also the value of an additional parameter.

# The area of parameterized complexity

Idea Measure the complexity of an algorithm in terms of the input size and an additional integer parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established area with hundreds of articles published every year in the most prestigious TCS journals and conferences.

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These three problems are NP-hard, but are they equally hard?

# They behave quite differently...

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   The problem is para-NP-hard

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Working hypothesis of parameterized complexity: $k$-CLIQUE is not FPT

(in classical complexity: 3-SAT cannot be solved in poly-time)

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Do all FPT problems admit polynomial kernels? NO!

**Theorem (Bodlaender, Downey, Fellows, Hermelin. 2009)**

*Deciding whether a graph has a* PATH *with* $\geq k$ *vertices* is FPT *but does not admit a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

# Typical approach to deal with a parameterized problem

Parameterized problem $L$

$k$-CLIQUE

$k$-VERTEX COVER

$k$-PATH

VERTEX $k$-COLORING

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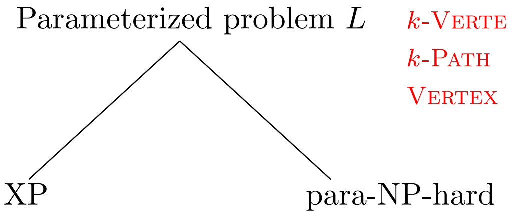


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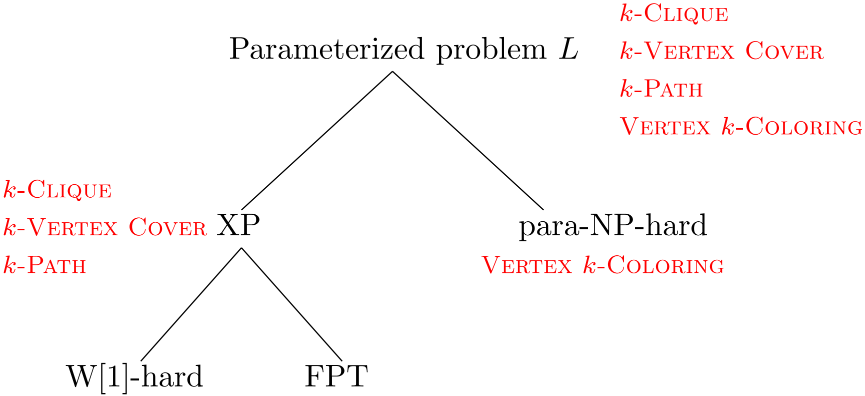

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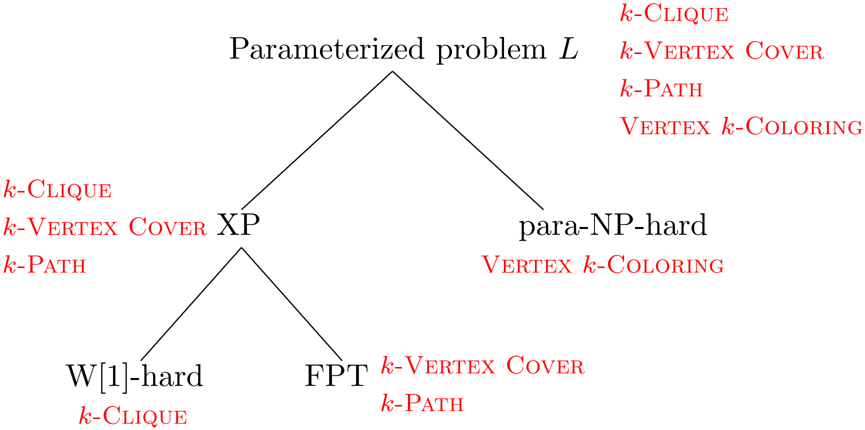

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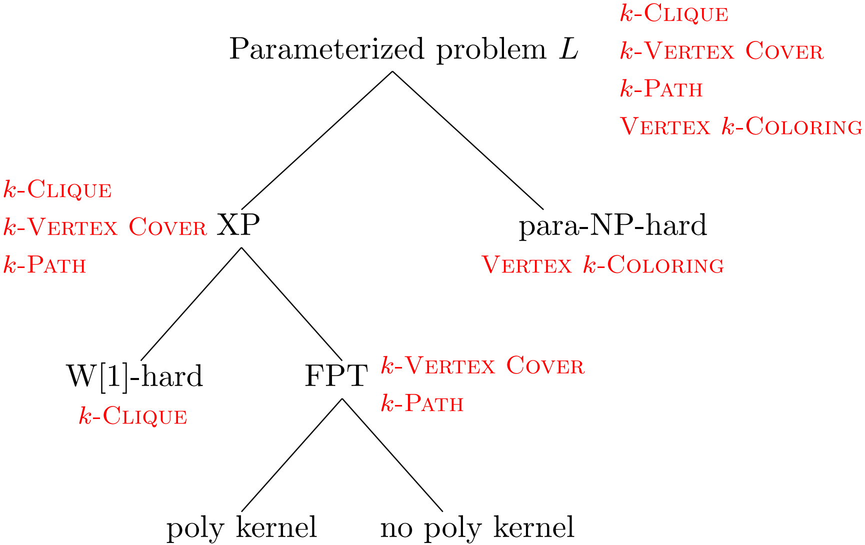

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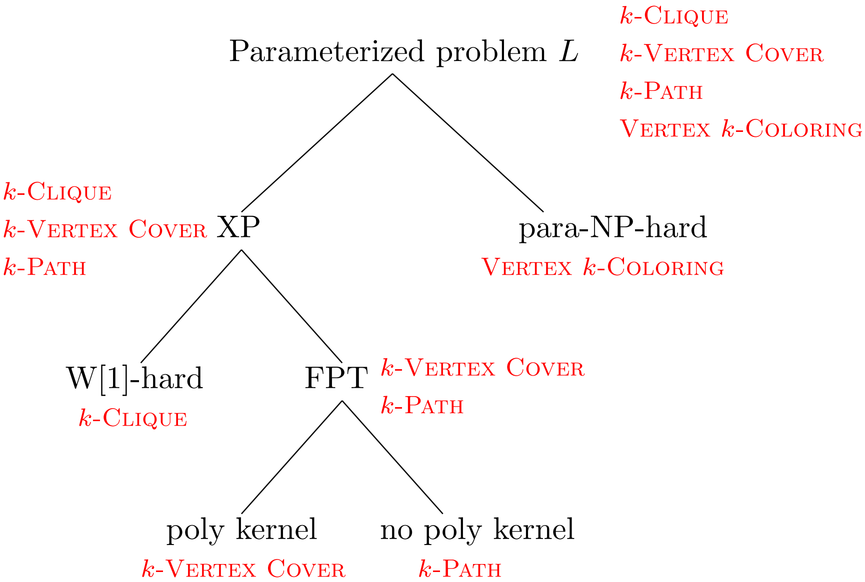

# Next subsection is...

1. Introduction
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   - Treewidth

2. FPT algorithms parameterized by treewidth

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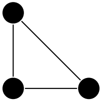
# Treewidth via $k$-trees

Example of a 2-tree:



[Figure by Julien Baste]

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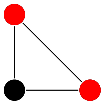

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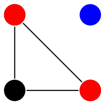

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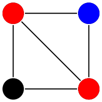

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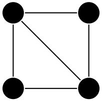

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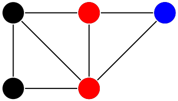

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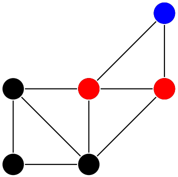

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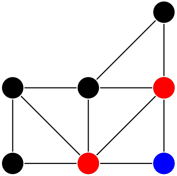

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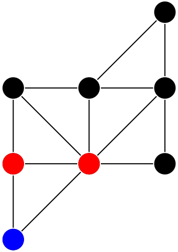

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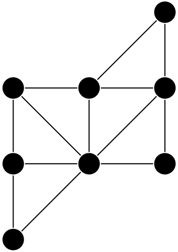

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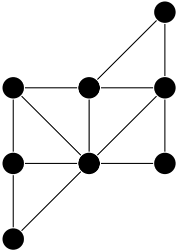

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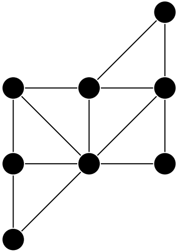

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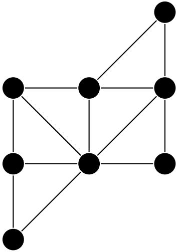

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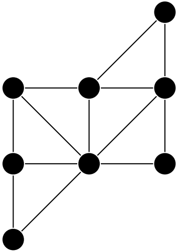

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Construction suggests the notion of tree decomposition: small separators.

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3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

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Monadic Second Order Logic (MSOL):
Graph logic that allows quantification over sets of vertices and edges.

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Typical statements:

ETH $\Rightarrow$ $k$-Vertex Cover cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$.

ETH $\Rightarrow$ Planar $k$-Vertex Cover cannot in time $2^{o(\sqrt{k})} \cdot n^{O(1)}$.

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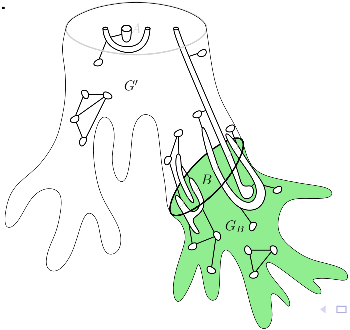
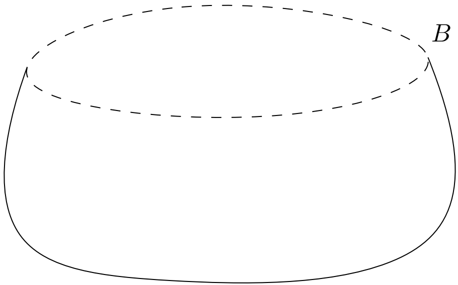
- Starting from the leaves of the tree decomposition, a set of appropriately defined partial solutions is computed recursively until the root, where a global solution is obtained.
- The way that these partial solutions are defined depends on each particular problem:




[Figure by Valentin Garnero]

# Two behaviors for problems parameterized by treewidth

Local problems: Vertex Cover, Dominating Set, Clique, Independent Set, $q$-Coloring for fixed $q$.



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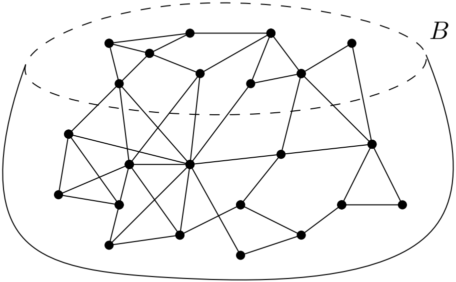

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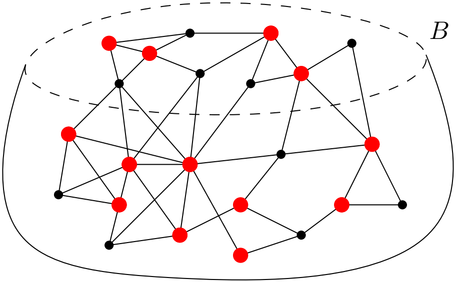

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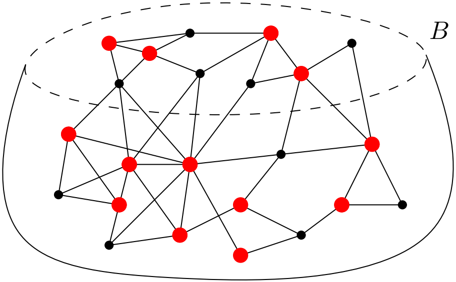


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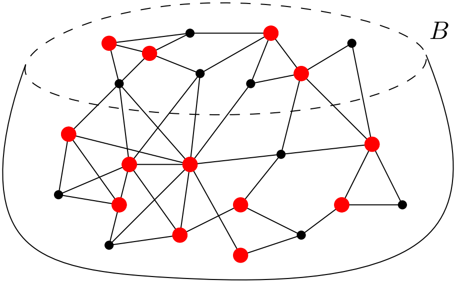


- It is sufficient to store, for each bag $B$, the subset of vertices of $B$ that belong to a partial solution: $2^{\mathrm{tw}}$ choices
- The "natural" DP algorithms lead to (optimal) single-exponential algorithms:

$$2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}.$$

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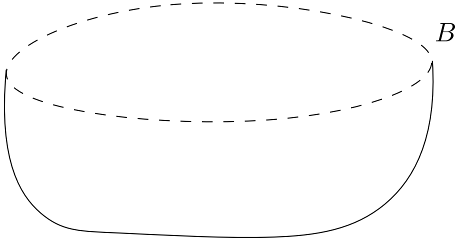
Connectivity problems HAMILTONIAN CYCLE, LONGEST PATH, STEINER TREE, CONNECTED VERTEX COVER.



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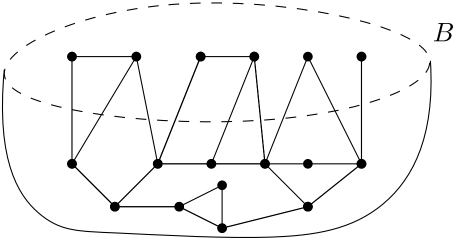

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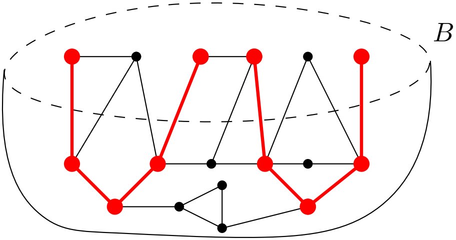

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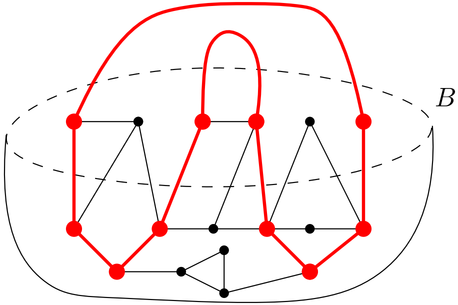

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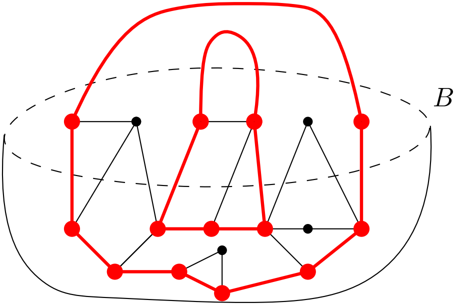

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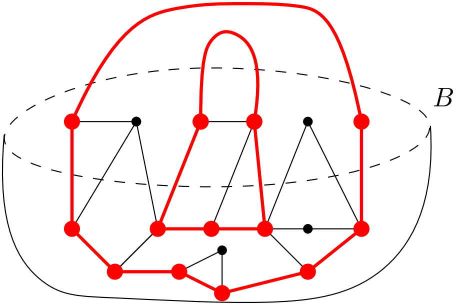


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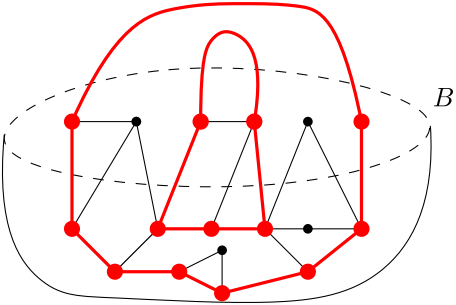


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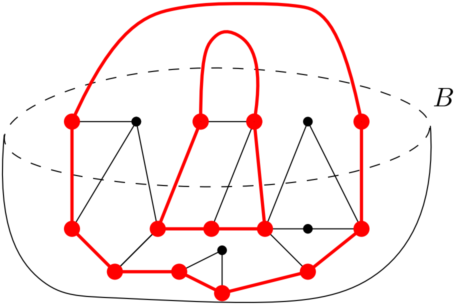


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# Two types of behavior

There seem to be two behaviors for problems parameterized by treewidth:

- Local problems:

$$2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}$$

Vertex Cover, Dominating Set, ...

- Connectivity problems:

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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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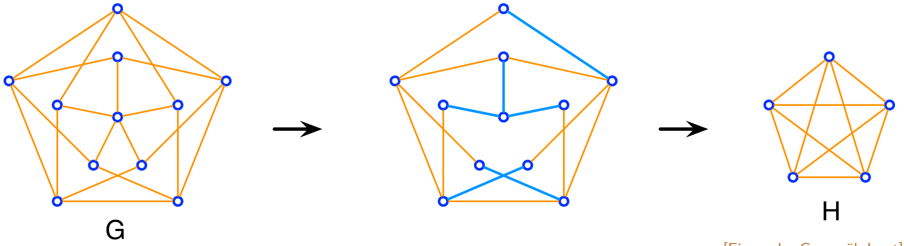
There are other examples of such problems...

# Next section is...

# Minors and topological minors




[Figure by Gwenaël Joret]

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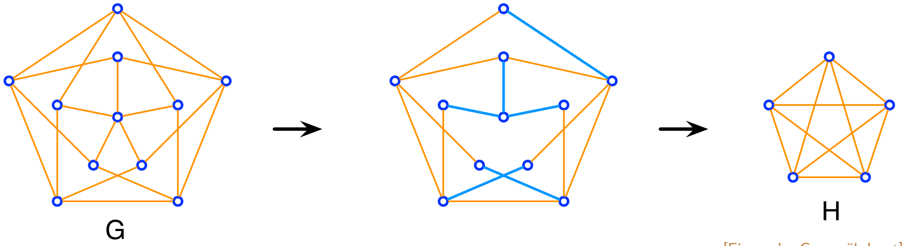


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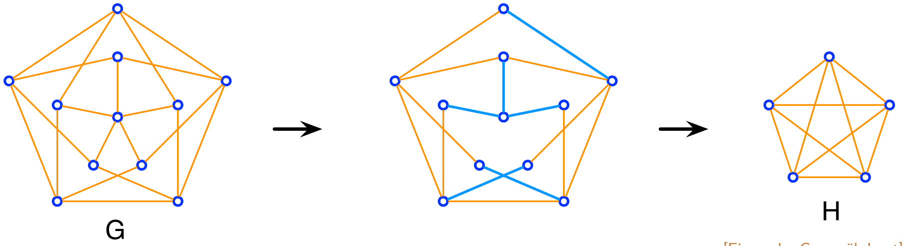


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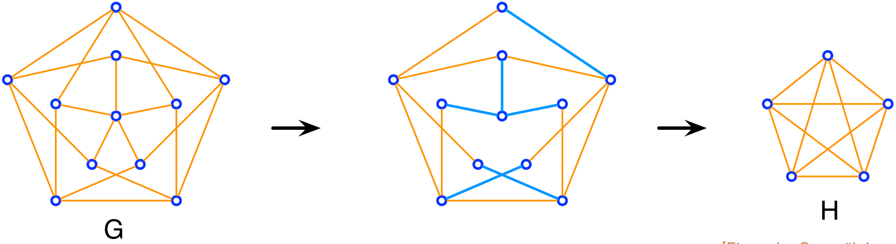


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|---|---|
| $\mathcal{F}$-M-Deletion | |
| **Input**: | A graph $G$ and an integer $k$. |
| **Parameter**: | The treewidth tw of $G$. |
| **Question**: | Does $G$ contain a set $S \subseteq V(G)$ with $\|S\| \leq k$ such that $G - S$ does not contain any of the graphs in $\mathcal{F}$ as a minor? |

- $\mathcal{F} = \{K_2\}$: Vertex Cover.
  Easily solvable in time $2^{\Theta(\text{tw})} \cdot n^{\mathcal{O}(1)}$.
- $\mathcal{F} = \{K_3\}$: Feedback Vertex Set.
  "Hardly" solvable in time $2^{\Theta(\text{tw})} \cdot n^{\mathcal{O}(1)}$. [Cut&Count. 2011]
- $\mathcal{F} = \{K_5, K_{3,3}\}$: Vertex Planarization.
  Solvable in time $2^{\Theta(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$. [Jansen, Lokshtanov, Saurabh. 2014 + Pilipczuk. 2015]

# Covering topological minors

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| **Input**: | A graph $G$ and an integer $k$. |
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Both problems are NP-hard if $\mathcal{F}$ contains some edge. [Lewis, Yannakakis. 1980]

FPT by Courcelle's Theorem.

# Work with Julien Baste and Dimitrios M. Thilikos (2016-)

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-Deletion/$\mathcal{F}$-TM-Deletion can be solved in time

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- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

# Summary of our results: arXiv 1704.07284+1907.04442

[1] Planar collection $\mathcal{F}$: contains at least one planar graph.

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- $\mathcal{F} = \{H\}$, $H$ connected: complete tight dichotomy...

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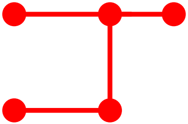
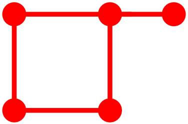
# A dichotomy for hitting a connected minor





# A dichotomy for hitting a connected minor

**Theorem**

*Let H be a connected graph.*

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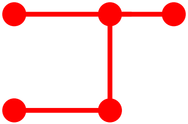

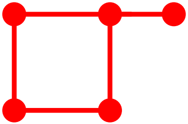

**Theorem**

*Let H be a connected graph.*
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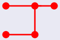
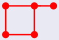
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{c}}$  *or* $H \preceq_{\mathrm{c}}$ .

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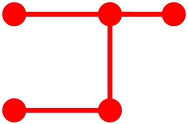
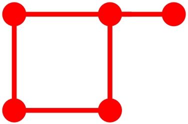

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*Let* $H$ *be a* *connected* *graph.*
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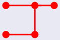
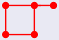
- $2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_c$  *or* $H \preceq_c$ .
- $2^{\mathcal{O}(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

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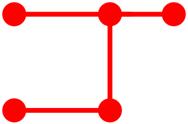

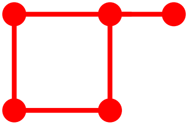

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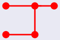
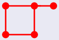
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_c$  *or* $H \preceq_c$ .
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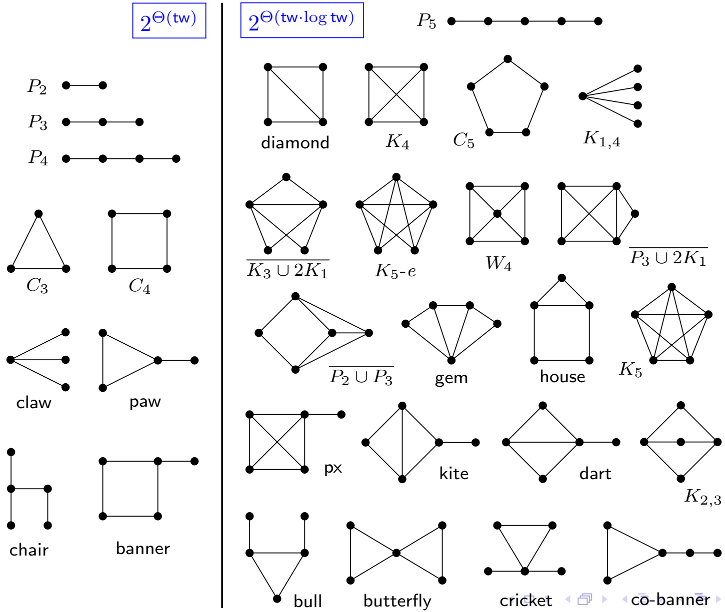
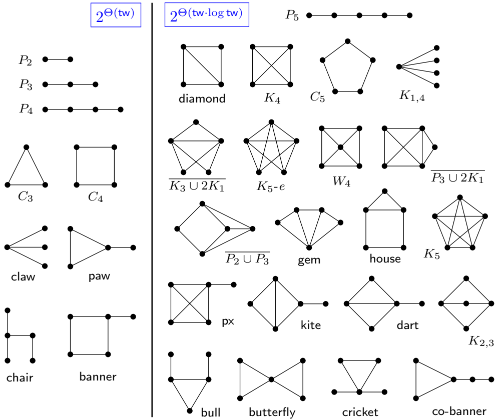
*In both cases, the running time is asymptotically optimal under the ETH.*

skip

# Complexity of hitting a single connected minor $H$



# A compact statement for a single connected graph




All these cases can be succinctly described as follows:

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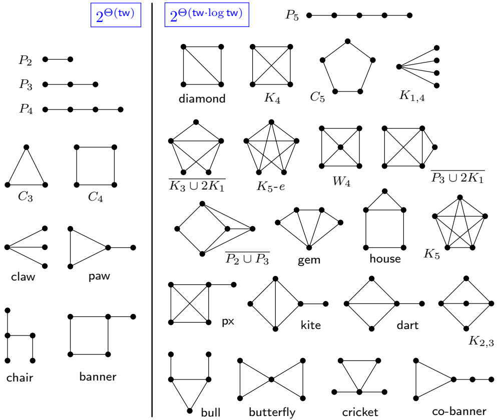


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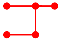
- All graphs on the left are contractions of  or 

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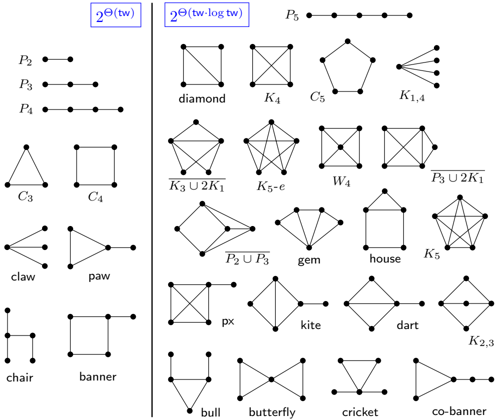


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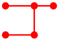
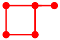
- All graphs on the left are contractions of  or 
- All graphs on the right are not contractions of  or 

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3. Lower bounds under the ETH

   - $2^{o(\text{tw})}$ is "easy".
   - $2^{o(\text{tw} \cdot \log \text{tw})}$ is much more involved and we get ideas from:

[Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

# Some ideas of the general algorithms

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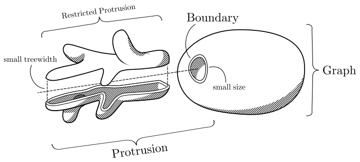


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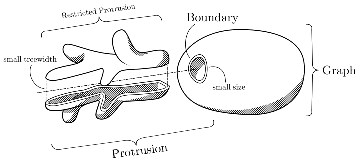


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  Extra: Bidimensionality, irrelevant vertices, protrusion decompositions...

skip

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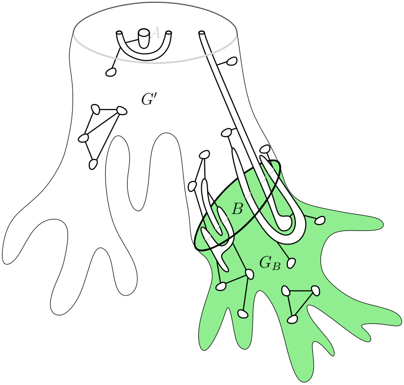

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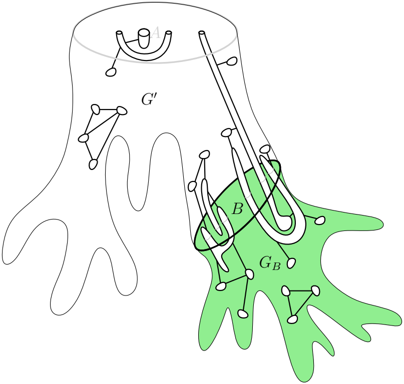

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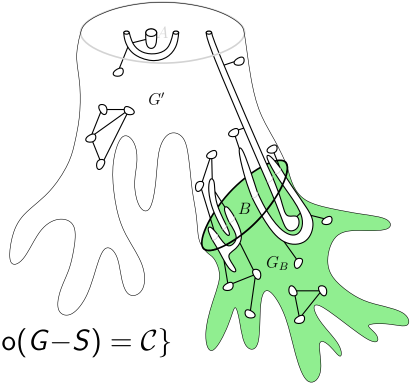

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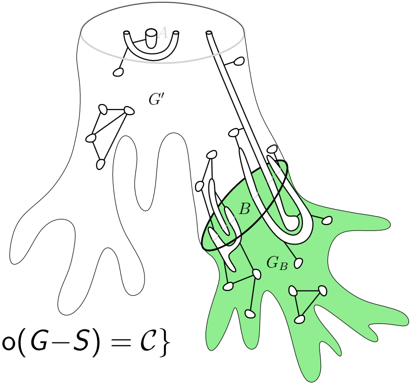


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- For every $t$-boundaried graph $G$,

$$|\text{folio}(G)| = \mathcal{O}_{\mathcal{F}}(1) \cdot \binom{t^2}{t} = 2^{\mathcal{O}_{\mathcal{F}}(t \cdot \log t)}$$

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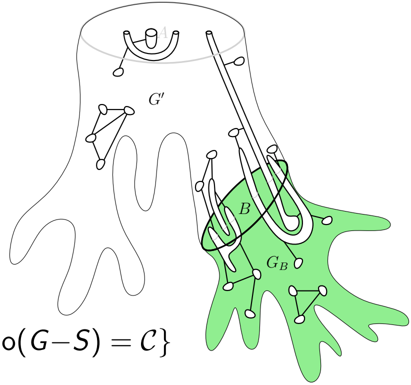


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- For every $t$-boundaried graph $G$,

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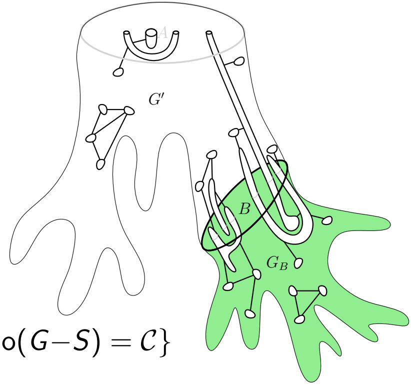


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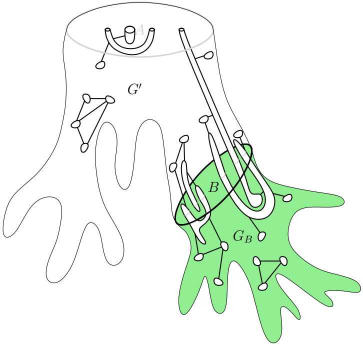

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- For a fixed $\mathcal{F}$, we define an equivalence relation $\equiv^{(\mathcal{F},t)}$ on $t$-boundaried graphs:

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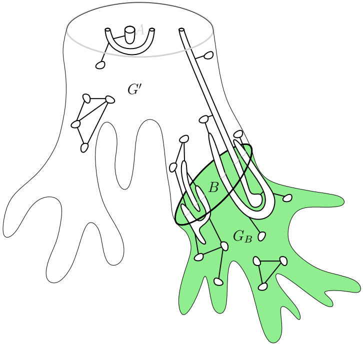

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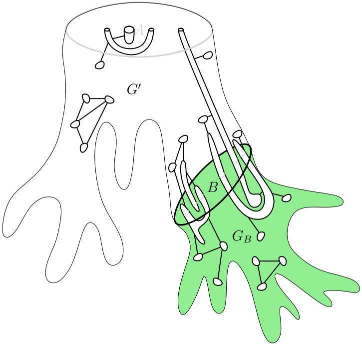

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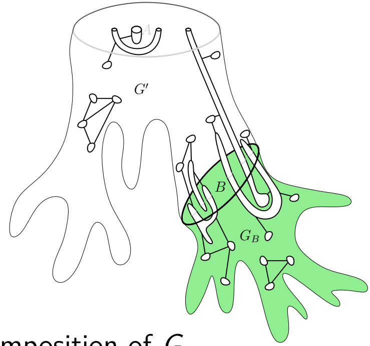


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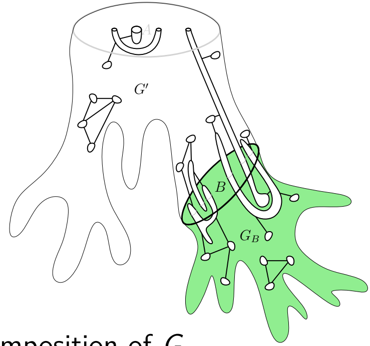


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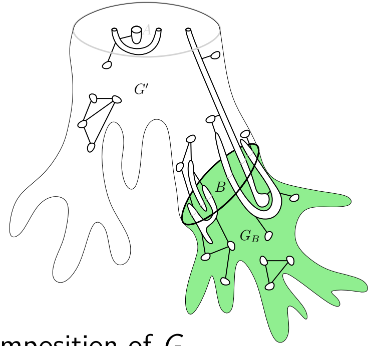


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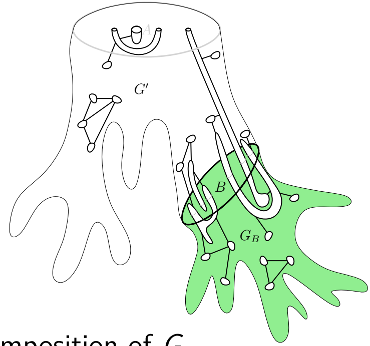


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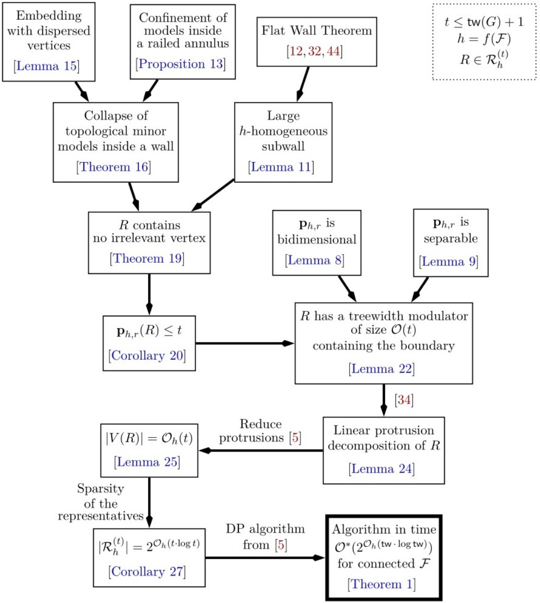


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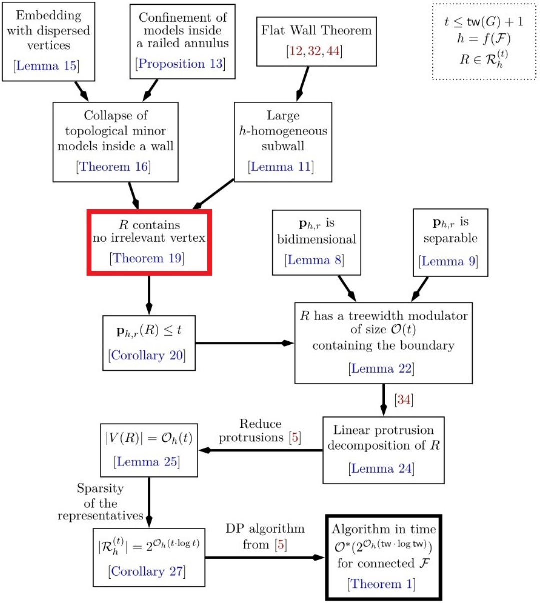


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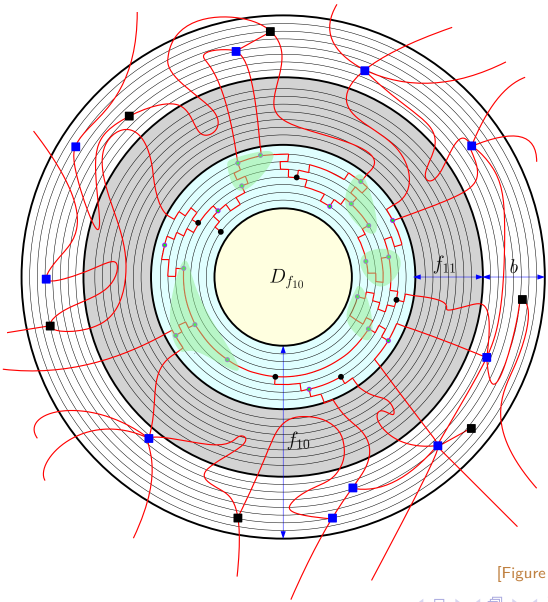


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[Figure by Dimitrios M. Thilikos]

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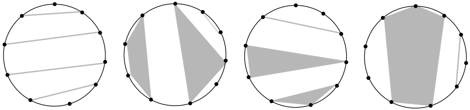

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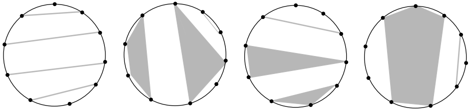

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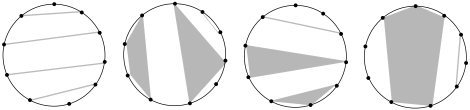

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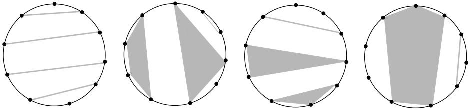

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- We can extend this algorithm to input graphs $G$ embedded in arbitrary surfaces by using surface-cut decompositions. skip [Rué, S., Thilikos. 2014]

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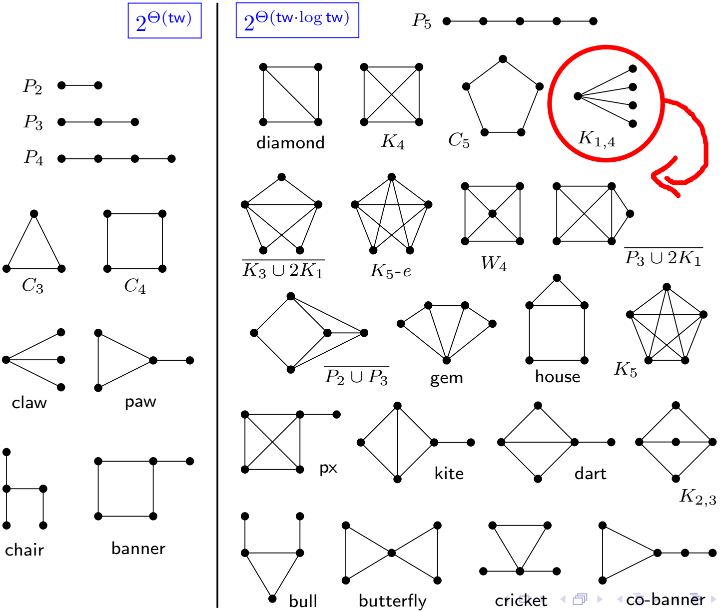
  - Conjecture For every family $\mathcal{F}$, the $\mathcal{F}$-TM-Deletion problem is solvable in time $2^{\mathcal{O}(\mathsf{tw} \cdot \log \mathsf{tw})} \cdot n^{\mathcal{O}(1)}$.

# For topological minors, there is (at least) one change



# Gràcies!