Degree-Constrained Subgraph Problems: Hardness and Approximation Results

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Outline of the talk

- Introduction
- Problem 1
  - Definition + results
  - An approximation algorithm
- Problem 2
  - Definition + results
  - A hardness result
- Problem 3
  - Definition + results
- Further research

Degree-Constrained Subgraph Problems

Broad family of problems

- A *typical* **DEGREE-CONSTRAINED SUBGRAPH PROBLEM**:

  Input:

  - a (*weighted* or *unweighted*) graph G, and
  - an integer d.

  Output:

  - a (*connected*) subgraph H of G,
  - satisfying some degree constraints ($\Delta(H) \leq d$ or $\delta(H) \geq d$),
  - and optimizing some parameter ($|V(H)|$ or $|E(H)|$).

- Several problems in this broad family are classical widely studied NP-hard problems.

- They have a number of applications in interconnection networks, routing algorithms, chemistry, ...

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1- MAXIMUM *d*-DEGREE-BOUNDED CONNECTED SUBGRAPH

Definition of the problem

- **MAXIMUM *d*-DEGREE-BOUNDED CONNECTED SUBGRAPH (MDBCS$_d$):**

  Input:
  - an undirected graph $G = (V, E)$,
  - an integer $d \geq 2$, and
  - a weight function $\omega : E \to \mathbb{R}^+$.

  Output:
  a subset of edges $E' \subseteq E$ of **maximum weight**, s.t. $G' = (V, E')$
  - is **connected**, and
  - has **maximum degree** $\leq d$.

- It is one of the classical **NP**-hard problems of *[Garey and Johnson, Computers and Intractability, 1979]*.

- If the output subgraph is not required to be connected, the problem is in **P** for any d (using matching techniques).

- For *fixed* $d = 2$ it is the well known **LONGEST PATH (OR CYCLE)** problem.

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- **Maximum d-Degree-Bounded Connected Subgraph (MDBCS_d):**

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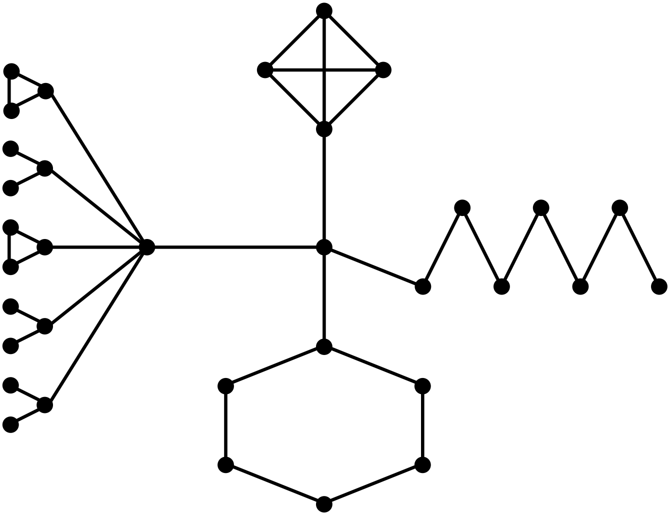
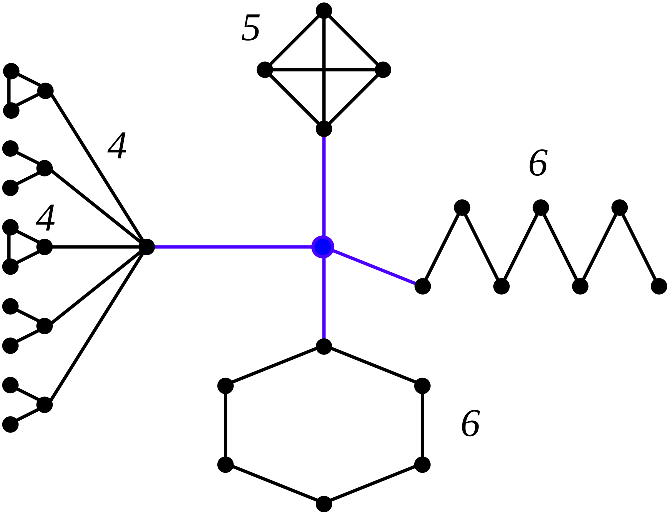
- If the output subgraph is not required to be connected, the problem is in **P** for any d (using matching techniques).
- For *fixed* $d = 2$ it is the well known **Longest Path (or Cycle)** problem.

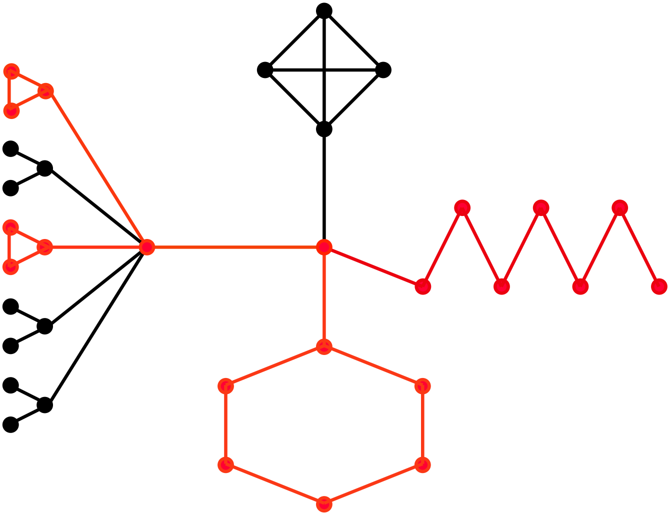
Example with $d = 3$, $\omega(e) = 1$ for all $e \in E(G)$



Example with $d = 3$ (II)



Example with $d = 3$ (III)



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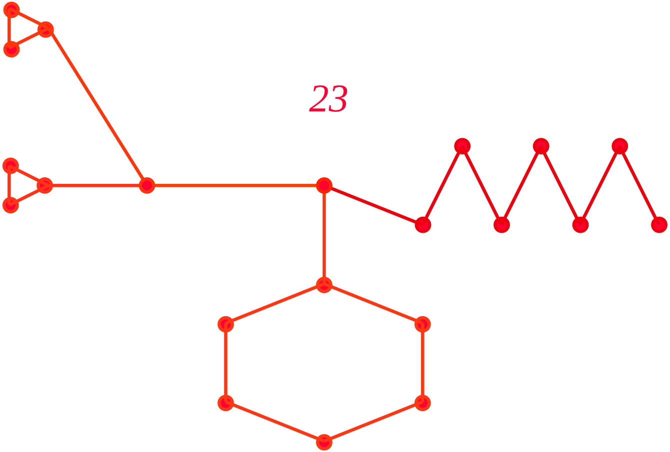

State of the art

To the best of our knowledge, there were no results in the literature except for the case $d = 2$, a.k.a. the **LONGEST PATH** problem:

- **Approximation algorithms**:
  $\mathcal{O}\left(\frac{n}{\log n}\right)$-approximation, using the **color-coding** method.
  [N. Alon, R. Yuster and U. Zwick, STOC'94].
  $\mathcal{O}\left(n\left(\frac{\log\log n}{\log n}\right)^2\right)$-approximation.
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- **Hardness results**:
  It does not accept *any* constant-factor approximation.
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Our results

- **Approximation algorithms** ($n = |V(G)|$, $m = |E(G)|$):
  - $\min\{\frac{n}{2}, \frac{m}{d}\}$-approximation algorithm for **weighted** graphs.
  - $\min\{\frac{m}{\log n}, \frac{nd}{2 \log n}\}$-approximation algorithm for **unweighted** graphs, using *color coding*.
  - when G **accepts a low-degree spanning tree**, in terms of d, then MDBCS$_d$ can be approximated within a **small constant factor**.

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  - For each fixed $d \geq 2$, MDBCS$_d$ does not accept *any* constant-factor approximation in general graphs.

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Approximation algorithm for weighted graphs

Input: undirected graph $G = (V, E)$, a weight function $\omega : E \to \mathbb{R}^+$, and an integer $d \geq 2$. Let $n = |V|$, $m = |E|$.

F: set of d heaviest edges in G, with weight $\omega(F)$.
W: set of endpoints of those edges. Let $H = (W, F)$.

Description of the algorithm: Two cases according to $H = (W, F)$:

(1) If $H = (W, F)$ is connected, the algorithm returns H.
Claim: this yields a $\min\{n/2, m/d\}$-approximation.

Proof.
Suppose an optimal solution consists of m^* edges of total weight ω^*. Then $ALG = \omega(F) \geq \frac{\omega^*}{m^*} \cdot d$, since by the choice of F the average weight of the edges in F can not be smaller than the average weight of the edges of an optimal solution. As $m^* \leq m$ and $m^* \leq dn/2$, we get that $ALG \geq \frac{\omega^*}{m} \cdot d = \frac{\omega^*}{m/d}$ and $ALG \geq \frac{\omega^*}{dn/2} \cdot d = \frac{\omega^*}{n/2}$.

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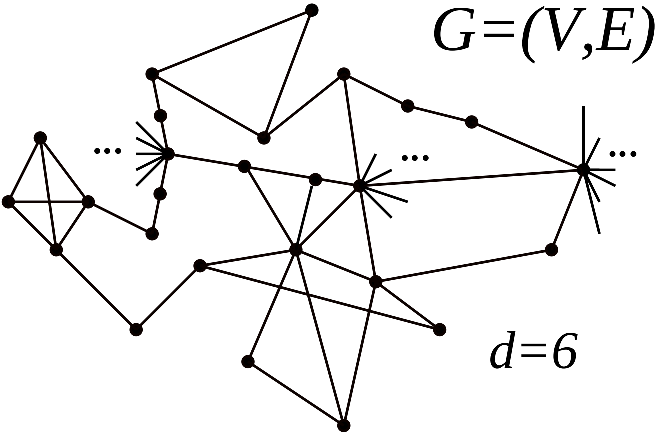
Example of the algorithm for weighted graphs



- Given a weighted graph $G = (V, E)$ and an integer d...

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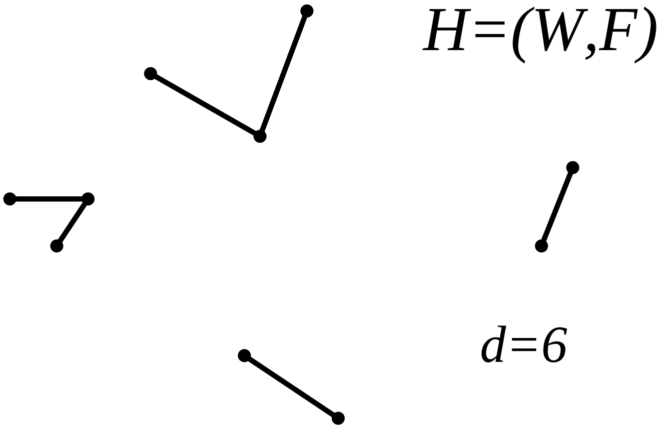

- Let $H = (W, F)$ be the graph induced by the d heaviest edges.

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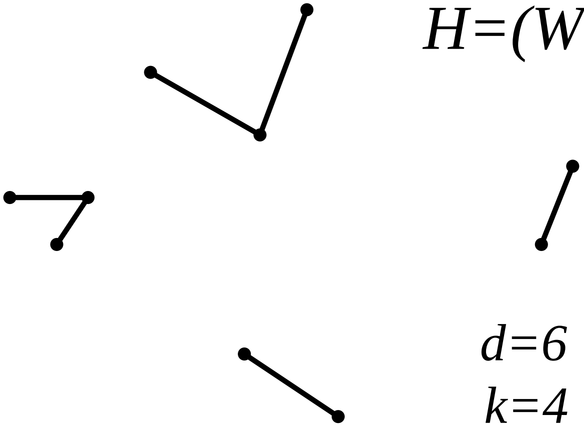

$H=(W,F)$

$d=6$

$k=4$

- Assume H has $k > 1$ connected components.

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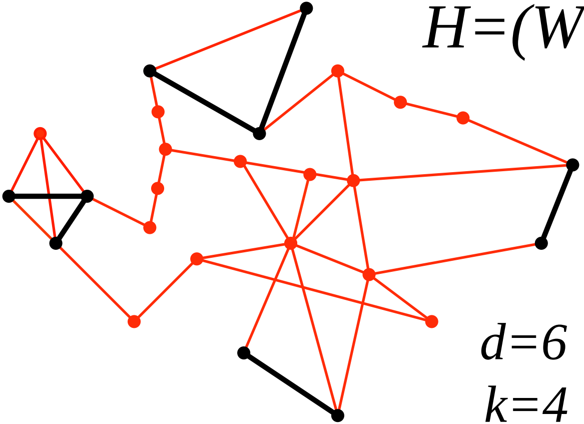

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- We compute the distance in G between each pair of components.

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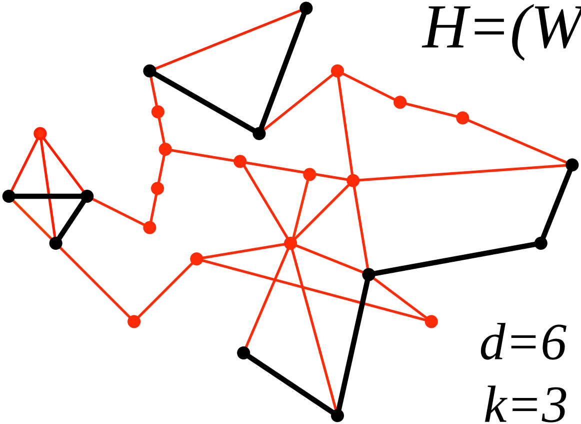

$H=(W,F)$

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- We add to H a path between a pair of closest vertices.

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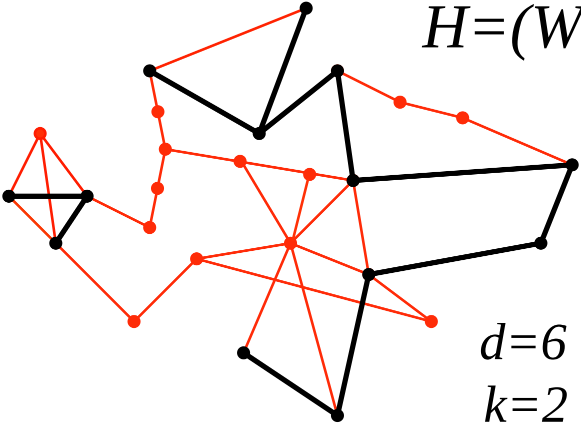

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- We repeat these two steps inductively...

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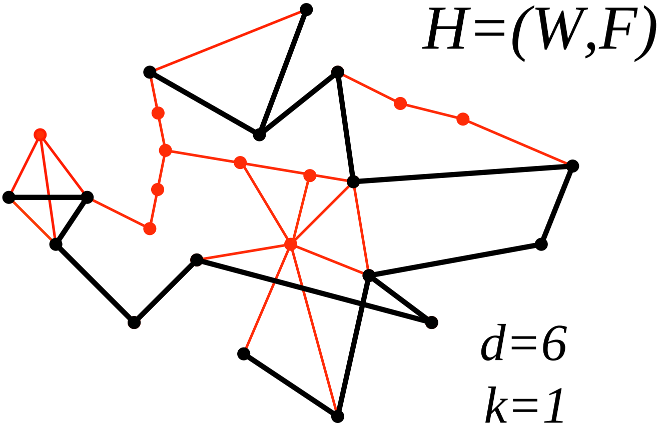

- Until the graph H is connected.

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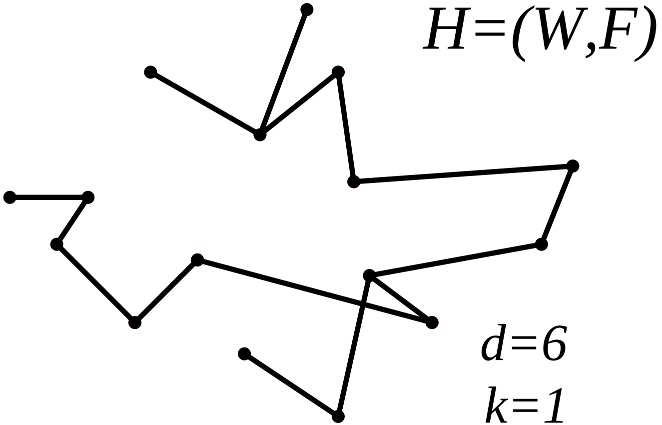

- The algorithm outputs this graph H.

Analysis of the algorithm

(a) Running time: clearly polynomial.

(b) Correctness:

- The output subgraph is connected.
- **Claim**: after i phases, $\Delta(H) \leq d - k + i + 1$.
  The proof is done by induction. When $i = k - 1$ we get $\Delta(H) \leq d$.

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- **Claim**: after i phases, $\Delta(H) \leq d - k + i + 1$.
  The proof is done by induction. When $i = k - 1$ we get $\Delta(H) \leq d$.

(c) **Approximation ratio**: follows from case (1).

2- MINIMUM SUBGRAPH OF MINIMUM DEGREE $\geq d$

Definition of the problem

- **MINIMUM SUBGRAPH OF MINIMUM DEGREE $\geq d$ (MSMD$_d$)**:

  Input: an undirected graph $G = (V, E)$ and an integer $d \geq 3$.

  Output: a subset $S \subseteq V$ with $\delta(G[S]) \geq d$, s.t. $|S|$ is minimum.

- For $d = 2$ it is the GIRTH problem (find the length of a shortest cycle), which is in P.

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State of the art + our results

- This problem was first introduced in [O. Amini, I. S. and S. Saurabh, IWPEC'08].
  - W[1]-hard in general graphs, for $d \geq 3$.
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Idea of the proof for $d = 3$

(1) First we will see that $\mathrm{MSMD}_3 \notin \mathrm{PTAS}$.

(2) Then we will see that $\mathrm{MSMD}_3 \notin \mathrm{APX}$.

(1) $MSMD_3$ is not in $PTAS$

- Reduction from VERTEX COVER:

  Instance H of VERTEX COVER $\rightarrow$ Instance G of MSMD$_3$

- We will see that

$$\text{PTAS for } G \Rightarrow \text{PTAS for } H$$

- And so,

$$\nexists \text{ PTAS for MSMD}_3$$

- We can suppose $|E(H)| = 3 \cdot 2^m$ and $\delta(H) \geq 3$.

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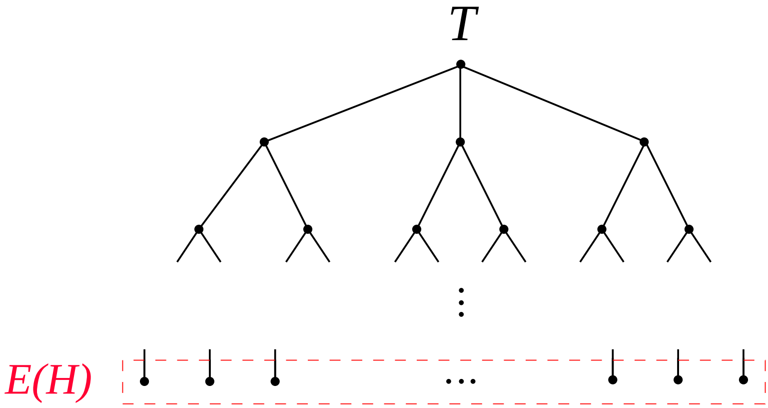
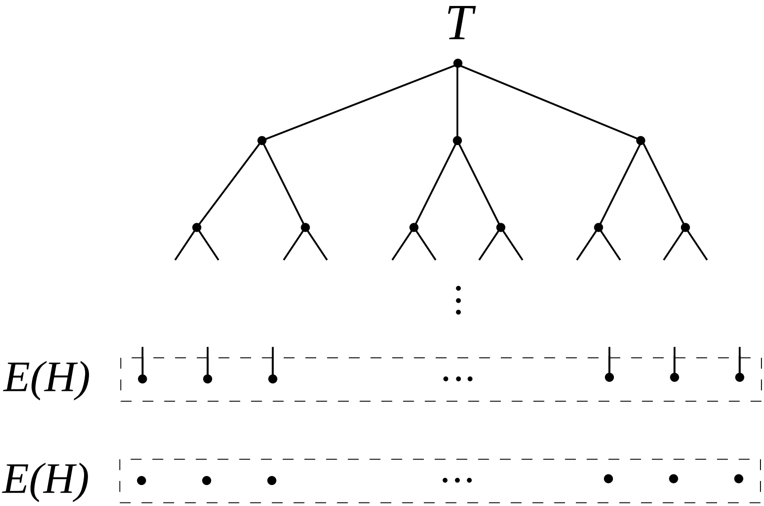
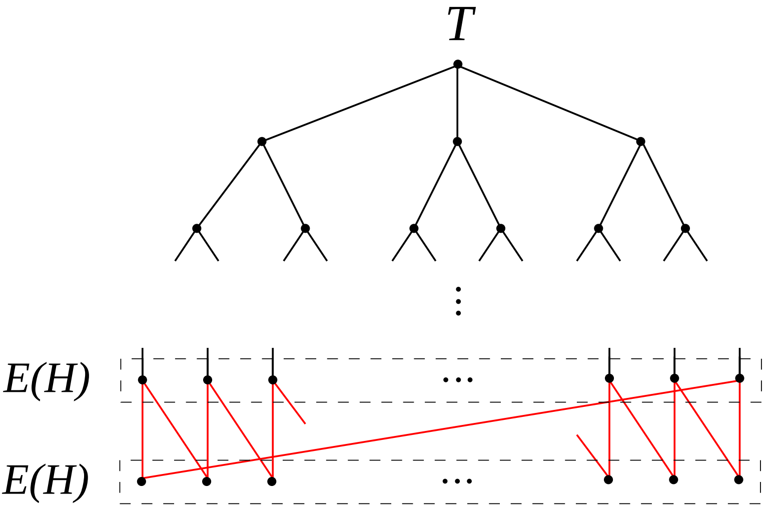
We build a complete ternary tree with $|E(H)| = 3 \cdot 2^m$ leaves:



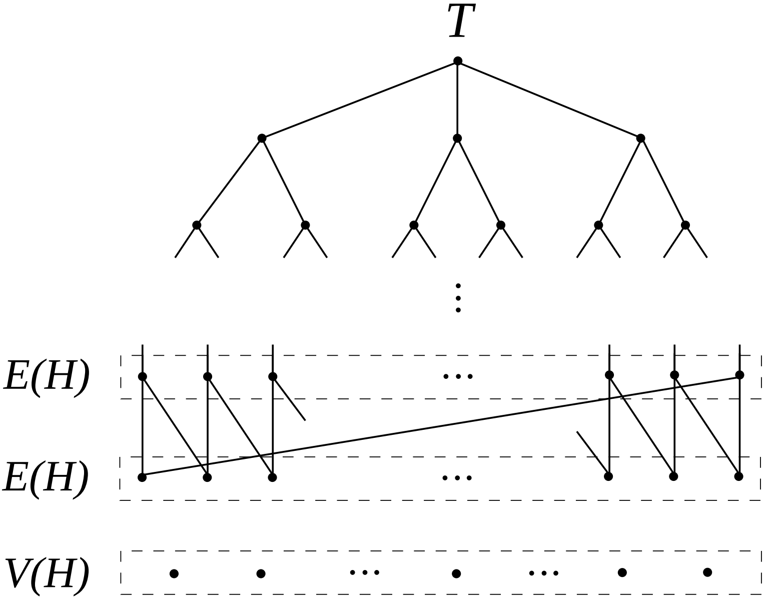
We add a copy of the set of leaves $E(H)$:



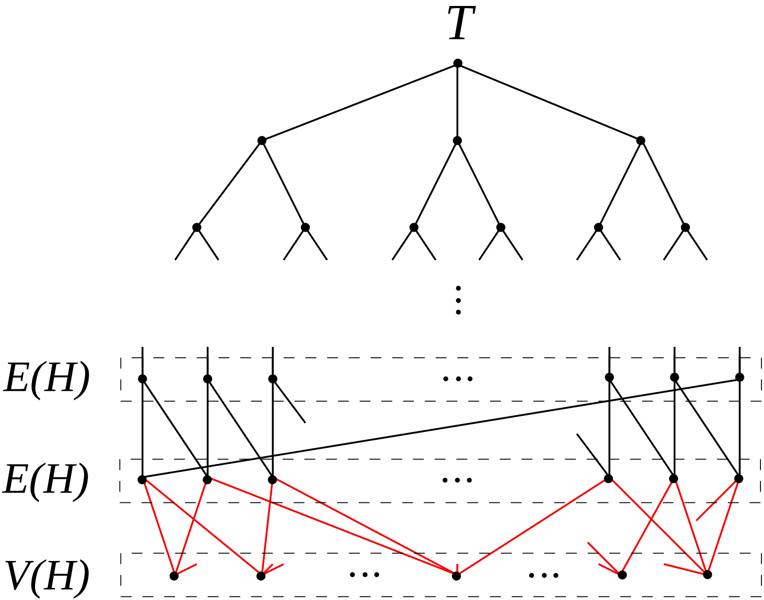
We join both sets with a Hamiltonian cycle (for technical reasons):



We add all the vertices of H:



We add the incidence relations between $E(H)$ and $V(H) \rightarrow G$:



(1) MSMD_3 is not in PTAS

- If we touch a vertex of $G \setminus V(H)$, we have to touch all the vertices of $G \setminus V(H)$

- Thus, MSMD_3 in G is equivalent to minimize the number of selected vertices in $V(H)$

  $\rightarrow$ this is **exactly** VERTEX COVER in H !!

- Thus,

$$OPT_{\text{MSMD}_3}(G) = OPT_{\text{VC}}(H) + |V(G \setminus V(H))| =$$
$$= OPT_{\text{VC}}(H) + 9 \cdot 2^m$$

- This clearly proves that

$$\text{PTAS for MSMD}_3 \Rightarrow \text{PTAS for VERTEX COVER}$$

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(2) MSMD_3 is not in APX

- Let $\alpha > 1$ be the factor of inapproximability of MSMD_3
- We use a technique called **error amplification**:
  - We build a sequence of families of graphs $\mathcal{G}_k$, such that MSMD_3 is hard to approximate in $\mathcal{G}_k$ within a factor α^k
  - This proves that the problem is not in APX

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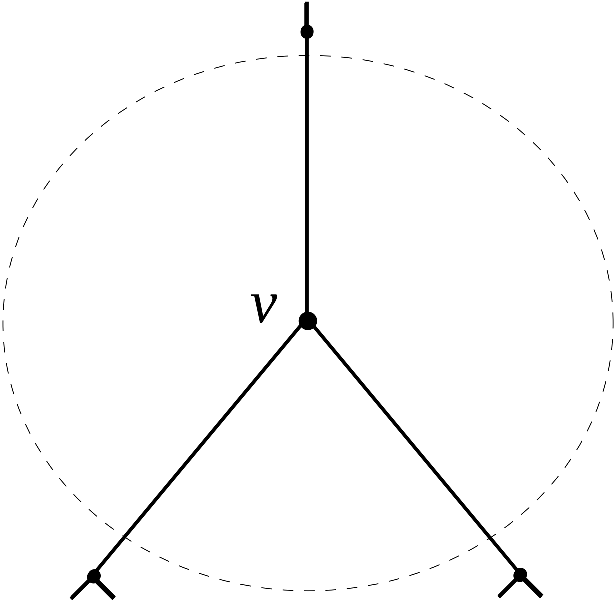
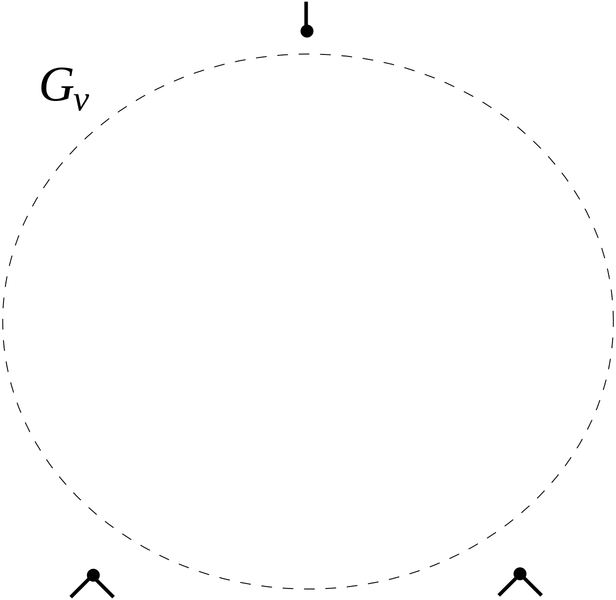
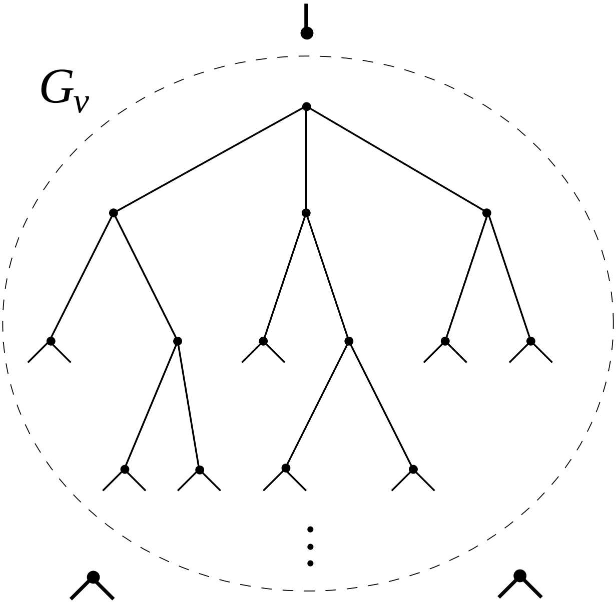
For any vertex v (note its degree by d_v):



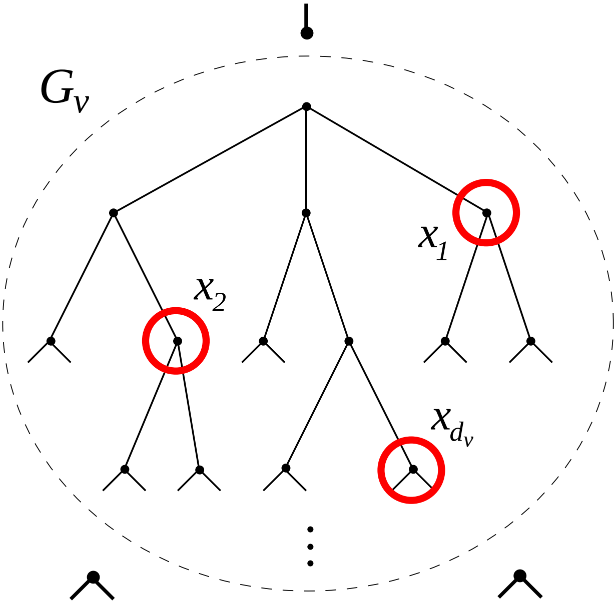
We will replace the vertex v with a graph G_v, built as follows:



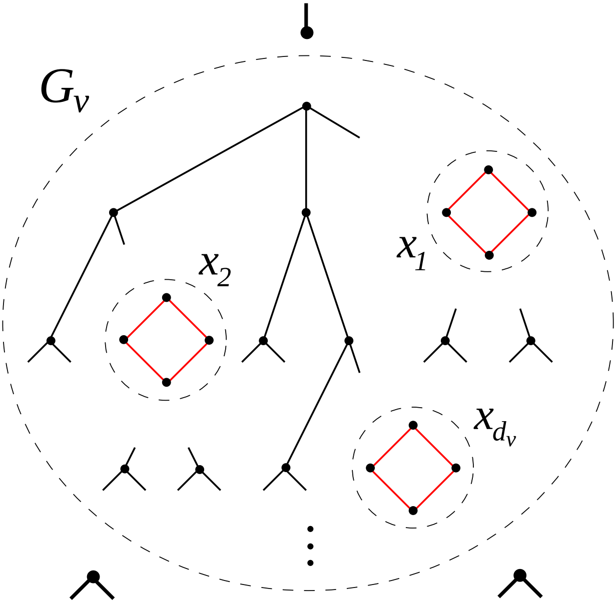
We begin by placing a copy of G (described before):



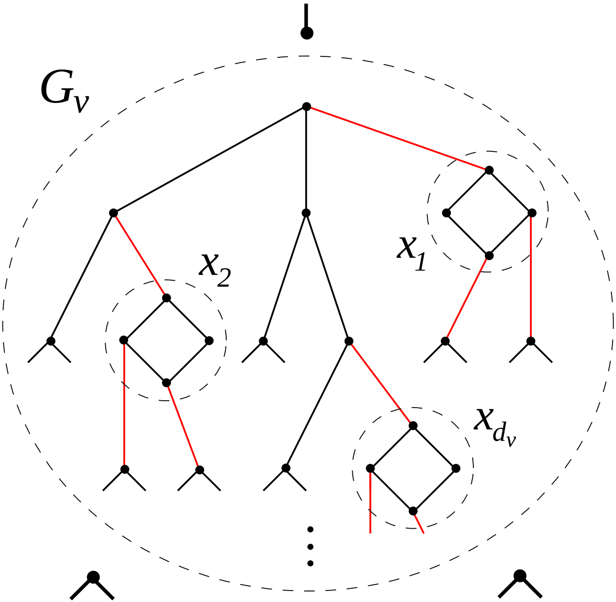
We select d_v vertices of degree 3 in $T \subset G$:



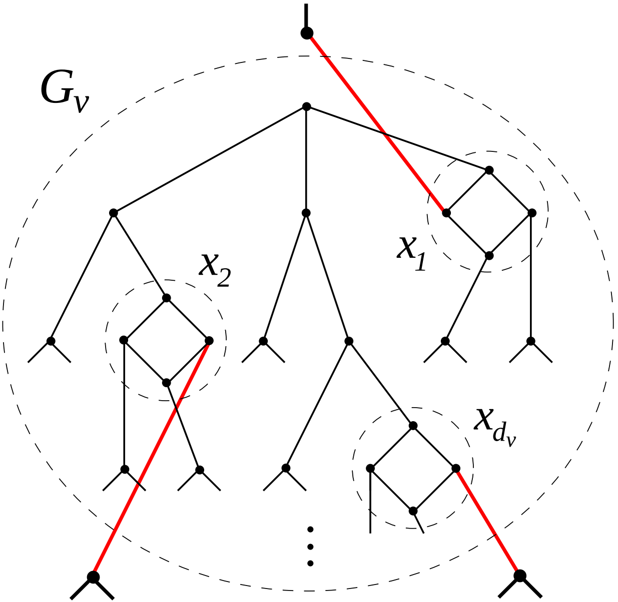
We replace each of these vertices x_i with a C_4:



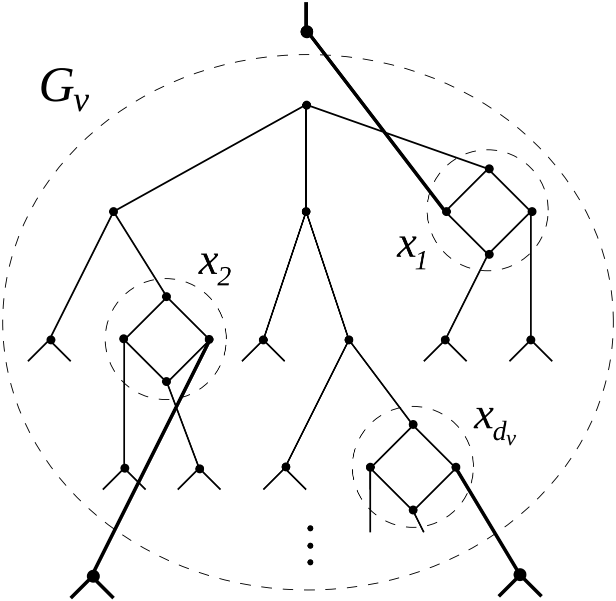
In each C_4, we join 3 of the vertices to the neighbors of x_i:



We join the d_v vertices of degree 2 to the d_v neighbors of v:



This construction for all $v \in G$ defines G_2:



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- Once a vertex in one G_v is chosen $\rightarrow$ $MSMD_3$ in G_v (which is hard up to a constant α)
- But minimize the number of v's for which we touch G_v $\rightarrow$ $MSMD_3$ in G (which is also hard up to a constant α)
- Thus, in G_2 the problem is hard to approximate up to a factor $\alpha \cdot \alpha = \alpha^2$
- Inductively we prove that in G_k the problem is hard to approximate up to a factor α^k

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3- DUAL DEGREE-DENSE k-SUBGRAPH (DDDkS)

Definition of the problem + results

- **DUAL DEGREE-DENSE k-SUBGRAPH (DDDkS)**:

  Input: an undirected graph $G = (V, E)$ and a positive integer k.

  Output: a subset $S \subseteq V$ with $|S| \leq k$, s.t. $\delta(G[S])$ is maximum.

- It is the natural *dual* version of the preceding problem.

- Our results:
  - Randomized $\mathcal{O}(\sqrt{n} \log n)$-approximation algorithm in general graphs.
  - Deterministic $\mathcal{O}(n^{\delta})$-approximation algorithm in general graphs, for some universal constant $\delta < 1/3$.

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  - Approximation algorithms and hardness results in general graphs.
  - **Open:** closing the *huge* complexity gap of MDBCS_d, $d \geq 2$.
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  - Hardness results and an approximation algorithm in minor-free graphs.
  - **Open:** finding approximation algorithms in general graphs for MSMD_d, $d \geq 3$.
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Thanks!