Traffic Grooming in Unidirectional WDM Rings with Bounded Degree Request Graph

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Outline of the talk

- Introduction to traffic grooming
- Definition of the problem
- Example
- **New model**
- Preliminaries
- Case of cubic request graphs
- Summary of results
- Conclusions and further research

Traffic Grooming

Introduction

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s
- Idea

  Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  $\longrightarrow$ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)
- Objectives
  - Better use of bandwidth
  - Reduce the equipment cost (mostly given by electronics)

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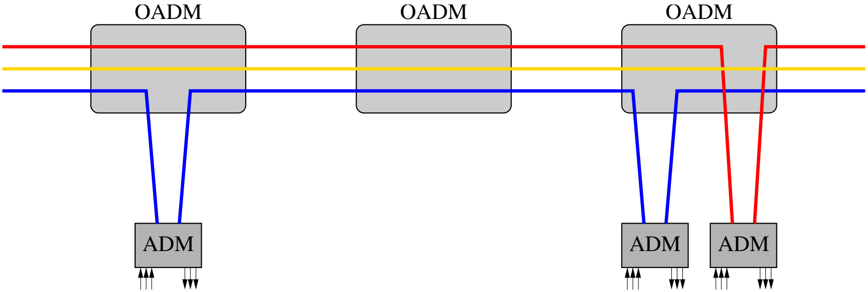
ADM and OADM

- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



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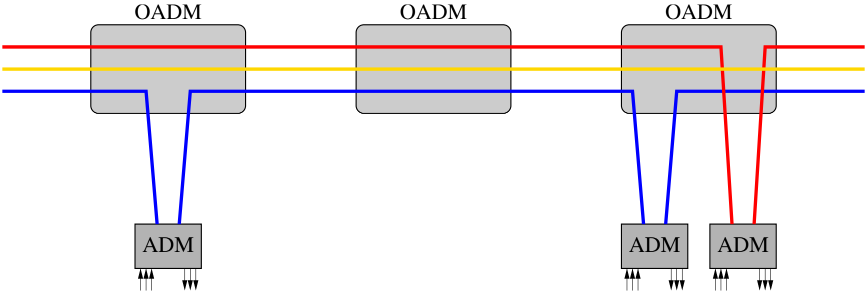

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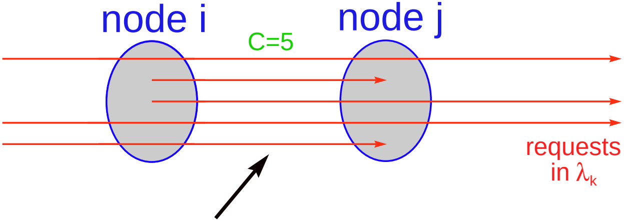

For each wavelenght and each arc between 2 nodes, there can be only C requests routed through this arc

Example:
Capacity of one wavelength $= 2400\ Mb/s$
Capacity used by a request $= 600\ Mb/s$ $\Rightarrow C = 4$

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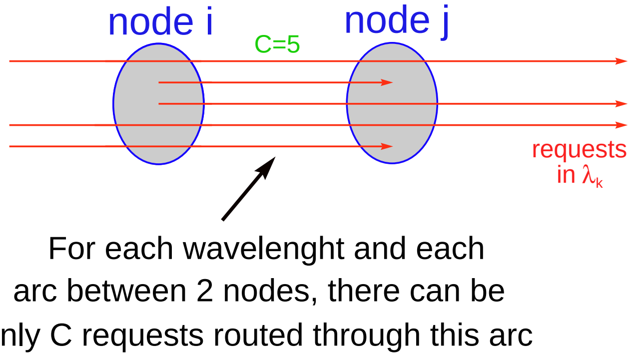

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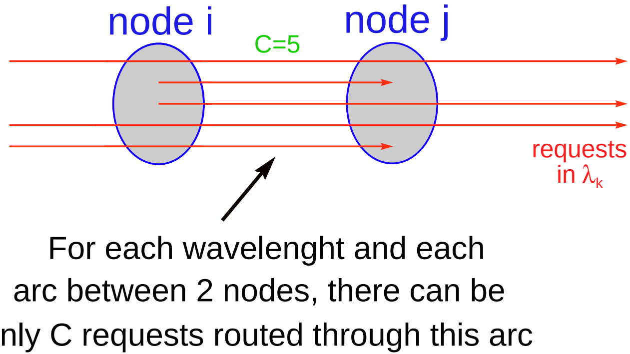

- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

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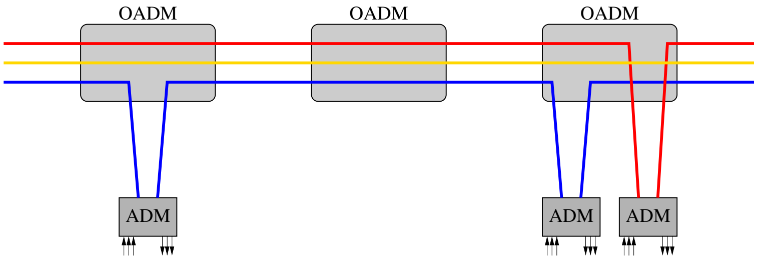

- **Idea:** Use an **ADM only at the endpoints of a request** (lightpaths) in order to save as many ADMs as possible

To fix ideas...

- Model:

Topology	$\rightarrow$	graph G
Request set	$\rightarrow$	graph R
Grooming factor	$\rightarrow$	integer C
Requests in a **wavelength**	$\rightarrow$	edges in a **subgraph** of R
ADM in a wavelength	$\rightarrow$	node in a subgraph of R

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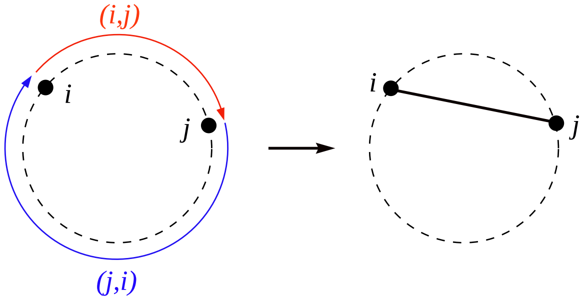
Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: whenever there is the request (i, j), there is also the request (j, i).



- W.l.o.g. the requests (i, j) and (j, i) can be in the same subgraph
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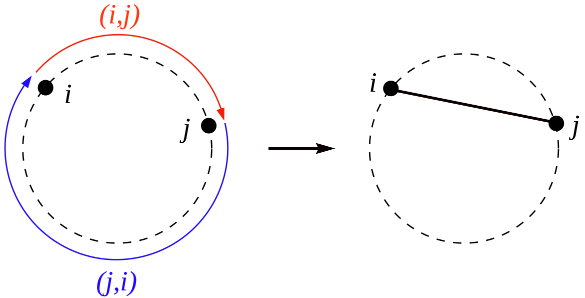

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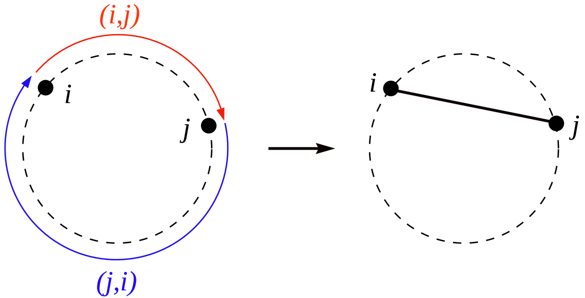

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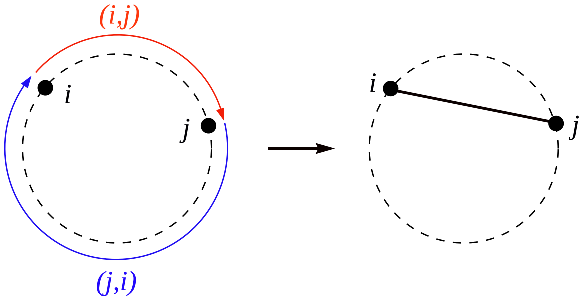

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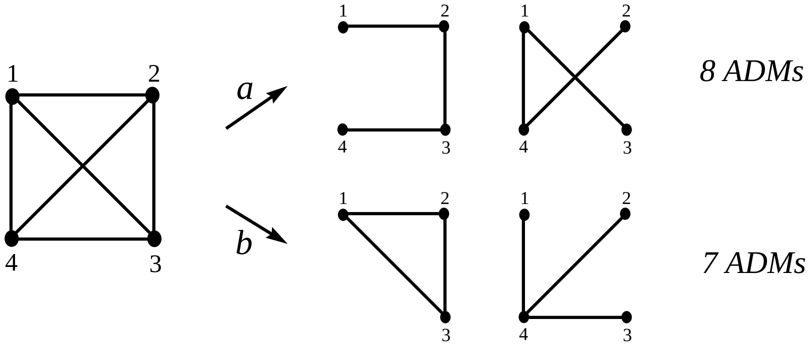
Statement of the "old" problem

Traffic Grooming in Unidirectional Rings

Input	A cycle C_n on n nodes (network); An *undirected* graph R on n nodes (request set); A grooming factor C.
Output	A partition of $E(R)$ into subgraphs $R_1, \ldots, R_W$ with $\|E(R_i)\| \leq C$, i=1...,W.
Objective	Minimize $\sum_{\omega=1}^{W} \|V(R_\omega)\|$.

Example: $n = 4$, $R = K_4$, and $C = 3$



New model

- **Non-exhaustive** previous work (a lot!):
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- In all of them: place ADMs at nodes for a **fixed request graph.**
  $\rightarrow$ placement of ADMs **a posteriori**.
- **New model**: place the ADMs at nodes such that the network can support **any request graph with maximum degree at most Δ.**
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Statement of the "new" problem

Traffic Grooming in Unidirectional Rings with Bounded-Degree Request Graph

Input An integer n (size of the ring);
An integer C (grooming factor);
An integer Δ (maximum degree).

Output An assignment of $A(v)$ ADMs to each node $v \in V(C_n)$, in such a way that **for any graph** R on n nodes with **maximum degree at most** Δ, it exists a partition of $E(R)$ into subgraphs $R_1, \ldots, R_W$ s.t.:

(i) $|E(B_i)| \leq C$ for all $i = 1, \ldots, W$; and
(ii) each vertex $v \in V(C_n)$ appears in $\leq A(v)$ subgraphs.

Objective Minimize $\sum_{v \in V(C_n)} A(v)$,
and the optimum is denoted $A(n, C, \Delta)$.

$M(C, \Delta)$

Definition

Let $M(C, \Delta)$ be the least positive number M such that, for any $n \geq 1$, the inequality $A(n, C, \Delta) \leq Mn$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** v, except for a subset whose size is **independent of** n.
- $M(C, \Delta)$ is always an **integer.**
- Equivalently:

  $M(C, \Delta)$ is the least integer M such that
  the edges of **any** graph with maximum degree at most Δ can be partitioned into subgraphs with at most C edges, in such a way that each vertex appears in at most M subgraphs.
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More formally...

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most Δ.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of partitions of $E(G)$ into subgraphs with at most C edges.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C,\Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- For a general graph G,

$$\min_{P \in \mathcal{P}_C(G)} occ(P) = \min_{P \in \mathcal{P}_C(G)} \left(\max_{v \in V(G)} |\{B \in P : v \in B\}| \right)$$

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Some properties of $M(C, \Delta)$

- W.l.o.g. we can assume that R has **regular degree** Δ.
- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all Δ.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all C.
- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.
- **Lower bound:** $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.
- Cases according to Δ:

$\Delta = 1$: $M(C, 1) = 1$ for all C (trivial).
$\Delta = 2$: $M(C, 2) = 2$ for all C (not difficult).
$\Delta = 3$: Cubic graphs. First "interesting" case...

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- Case $C = 3$:
  - If the request graph is **2-connected** (i.e. bridgeless), by **Petersen's Theorem** (1891) it can be partitioned into a perfect matching + disjoint cycles $\Rightarrow$ the graph can be partitioned into P_4's $\Rightarrow$ each vertex appears in **2** subgraphs.
  - If **not**, consider the following request graph (*next slide*):

    A cubic graph that **cannot** be edge-partitioned into subgraphs with at most 3 edges in such a way that each vertex appears in at most 2 subgraphs $\Rightarrow M(3,3) = 3$.

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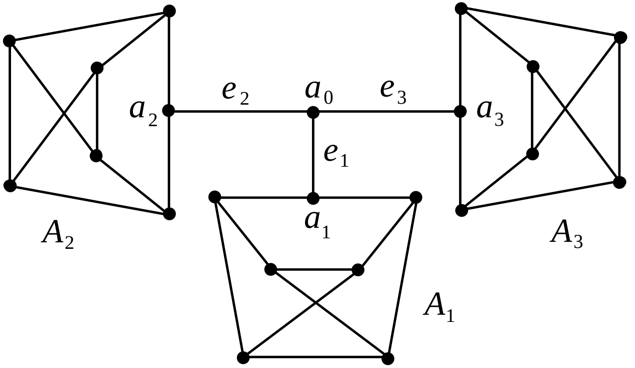

- Consider this cubic graph (which has girth 4).
- Suppose we can partition it into subgraphs with at most 3 edges in such a way that each vertex appears in at most 2 subgraphs.

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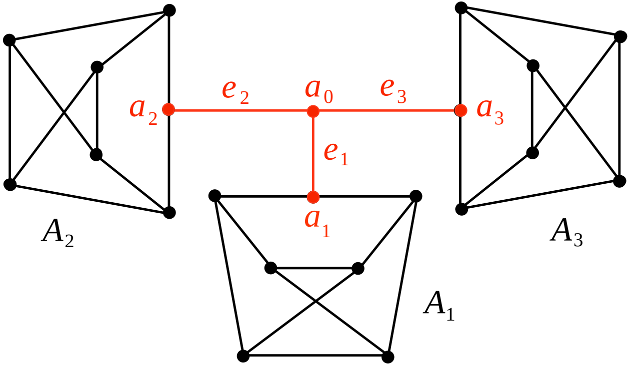

- Consider the subgraphs containing a_0 (i.e. containing the edges e_1, e_2, e_3).
- At least one of them "ends" in one of the vertices a_1, a_2, a_3.

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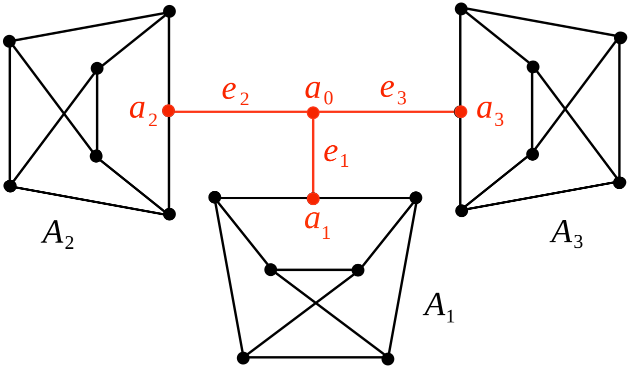

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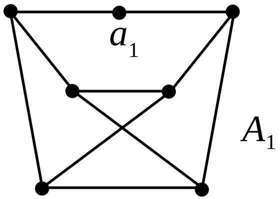
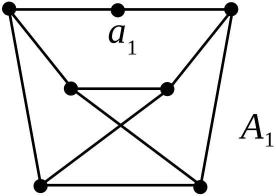


- Let w.l.o.g. a_1 be this vertex.
- Now we have to partition this graph into subgraphs with at most 3 edges in such a way that each vertex except a_1 appears in at most 2 subgraphs **and a_1 appears in exactly 1 subgraph.**

Case $\Delta = 3$, $C = 3$



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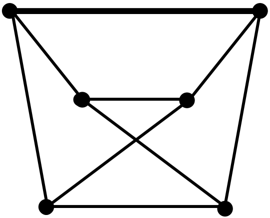
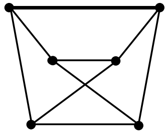


- Equivalently, we have to partition this graph H into subgraphs with at most 3 edges, in such a way that the **thick edge appears in a subgraph with at most 2 edges**, and each vertex appears in at most 2 subgraphs.

Case $\Delta = 3$, $C = 3$



- Let n_1 be the total number of vertices of degree 1 in all the subgraphs of the decomposition of H.
- Since each vertex of H can appear in at most 2 subgraphs and H is cubic, each vertex can appear with degree 1 in at most 1 subgraph. Thus, $n_1 \leq |V(H)| = 6$.
- Since we have to use at least 1 subgraph with at most 2 edges and $|E(H)| = 9$, there are at least $1 + \lceil \frac{9-2}{3} \rceil = 4$ subgraphs in the decomposition of H.
- But each subgraph involved in the decomposition of H has at least 2 vertices of degree 1 (because H is triangle-free). Therefore, $n_1 \geq 8$, a **contradiction**.

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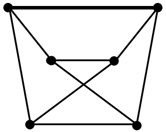

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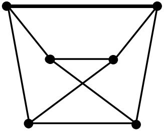

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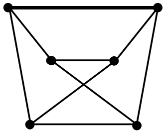

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Summary of results: values of $M(C,\Delta)$

$C \vert \Delta$	1	2	3	4	5	6	7	8	9	...	Δ even	Δ odd
1	1	2	3	4	5	6	7	8	9	...	Δ	Δ
2	1	2	3	3	4	5	6	6	7	...	$\boxed{\frac{3\Delta}{4}}$	$\boxed{\frac{3\Delta}{4}}$
3	1	2	3	3	4	4	5	6	≥ 6	...	$\boxed{\frac{2\Delta}{3}}$	$\geq \frac{2\Delta}{3}$
4	1	2	**2**	3	4	4	5	5	6	...	$\boxed{\frac{5\Delta}{8}}$	$\geq \frac{5\Delta}{8}$
5	1	2	2	3	≥ 3	4	5	5	6	...	$\boxed{\frac{3\Delta}{5}}$	$\geq \frac{3\Delta}{5}$
6	1	2	2	3	≥ 3	4	5	5	6	...	$\boxed{\frac{7\Delta}{12}}$	$\geq \frac{7\Delta}{12}$
7	1	2	2	3	≥ 3	4	≥ 4	5	6	...	$\boxed{\frac{4\Delta}{7}}$	$\geq \frac{4\Delta}{7}$
8	1	2	2	3	≥ 3	4	≥ 4	5	6	...	$\boxed{\frac{9\Delta}{16}}$	$\geq \frac{9\Delta}{16}$
9	1	2	2	3	≥ 3	4	≥ 4	5	≥ 5	...	$\boxed{\frac{5\Delta}{9}}$	$\geq \frac{5\Delta}{9}$
...	...	...	...	...	...	...	...	...	...	...	...	...
C	1	2	2	3	≥ 3	4	≥ 4	5	≥ 5	...	$\boxed{\frac{C+1}{C}\frac{\Delta}{2}}$	$\geq \frac{C+1}{C}\frac{\Delta}{2}$

- Red cases remain open. [case $M(4,3) = 2$ also with *Zhentao Li*].

Summary of results: values of $M(C, \Delta)$

$C \vert \Delta$	1	2	3	4	5	6	7	8	9	...	Δ even	Δ odd
1	1	2	3	4	5	6	7	8	9	...	Δ	Δ
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...	...	...	...	...	...	...	...	...	...	...	...	...
C	1	2	2	3	3	4	4	5	5	...	$\lceil \frac{C+1}{C} \frac{\Delta}{2} \rceil$	$\geq \frac{C+1}{C} \frac{\Delta}{2}$

- Red cases remain open.
- Blue cases only hold if the request graph has a perfect matching.

Conclusions and further research

- We have introduced a new model for **traffic grooming** from a graph-theoretical point of view → **graph partitioning.**
- This formulation allows the network to support dynamic traffic without reconfiguring the electronic equipment at the nodes.
- We have determined the value of $M(C, \Delta)$ when Δ is even, and when $\Delta = 3$ or $C = 2$.
- Also the case when both Δ and C are odd, provided that the request graph has a perfect matching.
- **Further Research:**
  - Determine $M(C, \Delta)$ for $\Delta \geq 5$ odd!
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Thanks!