The number of labeled graphs of bounded treewidth

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1. Introduction

2. The construction

3. Analysis

4. Further research
A \textit{k-tree} is a graph that can be built starting from a \((k + 1)\)-clique and then iteratively adding a vertex connected to a \(k\)-clique.

Example of a 2-tree:
$k$-trees and partial $k$-trees

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$k$-trees and partial $k$-trees

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A partial $k$-tree is a subgraph of a $k$-tree.

A graph has treewidth at most $k$ if and only if it is a partial $k$-tree.
What is known about the number of (partial) $k$-trees?

**Labeled $k$-trees**

![Diagram of labeled $k$-trees]

The number of $n$-vertex labeled trees is $n^{n-2}$.

[Cayley. 1889]

The number of $n$-vertex labeled $k$-trees is $\left(\frac{n}{k}\right) \left(\frac{n}{k} - 1\right)^{n-k-2}$.

[Beineke, Pippert. 1969]

Labeled partial $k$-trees for $k=1$: The number of $n$-vertex labeled forests is $\sim c \cdot n^{n-2}$ for some constant $c > 1$.

[Takács. 1990]

$k=2$: The number of $n$-vertex labeled series-parallel graphs is $\sim g \cdot n^{n-5/2}$ for some constants $g, \gamma > 0$.

[Bohman, Giménez, Kang, Noy. 2005]

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\[ \frac{4}{23} \]
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Let $T_{n,k}$ be the number of $n$-vertex labeled partial $k$-trees.

**Objective** We want to obtain accurate bounds for $T_{n,k}$. 

As an $n$-vertex $k$-tree has $kn - k(k+1)/2$ edges, we get the upper bound:

$$T_{n,k} \leq \binom{n}{k} \cdot (kn - k^2 + 1) n - k - 2 \cdot 2^{kn - k(k+1)/2} \leq \binom{k}{2} \cdot 2^{k} \cdot n^{2} - k(k+1)/2 \cdot 2^{k} - k$$
$T_{n,k}$ and an easy upper bound

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$$\leq (k \cdot 2^k \cdot n)^n \cdot 2^{-\frac{k(k+1)}{2}} \cdot k^{-k}$$
An easy lower bound

Take a forest on $n - (k - 1)$ vertices:

$(n - k + 1)^{n-k-1}$ possibilities
An easy lower bound

Take a forest on \( n - (k - 1) \) vertices:
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Add a vertex arbitrarily connected to the forest:
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T_{n,k} \geq (n - k + 1)^{(n-k-1)} \cdot 2^{(k-1)(n-k+1)} \geq \left(\frac{1}{4} \cdot 2^k \cdot n\right)^n \cdot 2^{-k^2}
\]
Our results

Summarizing, so far we have:

\[ T_{n,k} \leq (k \cdot 2^k \cdot n)^n \cdot 2^{\frac{k(k+1)}{2}} \cdot k^{-k} \]

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Theorem (Baste, Noy, S.)

For any two integers \(n, k\) with \(1 < k \leq n\), the number \(T_{n,k}\) of \(n\)-vertex labeled graphs with treewidth at most \(k\) satisfies

\[ T_{n,k} \geq \left( \frac{1}{128e} \cdot \frac{k \cdot 2^k \cdot n}{\log k} \right)^n \cdot 2^{-\frac{k(k+3)}{2}} \cdot k^{-2k-2}. \]
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A construction to get a “good” lower bound

Trade-off

creating many graphs vs. bounding the number of duplicates
A construction to get a “good” lower bound

Trade-off creating many graphs vs. bounding the number of duplicates

Some ingredients of the construction:

1. Labeling function $\sigma$: permutation of $\{1, \ldots, n\}$ with $\sigma(1) = 1$.

2. We will introduce vertices $\{v_1, v_2, \ldots, v_n\}$ one by one following the order $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}$.

3. If $j < i$, the vertex $v_{\sigma(j)}$ is said to be to the left of $v_{\sigma(i)}$. 
Another graph invariant: **proper-pathwidth**.

[Takahashi, Ueno, Kajitani. 1994]
Another graph invariant: proper-pathwidth.

Proper linear $k$-trees: graphs that can be constructed starting from a $(k + 1)$-clique and iteratively adding a vertex $v_i$ connected to a clique $K_{v_i}$ of size $k$ (called the active vertices), with the constraints that

- $v_{i-1} \in K_{v_i}$.
- $K_{v_i} \setminus \{v_{i-1}\} \subseteq K_{v_{i-1}}$.
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Proper-pathwidth of a graph $G$, denoted $\text{ppw}(G)$: smallest $k$ such that $G$ is a subgraph of a proper linear $k$-tree.
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**Proper linear** \( k \)-trees: graphs that can be constructed starting from a \((k + 1)\)-clique and iteratively adding a vertex \( v_i \) connected to a clique \( K_{v_i} \) of size \( k \) (called the **active vertices**), with the constraints that

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For any graph \( G \) it holds that

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\text{tw}(G) \leq \text{pw}(G) \leq \text{ppw}(G)
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$$\text{tw}(G) \leq \text{pw}(G) \leq \text{ppw}(G)$$

The graphs $G$ we will construct satisfy $\text{tw}(G) \leq \text{pw}(G) \leq \text{ppw}(G) \leq k$.
Ingredients of the construction

For every $i \geq k + 1$ we define:

1. A set $A_i \subseteq \{j \mid j < i\}$ with $|A_i| = k$ of active vertices (as in the definition of proper linear $k$-trees).
Ingredients of the construction

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2. A set $N(i) \subseteq A_i$ with $|N(i)| > \frac{k+1}{2}$: neighbors of $v_{\sigma(i)}$ to the left.
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3. An element \( f(i) \in A_i \cap N(i - 1) \), called the frozen vertex: a vertex that will not be active anymore.
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4. We insert the vertices by consecutive blocks of size $s = s(n, k)$. We will fix the value of $s$ later.
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5. A vertex $a_i \in A_i$, called the anchor: all vertices of the same block are adjacent to the same anchor $a_i$. 

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \ldots & \bullet & \bullet \\
\bullet & \bullet & \bullet & \ldots & \bullet & \bullet \\
\bullet & \bullet & \bullet & \ldots & \bullet & \bullet \\
\bullet & \bullet & \bullet & \ldots & \bullet & \bullet \\
\bullet & \bullet & \bullet & \ldots & \bullet & \bullet \\
\end{array}
\]

\[k + 1 \quad S \quad S \quad S \quad S\]
1. Choose $\sigma$, a permutation of $\{1, \ldots, n\}$ such that $\sigma(1) = 1$.

2. Choose the first $(k + 1)$-clique, with $1 \in N(i)$ for $2 \leq i \leq k + 1$.

3. Define $a_{k+1} = 1$. 
If $i \equiv k + 2 \pmod{s}$ (that is, at the beginning of a block):

- Define $f(i) = a_{i-1}$.
- Define $A_i = (A_{i-1} \setminus \{f(i)\}) \cup \{i - 1\}$.
- Define $a_i = \min A_i$.
- Choose $N(i) \subseteq A_i$ such that $a_i \in N(i)$ and $|N(i)| > \frac{k+1}{2}$.
If $i \not\equiv k + 2 \pmod{s}$ (that is, at the middle of a block):

- Choose $f(i) \in (A_{i-1} \setminus \{a_{i-1}\}) \cap N(i-1)$.
- Define $A_i = (A_{i-1} \setminus \{f(i)\}) \cup \{i-1\}$.
- Define $a_i = a_{i-1}$.
- Choose $N(i) \subseteq A_i$ such that $a_i \in N(i)$ and $|N(i)| > \frac{k+1}{2}$. 

\[
\begin{align*}
|V\sigma(i_1)| & \\
|V\sigma(i_2)| & \\
|V\sigma(i_3)| & \\
|V\sigma(i_4)| & \\
|V\sigma(i_5)| & \\
A_{i-1} & \\
\text{block of } s \text{ vertices} & \\
V\sigma(i) & \\
\end{align*}
\]
Active vertices
Example of the construction (by Julien Baste)

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\[ S \]
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\[ S \]
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$S$
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Active vertices
Active vertices
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Active vertices
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2 The construction

3 Analysis

4 Further research
Analysis of the construction

First note that the graphs $G$ we construct indeed satisfy $\text{ppw}(G) \leq k$. 

How many graphs are created by the construction?

The choices in the construction are the following:

Choices for the permutation $\sigma$:
$$(n-1)!$$

Choices for the neighbourhoods $N(i)$:
$$2^k(k-1)2^{n-(k+1)} \cdot 2^k(k-2)$$

Choices for the frozen vertices $f(i)$:
$$2^{(k-1)2^{n-(k+1)} - \lceil n-(k+1) \rceil}}$$

That is, we create $$(n-1)! \cdot 2^{(k-1)2^{n-(k+1)} - \lceil n-(k+1) \rceil}} \cdot 2^k(k-1)2^{n-(k+1)} \cdot 2^k(k-2)$$ graphs.
Analysis of the construction

- How many **graphs are created** by the construction?
- How many times the **same graph** may have been created?
Analysis of the construction

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- Choices for the frozen vertices $f(i)$: $\left(\frac{k-1}{2}\right)^{(n-(k+1)-\left\lfloor\frac{n-(k+1)}{s}\right\rfloor)}$
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That is, we create

$$(n - 1)! \cdot \left(\frac{k - 1}{2}\right)^{(n-(k+1)-\left\lfloor \frac{n-(k+1)}{s} \right\rfloor)} \cdot 2^{\frac{k(k-1)}{2}} \cdot 2^{(n-(k+1))(k-2)}$$

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How many times the same graph may have been created?

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So given an arbitrary constructible graph \(H\), we want to bound the number of triples \((\sigma, N, f)\) such that \(H = G(\sigma, N, f)\).
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First we reconstruct the permutation \(\sigma\):

- \(\sigma(1) = 1\) and \(f(k + 2) = 1\):
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So the number of possible permutations $\sigma$ that give rise to $H$ is at most

$$k! \cdot (s!)^\left\lceil \frac{n-(k+1)}{s} \right\rceil$$
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Secondly, we reconstruct the neighborhood $N(i)$:
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Secondly, we reconstruct the neighborhood $N(i)$:

uniquely determined once $\sigma$ is fixed.
Reconstruction of the frozen vertex $f(i)$

We define, for $i > 1$, $D_i$ as the set of neighbors of $i$ that will never have any neighbor among the non-introduced vertices. $f(i) \in D_i - 1$. $|D_i| \geq 1$ and $D_i \cap D_j = \emptyset$ for $i \neq j$.

$\sum_{n_i = k + 1}^{n} |D_i| \leq n$.

Let $I = \{i \in \{k + 1, \ldots, n\} || |D_i| \geq 2\}$, and note that $|I| \leq k$.

It holds that $\sum_{i \in I} |D_i| \leq 2k$.

The number of distinct functions $f$ is at most $n \prod_{i = k + 1}^{n} |D_i| = \prod_{i \in I} |D_i| \leq (\sum_{i \in I} |D_i|)^k \leq (2k)^k = 2k^k$.

So, the number of triples $(\sigma, N, f)$ such that $H = G(\sigma, N, f)$ is at most $2k \cdot k! \cdot (s!) \lceil n - (k + 1)s \rceil$. \[18/23\]
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18/23
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$$\prod_{i=k+1}^{n} |D_i|$$
Reconstruction of the frozen vertex \( f(i) \)

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$$2^{k} \cdot k! \cdot (s!)^{\lceil \frac{n-(k+1)}{s} \rceil}$$
The number of distinct graphs we have created is at least

\[
\frac{\text{number of created graphs}}{\text{number of duplicates}} \geq \frac{(n-1)!}{(k-1)^2 n - (k+1)^2 - \left\lceil n - (k+1) \right\rceil \cdot 2^k \cdot k \cdot (s!) \cdot \left\lfloor n - (k+1) \right\rfloor \cdot \left( n - (k+1) - s \left\lfloor n - (k+1) \right\rfloor \right)^{k-2} \cdot k!}}.
\]
Analysis of the construction

The number of distinct graphs we have created is at least

\[
\frac{\text{number of created graphs}}{\text{number of duplicates}} \geq (n - 1)! \cdot (\frac{k-1}{2})^{n-(k+1)} - \left\lfloor \frac{n-(k+1)}{s} \right\rfloor \cdot 2^{\frac{k(k-1)}{2}} \cdot 2^{(n-(k+1))(k-2)} \cdot 2^k \cdot k! \cdot (s!)^{\left\lfloor \frac{n-(k+1)}{s} \right\rfloor} \cdot (n - (k + 1) - s\left\lfloor \frac{n-(k+1)}{s} \right\rfloor)!.
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\]

\[
\geq \ldots \geq \left(\frac{1}{64e} \cdot \frac{k \cdot 2^{k} \cdot n}{k^s \cdot s}\right)^n \cdot 2^{-\frac{k(k+3)}{2}} \cdot k^{-2k-2}.
\]
Choice of the block size $s$

We want the value of $s = s(n, k)$ that minimizes $k^{\frac{1}{s}} \cdot s$. 

And the minimum of $1 + \log t(n, k) + \log \log k$ is reached for $t(n, k) = 1$. 

So $s(n, k) = \log k$ is the best choice for the block size.
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Next section is...

1. Introduction

2. The construction

3. Analysis

4. Further research
Further research

\[ T_{n,k} \leq (k \cdot 2^k \cdot n)^n \cdot 2^{-\frac{k(k+1)}{2}} \cdot k^{-k}. \]

\[ T_{n,k} \geq \left( \frac{1}{128e} \cdot \frac{k \cdot 2^k \cdot n}{\log k} \right)^n \cdot 2^{-\frac{k(k+3)}{2}} \cdot k^{-2k-2}. \]
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- We believe that there exist an absolute constant \( c > 0 \) and a function \( f(k) \), with \( k^{-2k-2} \leq f(k) \leq k^{-k} \) for every \( k > 0 \), such that

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- Improve the upper bound for pathwidth or proper-pathwidth?

- Other relevant parameters: branchwidth, cliquewidth, rankwidth, tree-cutwidth, booleanwidth, ...
Gràcies!